Faculty of Applied Mathematics

University of Twente

University for Technical and Social Sciences

P.O. Box 217
7500 AE Enschede
The Netherlands
Phone: +31-53-4893400
Fax: +31-53-4893114
Email:
memo@math.utwente.nl

Memorandum No. 1368

Long D_3-cycles in graphs with large minimum degree

H. Trommel

February 1997

ISSN 0169-2690

Long D_3-cycles in graphs with large minimum degree

H. Trommel

Faculty of Applied Mathematics, University of Twente
P.O. Box 217, 7500 AE Enschede, The Netherlands

Abstract

It is shown that if G is a 2-connected graph on n vertices, with minimum degree δ such that $n \leq 4\delta - 6$, and with a maximum independent set of cardinality α, then G contains a cycle of length at least $\min \{n, n + 2\delta - 2\alpha - 2\}$ or $G \in \mathcal{F}$, where $\mathcal{F}$ denotes a well-known class of nonhamiltonian graphs of connectivity 2.

Keywords: graph, (long, Hamilton) cycle

AMS Subject Classifications (1991): 05C45, 05C38, 05C35

We use [1] for terminology and notation not defined here and consider finite simple graphs only.

Let G be a graph and k a positive integer. We denote by $n(G)$ the number of vertices of G, by $\alpha(G)$ the number of vertices in a maximum independent set of G, and by $\omega(G)$ the number of components of G. The number of components of G with at least k vertices is denoted by $\omega_k(G)$. When no confusion can arise, we will often write $n, \alpha, V, \ldots$ instead of $n(G), \alpha(G), V(G), \ldots$

Let C be a cycle of G and $u, v \in V(C)$. We denote by $\overrightarrow{C}$ the cycle C with a given orientation. By $u\overrightarrow{C}v$ we denote the set of consecutive vertices of C from u to v in the direction specified by $\overrightarrow{C}$. We use u^+ to denote the successor of u on $\overrightarrow{C}$ and u^- to denote its predecessor. We call C a *D_k-cycle* if $\omega_k(G - V(C)) = 0$. Similar notation is used for paths.

If $v \in V(G)$ and $A \subseteq V(G)$, then $\varepsilon(v, A)$ denotes the number of vertices in A which are adjacent to v. If H is a subgraph of G and $v \in V(G)$ then $N_H(v) = \{x \in V(H) \mid xv \in E(G)\}$.

We now define a number of specific graphs and classes of graphs. For a positive integer k we denote by $\mathcal{K}_k$ the set of all graphs consisting of three disjoint complete graphs, where each of the components has order at least k. The class $\mathcal{G}$ will be the set of all spanning subgraphs of graphs that can be obtained as the join of K_2 and a graph in $\mathcal{K}_1$. The class $\mathcal{H}$ is the set of all spanning subgraphs of graphs that can be obtained from the join of K_1 and a graph H

in $\mathcal{K}_2$ by adding the edges of a triangle between three vertices from distinct components of H. The class $\mathcal{J}$ is the set of all spanning subgraphs of graphs that can be obtained from a graph H in $\mathcal{K}_3$ by adding the edges of two triangles between two disjoint triples of vertices, each containing one vertex of each component of H. We set $\mathcal{F} = \mathcal{G} \cup \mathcal{H} \cup \mathcal{J}$. The class $\mathcal{F}$ is a well-known class of nonhamiltonian graphs. Note that all graphs in $\mathcal{F}$ have connectivity at most 2.

A generalization of the following result occurs in [3].

Theorem 1 ([3])
If G is a 2-connected graph with $n \leq 3\delta - 2$, then G contains a cycle of length at least $min\{n, n + \delta - \alpha\}$.

In a sense, Theorem 1 is best possible. The complete bipartite graph $K_{p,q}$ $(p \geq 2, q \geq p + 1)$ contains a longest cycle of length $2p = n + \delta - \alpha$. Theorem 1 is also best possible in the sense that the upper bound $3\delta - 2$ imposed on n is tight, as shown by the graph $K_2 \vee 3K_p$ $(p \geq 2)$ with $n = 3\delta - 1$, which has a longest cycle of length $2p + 2 < \min\{n, n + \delta - \alpha\} = \min\{3p + 2, 4p\}$. Nevertheless it is possible to obtain a lower bound for the length of a longest cycle in graphs with $n > 3\delta - 2$ outside the class $\mathcal{F}$. We will prove the following result.

Theorem 2
If G is a 2-connected graph with $n \leq 4\delta - 6$, then G contains a cycle of length at least $min\{n, n + 2\delta - 2\alpha - 2\}$ or $G \in \mathcal{F}$.

In a sense, Theorem 2 is best possible. The graph $K_p \vee qK_2$ $(p \geq 2, q \geq p + 1)$ has a longest cycle of length $3p = n + 2\delta - 2\alpha - 2$. Again, the upper bound $4\delta - 6$ imposed on n is tight, as shown by the graph $K_3 \vee 4K_p$ $(p \geq 3)$ with $n = 4\delta - 5$, which has a longest cycle of length $3p + 3 < \min\{n, n + 2\delta - 2\alpha - 2\} = \min\{4p + 3, 6p - 3\}$.

Theorem 2 generalizes the following result from [5].

Corollary 3 ([5])
If G is a 2-connected graph with $n \leq 4\delta - 6$ and $\alpha \leq \delta - 1$, then G is hamiltonian or $G \in \mathcal{F}$.

It should be noted that for nonhamiltonian graphs with $n \leq 3\delta - 2$ the lower bound for the length of a longest cycle in Theorem 1 is always strictly greater than the one in Theorem 2.

To prove Theorem 2, we use the following result.

Theorem 4 ([5])
If G is a 2-connected graph with $n \leq 4\delta - 6$, then G contains a D_3-cycle or $G \in \mathcal{F}$.

Proof of Theorem 2
Let G be a 2 connected graph with $n \leq 4\delta - 6$ and assume $G \notin \mathcal{F}$. By Theorem 4 we know that G contains a D_3-cycle. Let C be a longest D_3-cycle such that $\omega_2(G - V(C)) \geq \omega_2(G - V(C'))$ for every longest D_3-cycle C'. Fix an orientation on C. We distinguish two cases.

Case 1 C is not a D_2-cycle.
Then $G - V(C)$ contains at least one component of cardinality 2. Let x and y be the vertices of such a component and let $u_1, \dots, u_k$ be the vertices in $N_C(x) \cup N_C(y)$, ordered cyclically around C. For every segment $u_i\overrightarrow{C}u_{i+1}$ (subscripts taken modulo k) we will mark two or three vertices and we will call this set of marked vertices S_i.

If u_i has only one neighbor in $\{x, y\}$ we will call the segment $u_i\overrightarrow{C}u_{i+1}$ a 1-segment. In that case we will mark two vertices according to the first feasible rule out of the following.

1. $s_{i1} = u_i^+$, $s_{i2} \in N(s_{i1})\backslash V(C)$.

2. $s_{i1} = u_i^+$, $s_{i2} = u_i^{++}$.

If u_i has both x and y as neighbors we will call $u_i\overrightarrow{C}u_{i+1}$ a 2-segment. In that case we will mark three vertices according to the first feasible rule out of the following.

3. $s_{i1} = u_i^+$, $s_{i2} \in N(s_{i1})\backslash V(C)$, $s_{i3} \in N(s_{i2})\backslash V(C)$.

4. $s_{i1} = u_i^+$, $s_{i2} \in N(s_{i1})\backslash V(C)$, $s_{i3} = u_i^{++}$.

5. $s_{i1} = u_i^+$, $s_{i2} = u_i^{++}$, $s_{i3} \in N(s_{i2})\backslash V(C)$.

6. $s_{i1} = u_i^+$, $s_{i2} = u_i^{++}$, $s_{i3} = u_i^{+++}$.

By the way C and the marked sets are chosen, $S_i \subseteq u_i^+\overrightarrow{C}u_{i+1} \cup (V(G)\backslash V(C))$ for every marked set S_i.

In the proof we will often use the following path to construct a cycle which contradicts the choice of C. Let Q_{ij} be a longest (u_i, u_j)-path which has all its internal vertices in $\{x, y\}$. Note that if $u_i\overrightarrow{C}u_{i+1}$ or $u_j\overrightarrow{C}u_{j+1}$ is a 2-segment, then Q_{ij} always contains both x and y.

Claim 1 *$S_i \cap S_j = \emptyset$ for all $i, j = 1, \dots, k$ with $i \neq j$.*

Proof Assuming the contrary, let $v \in S_i \cap S_j$. Since $S_i \subseteq u_i^+\overrightarrow{C}u_{i+1} \cup (V(G)\backslash V(C))$ for every marked set S_i, we have $v \in V(G)\backslash V(C)$. If there exists a (u_i^+, u_j^+)-path P internally disjoint from C, then clearly $x, y \notin V(P)$ and the cycle $u_i\overrightarrow{Q}_{ij}u_j\overleftarrow{C}u_i^+\overrightarrow{P}u_j^+\overrightarrow{C}u_i$ contradicts the choice of C (since it is a longer D_3-cycle than C).

Now we are left with the case that S_i or S_j is of type 5. By symmetry we may assume that S_j is of type 5. If S_i is of type 1, 3 or 4, and $v = s_{i2} = s_{j3}$, then the cycle $u_i\overrightarrow{Q}_{ij}u_j\overleftarrow{C}u_i^+s_{i2}u_j^{++}\overrightarrow{C}u_i$ contradicts the choice of C. If S_i is of type 3 and $v = s_{i3} = s_{j3}$, then the cycle $u_i\overrightarrow{Q}_{ij}u_j\overleftarrow{C}u_i^+s_{i2}s_{i3}u_j^{++}\overrightarrow{C}u_i$ contradicts the choice of C. If S_i and S_j are both of type 5, then $v = s_{i3} = s_{j3}$ and the cycle $u_i\overrightarrow{Q}_{ij}u_j\overleftarrow{C}u_i^{++}s_{i3}u_j^{++}\overrightarrow{C}u_i$ contradicts the choice of C. □

Claim 2 *$N(u_i^+) \subseteq V(C)$ for all $i = 1, \dots, k$.*

Proof Assume that u_1^+ has a neighbor $v \in V(G)\backslash V(C)$. By showing that v does not have neighbors in the marked sets $S_2, \ldots, S_k$, we will get a contradiction with the degree condition. Since $S_1 \cap S_j = \emptyset$ for all $j = 2, \ldots, k$ and since there does not exist a (u_1^+, u_j^+)-path internally disjoint from C for all $j = 2, \ldots, k$, we are left with the following cases.

a. S_j of type 2, $vs_{j2} \in E(G)$.

b. S_j of type 4, $vs_{j3} \in E(G)$.

c. S_j of type 5, $vs_{j2} \in E(G)$.

d. S_j of type 5, $vs_{j3} \in E(G)$.

e. S_j of type 6, $vs_{j3} \in E(G)$.

Note that the case 'S_j of type 6 with $vs_{j2} \in E(G)$' does not occur, since then S_j would have been a marked set of type 5.

In Case a the cycle $C' = u_1\overrightarrow{Q}_{1j}u_j\overleftarrow{C}u_1^+vs_{j2}\overrightarrow{C}u_1$ contradicts the choice of C, since C' is a longer D_3-cycle. In Case b, c and d we can find similar cycles. In Case e the cycle $C' = u_1\overrightarrow{Q}_{1j}u_j\overleftarrow{C}u_1^+vs_{j3}\overrightarrow{C}u_1$ contradicts the choice of C, since Q_{1j} contains both x and y (since S_j is a 2-segment), and thus C' is a longer D_3-cycle.

Now let p be the number of 2-segments. Then C has $d(x) + d(y) - 2 - 2p$ 1-segments and p is at most $d(x) - 1$. Since $S_i \cap S_j = \emptyset$ for all $i, j = 1, \ldots, k$ with $i \neq j$, we get

$$\begin{aligned} |\cup_{i=2}^k S_i| &\geq |\cup_{i=1}^k S_i| - 3 = 3p + 2(d(x) + d(y) - 2 - 2p) - 3 \\ &\geq d(x) + 2d(y) - 6 \geq 3\delta - 6. \end{aligned}$$

Using the fact that v does not have neighbors in $S_2, \ldots, S_k$ and $v \notin \cup_{i=2}^k S_i$ (since we may assume that $v = s_{12}$), and since $n \leq 4\delta - 6$ we get

$$d(v) \leq n - 1 - |\cup_{i=2}^k S_i| \leq n - 3\delta + 5 \leq \delta - 1,$$

a contradiction. □

Now consider the set consisting of a vertex from each component of $G - V(C)$ and the vertices $u_1^+, \ldots, u_k^+$. This set is an independent set. Since C is a D_3-cycle, $G - V(C)$ has at least $(n - |V(C)|)/2$ components. Since $k \geq \delta - 1$, we get

$$\alpha \geq \delta - 1 + \frac{n - |V(C)|}{2},$$

from which it follows that

$$|V(C)| \geq n + 2\delta - 2\alpha - 2.$$

Case 2 C is a D_2-cycle.
Assuming G is nonhamiltonian, let $x \in V(G)\backslash V(C)$ and let $u_1, \ldots, u_k$ be the vertices in $N_C(x)$, ordered cyclically around C. As in Case 1 we will mark two or three vertices for every segment $u_i\overrightarrow{C}u_{i+1}$. If $u_i^{++} = u_{i+1}$ and $N(u_i^+) \subseteq V(C)$ we define $S_i = \{u_i^+, u_i^{++}\}$ and call this set a set of type 1. In all other cases we will mark three vertices according to the first feasible rule out of the following.

2. $s_{i1} = u_i^+$, $s_{i2} \in N(s_{i1})\backslash V(C)$, $s_{i3} = u_i^{++}$.
3. $s_{i1} = u_i^+$, $s_{i2} = u_i^{++}$, $s_{i3} \in N(s_{i2})\backslash V(C)$.
4. $s_{i1} = u_i^+$, $s_{i2} = u_i^{++}$, $s_{i3} = u_i^{+++}$.

Claim 3 *If S_i and S_j $(i \neq j)$ are not both of type 3, then $S_i \cap S_j = \emptyset$*

The proof of Claim 3 is similar to the proof of Claim 1.

Claim 4 *$N(u_i^+) \subseteq V(C)$ for all $i = 1, \ldots, k$.*

Proof Assume that u_1^+ has a neighbor $v \in V(G)\backslash V(C)$. Showing that v does not have neighbors in the marked sets $S_2, \ldots, S_k$ will not be enough, since there may be an overlap between marked sets of type 3. However, given the vertex v, we can modify $S_2, \ldots, S_k$ to a collection $S'_2, \ldots, S'_k$ of marked sets, such that v does not have neighbors in $S'_2, \ldots, S'_k$, $|S'_i| = |S_i|$ for all $i = 1, \ldots, k$ and $S'_i \cap S'_j = \emptyset$ for all $i, j = 1, \ldots, k$ with $i \neq j$.

Consider a maximal collection of t marked sets of type 3, with $t \geq 2$, which all have the same vertex $z \in V(G)\backslash V(C)$ in common. Let $S'_i = (S_i \cup \{u_i^{+++}\})\backslash\{z\}$ for the first $t-1$ sets and $S'_i = S_i$ for the remaining set. Let u^* be the vertex in $N_C(x)$ corresponding to this remaining set. Note that $u_i^{+++} \notin S_{i+1}$, otherwise S_i would have been a set of type 1. We also see that $vu_i^{+++} \notin E(G)$, otherwise the cycle $u_1xu_*\overleftarrow{C}u_i^{+++}vu_1^+\overrightarrow{C}u_i^{++}zu_*^{++}\overrightarrow{C}u_1$ would be a longer D_3-cycle than C.

We repeat this procedure for all maximal collections of at least two marked sets of type 3, with the same vertex in $V(G)\backslash V(C)$ in common. Finally, for all marked sets which have not been modified yet, we define $S'_i = S_i$.

Using similar arguments as in the proof of Claim 2 we can prove that v does not have neighbors in $S'_2, \ldots, S'_k$. If S_i is of type 4 and if $vs_{i3} \in E(G)$, the cycle $u_1xu_i\overleftarrow{C}u_1^+vs_{i3}\overrightarrow{C}u_1$ would be a D_3-cycle of the same length as C, but with more components of order 2 outside of it (one instead of zero). In all other cases we can construct longer D_3-cycles.

Let $T_1, \dots, T_m$ be the segments of C remaining after deleting $S'_2, \dots, S'_k$ and let $T = \cup_{i=1}^m T_i$. Since v does not have two consecutive vertices of C as its neighbors, we have $\varepsilon(v, T_i) \le \frac{1}{2}(|T_i| + 1)$. Let p be the number of segments of type 1. Since $x, v \notin V(C)$, we get

$$|T| \le n - 2p - 3(d(x) - p - 1) - 2.$$

Thus, using that $n \le 4\delta - 6$, we have

$$\begin{aligned} d(v) &= \varepsilon(v, T) = \sum_{i=1}^{m} \varepsilon(v, T_i) \le \tfrac{1}{2} \sum_{i=1}^{m} (|T_i| + 1) = \tfrac{1}{2}|T| + \tfrac{1}{2}m \\ &\le \tfrac{1}{2}(n - 2p - 3(d(x) - p - 1) - 2) + \tfrac{1}{2}(d(x) - p) \\ &= \tfrac{1}{2}n - d(x) + \tfrac{1}{2} \le \tfrac{1}{2}n - \delta + \tfrac{1}{2} \le \delta - \tfrac{5}{2}, \end{aligned}$$

a contradiction. □

Using Claim 4, we know that the set consisting of all vertices of $G - V(C)$, together with $u_1^+, \dots, u_k^+$, is an independent set. Since $k \ge \delta$ we get

$$\alpha \ge \delta + n - |V(C)| \ge \delta - 1 + \frac{n - |V(C)|}{2},$$

from which it follows that

$$|V(C)| \ge n + 2\delta - 2\alpha - 2.$$

■

The proof of Theorem 2 is quite long and can be considerable shortened if we use the following result independently obtained by Brandt and Jung.

Theorem 5 ([2] , [4])
If G is a 2-connected graph with $n \le 4\delta - 6$, then every longest cycle of G is a D_3-cycle or $G \in \mathcal{F}$.

If we use Theorem 5, we can distinguish the cases 'G has a longest cycle which is a D_3-cycle and not a D_2-cycle' and 'every longest cycle of G is a D_2-cycle'. Instead of marking two or three vertices as in our proof of Theorem 2, we can now simply mark u_i^+ and u_i^{++}, or u_i^+, u_i^{++} and u_i^{+++}. The rest of the proof is similar to ours.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*. Macmillan, London and Elsevier, New York (1976).

[2] S. Brandt, Personal communication.

[3] D. Bauer, A. Morgana, E. F. Schmeichel and H. J. Veldman, *Long cycles in graphs with large degree sums*. Discrete Math. **79** (1989/1990) 59–70.

[4] H. A. Jung, Personal communication.

[5] H. J. Veldman, *D_λ-cycles in 3-cyclable graphs*. Graphs and Combinatorics **11** (1995) 379–388.