Test regions using two or more correlated product characteristics

Willem Albers, Gerda R.J. Arts and Wilbert C.M. Kallenberg
Department of Applied Mathematics
University of Twente
P.O. Box 217, 7500 AE Enschede
The Netherlands

Abstract

For inspection of manufactured parts, one can use the information of two or more product characteristics that are strongly related to the characteristic of interest. Under the condition that at most a given, typically very small, fraction of the accepted parts does not satisfy the specification limit, test regions are determined such that the number of accepted products is maximized. The methods are illustrated by Monte Carlo results and a numerical example from semiconductor industry.

Keywords and phrases: consumer risk, yield, inspection, simulations

1991 Mathematics Subject classification: 62 H 20, 62 N 10

1 Introduction

In many large scale production processes all products are subjected to a test procedure. In principle, the characteristic which has to satisfy some specification limit, should be measured itself, but it might be (much) cheaper to measure instead one or more correlated characteristics. In that case one uses the correlated measurements to control the characteristic of interest.

A natural criterion for setting a test limit is its consumer risk (CR), the probability that an accepted product is nonconforming. For instance, requiring that $CR \leq 100$ ppm (parts per million) means that of all products the consumer will get about 1 in ten thousand is nonconforming the specification limit. Another classical criterion is the so-called consumer loss (CL), the probability that a product is nonconforming and accepted. When the yield (YD), which is the probability of accepting a product, is large, there is not much difference between CL and CR ($=CL/YD$). For direct measurements, as a rule YD is pretty high and it does not matter which criterion is applied and we may simply use CL. However, in case of correlated measurements a lower YD may be acceptable if direct measurements are much more expensive. Therefore, in that case the slightly more complicated CR is considered as well, cf. Albers, Arts and Kallenberg (1995a,b).

Due to the large scale of the production processes considered here, a gain in yield is very profitable. Such a gain of yield may be obtained by using not only one, but two or more correlated measurements. Using CR as criterion we construct a test region such that the yield is maximized. The construction of such a test region consists of two parts. First we show what kind of region should be used, secondly we determine specific limits for that region.

In section 2 we determine the test region using a model with known parameters. This is the starting point for the more realistic situation where parameters are unknown. Then we have to estimate them, which leads to stochastic test regions. This in its turn results in a stochastic consumer risk $\widehat{CR}$. In section 3 we show how the test region determined in section 2, should be modified if parameters are unknown. This modification is needed to ensure that the consumer risk is unbiased to high precision. Monte Carlo results show that the modification works very well and that an important gain in yield can be obtained. In section 4 a numerical example from semiconductor industry is presented, to show in a practical situation that substantial improvements can be made, using two instead of one correlated characteristic.

2 Test region if parameters are known

The characteristic of interest can be measured by

$$\tilde{X} = X + U,$$

where X is the true measurement and U the measurement error. We assume that X and U are independent and normally distributed: $X \sim N(\mu_X, \sigma_X^2)$ and $U \sim N(0, \sigma_U^2)$. A product is called defective if $X > s$, where s is the specification limit. In general $\pi = P(X > s)$ will lie in $(0, 0.15]$. The k correlated characteristics can be measured by Y_1, Y_2,..., Y_k and we assume that these measurements depend linearly on X,

$$Y_l = \alpha_l + \beta_l X + Z_l, \;\; \forall l = 1, ..., k. \tag{2.1}$$

This model is a so-called measurement error model, cf. Fuller (1987). The variable Y_l depends on a variable which cannot be observed itself, but can only be measured with a measurement error. We assume that the error term Z_l in (2.1) is independent of X and U for all $l = 1, ..., k$ and that the Z_l are independent normally distributed: $Z_l \sim N(0, \sigma_{Z_l}^2)$. So $(Y_1, Y_2, ..., Y_k)$ is jointly normally distributed with

$$\begin{array}{lcll} \mathrm{E}Y_l & = & \mu_{Y_l} = \alpha_l + \beta_l \mu_X & \forall l = 1, ..., k, \\ \mathrm{Var}(Y_l) & = & \sigma_{Y_l}^2 = \beta_l^2 \sigma_X^2 + \sigma_{Z_l}^2 & \forall l = 1, ..., k, \\ \mathrm{Cov}(Y_l, Y_{l'}) & = & \beta_l \beta_{l'} \sigma_X^2 & \forall l \neq l'. \end{array}$$

If $(Y_1, Y_2 ..., Y_k) \in T$, where $T \subset \mathbb{R}^k$ is the test region, we accept the corresponding item. The consumer risk is equal to

$$CR = P(X > s | (Y_1, Y_2, ..., Y_k) \in T)$$

and the yield is equal to

$$YD = P((Y_1, Y_2, ..., Y_k) \in T).$$

The CR should fall below a prescribed bound γ, which is typically quite small (10-100 ppm). The following lemma presents the test region which gives the optimal yield under the condition $CR \leq \gamma$. It turns out that for a specific linear combination of $(Y_1, Y_2, ..., Y_k)$ the optimal test region is the region where this linear combination falls below some test limit t.

Lemma 2.1. Define the test region

$$T^* = \left\{ (y_1, y_2, ..., y_k) \in \mathbb{R}^k \middle| \sum_{l=1}^{k} \frac{\beta_l}{\sigma_{Z_l}^2} y_l < t \right\} \tag{2.2}$$

with $CR^* = P(X > s | (Y_1, Y_2, ..., Y_k) \in T^*) = \gamma$. For all regions $T \subset \mathbb{R}^k$ that satisfy $CR = P(X > s | (Y_1, Y_2, ..., Y_k) \in T) \leq \gamma$, we have

$$YD = P((Y_1, Y_2, ..., Y_k) \in T) \leq P((Y_1, Y_2, ..., Y_k) \in T^*) = YD^*. \tag{2.3}$$

Proof. See the Appendix. □

This lemma shows that we should accept an item if the linear combination $Y = \sum_{l=1}^{k} \frac{\beta_l}{\sigma_{Z_l}^2} Y_l$ falls below a test limit t. The next step is to determine the test limit t such that the consumer risk equals γ. Define

$$\begin{aligned} \alpha &= \sum_{l=1}^{k} \frac{\beta_l}{\sigma_{Z_l}^2}\alpha_l, \;\; \beta = \sum_{l=1}^{k} \frac{\beta_l}{\sigma_{Z_l}^2}\beta_l, \;\; Z = \sum_{l=1}^{k} \frac{\beta_l}{\sigma_{Z_l}^2} Z_l \\ \sigma_Z^2 &= \mathrm{Var}(Z) = \sum_{l=1}^{k} \frac{\beta_l^2}{\sigma_{Z_l}^2} \text{ and } \sigma = \frac{\sigma_Z}{\beta\sigma_X}, \end{aligned} \tag{2.4}$$

so Z and $Y = \alpha + \beta X + Z$ are normally distributed: $Z \sim N(0, \sigma_Z^2)$ and $Y \sim N(\alpha + \beta\mu_X, \beta^2\sigma_X^2 + \sigma_Z^2)$. Now we can use the results of Albers, Arts and Kallenberg (1995b), where test limits are determined for one correlated measurement. In the present case the linear combination Y can be seen as the correlated measurement. The assumption made is that the error term Z has a small variance relative to the variance of βX. In terms of parameters that means that σ is small, e.g. $\sigma \in (0, 0.5]$. Note that by (2.4),

$$\frac{1}{\sigma^2} = \frac{\beta^2\sigma_X^2}{\sigma_Z^2} = \sum_{l=1}^{k} \frac{\beta_l^2\sigma_X^2}{\sigma_{Z_l}^2} = \sum_{l=1}^{k} \frac{1}{\sigma_l^2},$$

with $\sigma_l = \sigma_{Z_l}/(\beta_l\sigma_X)$. This illustrates that (and how) σ improves on σ_1, σ_2,..., by being smaller. Define

$$\rho = \frac{1}{\sqrt{1+\sigma^2}}, \;\; E = -\frac{Z}{\sigma_Z}, \;\; \bar{X} = \frac{X - \mu_X}{\sigma_X} \text{ and } \bar{s} = \frac{s - \mu_X}{\sigma_X}, \tag{2.5}$$

then for the test limit $t = \alpha + \beta s - a\sigma_Z$ it follows that

$$\begin{aligned} YD &= P(Y < t) = P(\alpha + \beta X + Z < \alpha + \beta s - a\sigma_Z) \\ &= P(\bar{X} - \sigma E < \bar{s} - a\sigma) = \Phi[\rho(\bar{s} - a\sigma)]. \end{aligned}$$

The assumption that σ is small ensures that the yield is not too far from $\Phi(\bar{s})$, which is the probability that a product is conforming the specification limit. Otherwise, the yield would be too small and it is not worthwhile to use the correlated characteristics.

The consumer risk is equal to

$$\begin{aligned} CR = P(X > s|Y < t) &= \frac{P(X > s, Y < t)}{P(Y < t)} = \frac{P(\bar{s} < \bar{X} < \bar{s} + \sigma(E - a), E > a)}{P(\bar{X} - \sigma E < \bar{s} - a\sigma)} \\ &= \frac{\int_a^\infty [\Phi[\bar{s} + \sigma(e - a)] - \Phi(\bar{s})]\phi(e)de}{\Phi[\rho(\bar{s} - a\sigma)]}. \end{aligned} \tag{2.6}$$

Although in setting $CR = \gamma$, the exact solution for a can be determined numerically, a relatively simple approximation is needed if parameters have to be estimated, since correction for plugging in the estimators is required. As σ is small, we can approximate CR using expansion in powers of σ. By setting the approximated CR equal to γ, we find approximations of a. Define the function $g_m(x)$ for $m = 1, 2, ...$ as

$$g_m(x) = \int_x^\infty (e - x)^m \phi(e) de.$$

A first order approximation of CR in (2.6) is $\sigma \frac{\phi(\bar{s})}{\Phi(\bar{s})} g_1(a)$. Setting this approximation equal to γ we find a first order approximation of a:

$$a_1 = g_1^{-1}\Big(\frac{\gamma\Phi(\bar{s})}{\sigma\phi(\bar{s})}\Big). \tag{2.7}$$

A second order approximation of a is

$$a_2 = a_1 - \frac{1}{2}\sigma\bar{s}[a_1^2 + 1 - a_1 k(a_1)] + \frac{\Phi(\bar{s}) - \Phi[\rho(\bar{s} - a_1\sigma)]}{\Phi(\bar{s})}[k(a_1) - a_1], \tag{2.8}$$

where $k(x) = \phi(x)/(1 - \Phi(x))$. More details about these approximations can be found in Albers, Arts and Kallenberg (1995b). The test limit $t_2 = \alpha + \beta s - a_2\sigma_Z$ is a second order test limit, i.e.

$$CR_2 = P(X > s|Y < t_2) = \gamma[1 + O(\sigma^2)].$$

The resulting test region is

$$T_2 = \Big\{(y_1, ..., y_k) \in \mathbb{R}^k \Big| \sum_{l=1}^k \frac{\beta_l}{\sigma_{Z_l}^2} y_l < \alpha + \beta s - a_2\sigma_Z\Big\}. \tag{2.9}$$

3 Test regions if parameters are unknown

In general, before we can apply the results from the previous section in practice, we need to estimate the parameters. We assume that we have observations $(\tilde{X}_{i1}, \tilde{X}_{i2}, Y_{1i}, ..., Y_{ki})$, $i = 1, ..., n$, where

$$\tilde{X}_{ij} = X_i + U_{ij},$$

are repeated measurements on $\tilde{X} = X + U$. These repeated measurements make it possible to estimate σ_U^2 by

$$\hat{\sigma}_U^2 = \frac{1}{n}\sum_{i=1}^n \sum_{j=1}^2 (\tilde{X}_{ij} - \tilde{X}_{i\bullet})^2 = \frac{1}{n}\sum_{i=1}^n \sum_{j=1}^2 (U_{ij} - U_{i\bullet})^2, \tag{3.1}$$

where $\tilde{X}_{i\bullet} = \frac{1}{2}(\tilde{X}_{i1} + \tilde{X}_{i2})$ and $U_{i\bullet} = \frac{1}{2}(U_{i1} + U_{i2})$. For any sequence

$(A_i, B_i, C_{ij}, D_{ij})$, $i = 1, ..., n$ and $j = 1, 2$, let

$$\begin{aligned} A_{\bullet} &= \frac{1}{n}\sum_{i=1}^{n} A_i, \quad S_{AB} = \frac{1}{n-1}\sum_{i=1}^{n}(A_i - A_{\bullet})(B_i - B_{\bullet}), \\ C_{i\bullet} &= (C_{i1} + C_{i2})/2, \quad C_{\bullet\bullet} = \frac{1}{n}\sum_{i=1}^{n} C_{i\bullet}, \\ S_{AC\bullet} &= \frac{1}{n-1}\sum_{i=1}^{n}(A_i - A_{\bullet})(C_{i\bullet} - C_{\bullet\bullet}), \\ S_{C\bullet D\bullet} &= \frac{1}{n-1}\sum_{i=1}^{n}(C_{i\bullet} - C_{\bullet\bullet})(D_{i\bullet} - D_{\bullet\bullet}). \end{aligned}$$

Using this notation, the estimators of the parameters are

$$\begin{aligned} &\hat{\mu}_X = \tilde{X}_{\bullet\bullet}, \quad \hat{\sigma}_X^2 = S_{\tilde{X}_\bullet\tilde{X}_\bullet} - \hat{\sigma}_U^2/2, \quad \hat{\beta}_l = S_{\tilde{X}_\bullet Y_l}/\hat{\sigma}_X^2, \\ &\hat{\alpha}_l = Y_{l\bullet} - \hat{\beta}_l\hat{\mu}_X, \quad \hat{\sigma}_{Z_l}^2 = S_{Y_lY_l} - \hat{\beta}_l^2\hat{\sigma}_X^2. \end{aligned} \tag{3.2}$$

These are essentially the Maximum Likelihood Estimators, with mainly some modifications in connection with unbiasedness. Since $\sigma_X^2 > 0$ and $\sigma_{Z_l}^2 \geq 0$ for all l, the corresponding estimators should be positive as well. This will hold if

$$S_{Y_lY_l}(S_{\tilde{X}_\bullet\tilde{X}_\bullet} - \hat{\sigma}_U^2/2) - S_{\tilde{X}_\bullet Y_l}^2 > 0 \ \forall l = 1, ..., k. \tag{3.3}$$

If (3.3) is violated for at least one l and if for $q \in \{1, ..., k\}$ and all $l = 1, ..., k$, we have

$$S_{Y_qY_q}(S_{\tilde{X}_\bullet\tilde{X}_\bullet} - \hat{\sigma}_U^2/2) - S_{\tilde{X}_\bullet Y_q}^2 \leq S_{Y_lY_l}(S_{\tilde{X}_\bullet\tilde{X}_\bullet} - \hat{\sigma}_U^2/2) - S_{\tilde{X}_\bullet Y_l}^2,$$

then the estimators of σ_X^2, β_q and $\sigma_{Z_q}^2$ become

$$\hat{\sigma}_X^2 = S_{\tilde{X}_\bullet Y_q}^2/S_{Y_qY_q}, \quad \hat{\beta}_q = S_{Y_qY_q}/S_{\tilde{X}_\bullet Y_q}, \quad \hat{\sigma}_{Z_q}^2 = 0.$$

Since now $\hat{\sigma}_{Z_q}^2 = 0$, it makes no sense to use any other correlated measurements but Y_q, to control the characteristic of interest. The corresponding test limit will be $\hat{t}_q = \hat{\alpha}_q + \hat{\beta}_q s$. Because negative values of $\hat{\sigma}_X^2$ or $\hat{\sigma}_Z^2$ in (3.2) only occur with exponentially small probability, we will ignore this situation in the sequel and we assume that (3.3) holds true.

If we have a test limit for an observable correlated measurement Y, we only have to estimate the parameters in the test limit. Together with (2.4), (2.5), (2.7) and (2.8), (3.2) leads to an estimated version of the test limit $\hat{\alpha} + \hat{\beta}s - \hat{a}_2\hat{\sigma}_Z$. However, since Y is an unknown linear combination of Y_1, Y_2,..., Y_k, the test region is still unknown. Although the test limit is known, we don't know when an item should be accepted. This is a new complication, due to the fact that we now have more than one correlated measurement. Beside the test limit, the linear combination is estimated too and we get an estimated version of the test region

$$\Big\{(y_1,...,y_k)\in\mathbb{R}^k\Big|\sum_{l=1}^k \frac{\hat\beta_l}{\hat\sigma^2_{Z_l}}y_l<\hat\alpha+\hat\beta s-\hat a_2\hat\sigma_Z\Big\}. \tag{3.4}$$

At first sight one might expect that the expected consumer risk is close to γ, at least for moderately large n. Unfortunately, just plugging in estimates in the test limit leads to an expected consumer risk which is seriously biased (cf. Albers, Arts and Kallenberg (1995a)). To ensure that the expected consumer risk equals γ up to high precision, the test limit $\hat\alpha+\hat\beta s-\hat a_2\hat\sigma_Z$ has to be modified. The modified test limit is $\hat t_2=\hat\alpha+\hat\beta s-(\hat a_2+\hat c)\hat\sigma_Z$, where $\hat c$ is a small correction term of order $1/n$. The correction term $\hat c$ is determined such that the expected consumer risk equals γ up to second order w.r.t. σ and $1/n$. Define

$$\alpha^*=\sum_{l=1}^k \frac{\hat\beta_l}{\hat\sigma^2_{Z_l}}\alpha_l,\ \ \beta^*=\sum_{l=1}^k \frac{\hat\beta_l}{\hat\sigma^2_{Z_l}}\beta_l,\ \ Z^*=\sum_{l=1}^k \frac{\hat\beta_l}{\hat\sigma^2_{Z_l}}Z_l\ \ \text{and}\ \ Y^*=\sum_{l=1}^k \frac{\hat\beta_l}{\hat\sigma^2_{Z_l}}Y_l,$$

so $Y^*=\alpha^*+\beta^*X+Z^*$ is the estimated linear combination. Furthermore define

$$\sigma^2_{Z^*}=\sum_{l=1}^k \frac{\hat\beta_l^2}{\hat\sigma^4_{Z_l}}\sigma^2_{Z_l},\quad \sigma^*=\frac{\sigma_{Z^*}}{\beta^*\sigma_X},\quad \rho^*=\frac{1}{\sqrt{1+\sigma^{*2}}}\ \ \text{and}\ \ a_1^*=g_1^{-1}\Big(\frac{\gamma\Phi(\bar s)}{\sigma^*\phi(\bar s)}\Big).$$

The consumer risk, $\widehat{CR}=P(X>s|Y^*<\hat t_2)$, can be obtained from CR in (2.6) by replacing σ by σ^*, ρ by ρ^* and a by

$$\tilde a=\frac{\hat\sigma_Z}{\sigma_{Z^*}}(\hat a_2+\hat c)-\frac{\hat\alpha-\alpha^*}{\sigma_{Z^*}}-\frac{\hat\beta-\beta^*}{\sigma_{Z^*}}s. \tag{3.5}$$

Let '$\doteq$' denote equality to the order considered. It can be shown that expansion of $\mathrm{E}\widehat{CR}$ in powers of σ^* and $1/n$ results in

$$\mathrm{E}\widehat{CR}\doteq\mathrm{E}\{\sigma^*\frac{\phi(\bar s)}{\Phi(\bar s)}g_1(\tilde a_1)\}+\sigma\frac{\phi(\bar s)}{\Phi(\bar s)}cg_1'(a_1), \tag{3.6}$$

where $\tilde a_1=\frac{\hat\sigma_Z}{\sigma_{Z^*}}\hat a_1-\frac{\hat\alpha-\alpha^*}{\sigma_{Z^*}}-\frac{\hat\beta-\beta^*}{\sigma_{Z^*}}s$. For details we refer to Albers, Arts and Kallenberg (1995b). Introduce the following notation:

$$\xi^*=\frac{\hat\sigma_Z}{\sigma_{Z^*}},\ \eta^*=\frac{\hat\beta}{\beta^*}\frac{\hat\sigma_X}{\sigma_X}\frac{\phi(\bar s)}{\Phi(\bar s)}\frac{\Phi[(s-\hat\mu_X)/\hat\sigma_X]}{\phi[(s-\hat\mu_X)/\hat\sigma_X]}\ \text{and}\ \zeta^*=-\frac{\hat\alpha-\alpha^*}{\sigma_{Z^*}}-\frac{\hat\beta-\beta^*}{\sigma_{Z^*}}s. \tag{3.7}$$

Using Taylor expansion of $m(\xi^*,\eta^*,\zeta^*)=g_1\Big(\xi^*g_1^{-1}\Big(\frac{\gamma\Phi(\bar s)}{\sigma^*\phi(\bar s)}\frac{\eta^*}{\xi^*}\Big)+\zeta^*\Big)=g_1(\tilde a_1)$ around $m(1,1,0)=g_1(a_1^*)=\frac{\gamma\Phi(\bar s)}{\sigma^*\phi(\bar s)}$, we have

$$\sigma^*\frac{\phi(\bar s)}{\Phi(\bar s)}g_1(\tilde a_1)-\gamma=\sigma^*\frac{\phi(\bar s)}{\Phi(\bar s)}g_1'(a_1^*)\{(\xi^*-1)k(a_1^*)-(\eta^*-1)\{k(a_1^*)-a_1^*\}+\zeta^* \tag{3.8}$$
$$+\tfrac12 k(a_1^*)[(\xi^*-1)a_1^*+\zeta^*][(\xi^*-1)(a_1^*-2k(a_1^*))-\zeta^*+2(\eta^*-1)(k(a_1^*)-a_1^*)]\}.$$

So if we take the correction term such that $-\sigma\dfrac{\phi(\bar{s})}{\Phi(\bar{s})}cg'(a_1)$ equals the expectation of (3.8), the expected consumer risk from (3.6) will equal γ up to the order considered. Since σ_{Z^*}, β^* and α^* depend on the estimators $\hat{\beta}_l$ and $\hat{\sigma}^2_{Z_l}$, $l = 1, ..., k$, these quantities are stochastic. That means that also σ^* and a_1^* are stochastic. Therefore we will expand σ^* and a_1^* around σ and a_1. As the right-hand side of (3.8) only consists of first and second order terms w.r.t. $(\xi^* - 1)$, $(\eta^* - 1)$ and ζ^*, we only need first order expansion of σ^* and a_1^*. The derivatives of

$$\sigma^* = \frac{\sigma_{Z^*}}{\beta^*\sigma_X} = \sqrt{\sum_{l=1}^{k}\frac{\hat{\beta}_l^2}{\hat{\sigma}^4_{Z_l}}\sigma^2_{Z_l}}\Big/\left(\sigma_X\sum_{l=1}^{k}\frac{\hat{\beta}_l}{\hat{\sigma}^2_{Z_l}}\beta_l\right)$$

w.r.t. $\hat{\sigma}^2_{Z_l}$ and $\hat{\beta}_l$, $l = 1, ..., k$, are zero in $(\hat{\sigma}^2_{Z_l}, \hat{\beta}_l) = (\sigma^2_{Z_l}, \beta_l)$, so σ^* equals σ up to first order. It is obvious that also the derivatives of $a_1^* = g_1^{-1}\left(\dfrac{\gamma\Phi(\bar{s})}{\sigma^*\phi(\bar{s})}\right)$ are zero, so a_1^* equals a_1 up to first order. Therefore we can replace σ^* and a_1^* by σ and a_1 in (3.8) and the correction term should be equal to

$$\begin{aligned}&-\mathrm{E}\{(\xi^*-1)k(a_1) - (\eta^*-1)\{k(a_1)-a_1\} + \zeta^* \\ &\quad+\tfrac{1}{2}k(a_1)[(\xi^*-1)a_1+\zeta^*][(\xi^*-1)(a_1-2k(a_1)) - \zeta^* + 2(\eta^*-1)(k(a_1)-a_1)]\}.\end{aligned} \tag{3.9}$$

Lemma 3.1. Let $\kappa_l = \beta_l\sigma_U/\sigma_{Z_l}$ and $\kappa^2 = \beta^2\sigma^2_U/\sigma^2_Z = \sum_{l=1}^{k}\kappa_l^2$, $l = 1, ..., k$, then

$$\begin{aligned}
n\mathrm{E}(\xi^*-1) &\doteq -\tfrac{3}{4} - \tfrac{1}{2}\kappa^2 - \tfrac{1}{8}\kappa^4 - \frac{1}{\beta^2}\sum_{l=1}^{k}\sum_{l'\neq l}^{k}\frac{\beta_l^2}{\sigma^2_{Z_l}}\frac{\beta_{l'}^2}{\sigma^2_{Z_{l'}}}\kappa_l^2(\tfrac{7}{4}-\kappa^2)\\
&\quad -\frac{1}{\beta^2}\sum_{l=1}^{k}\sum_{l'\neq l}^{k}\frac{\beta_l^2}{\sigma^2_{Z_l}}\frac{\beta_{l'}^2}{\sigma^2_{Z_{l'}}}\left[\tfrac{7}{4} - \tfrac{1}{2}\kappa^2 - \tfrac{1}{8}\kappa^4 + \tfrac{7}{8}\sum_{l=1}^{k}\kappa_l^4\right],\\
n\mathrm{E}(\xi^*-1)^2 &\doteq \tfrac{1}{2} + \tfrac{1}{2}\kappa^2 + \tfrac{1}{4}\kappa^4 - \frac{1}{2\beta^2}\sum_{l=1}^{k}\sum_{l'\neq l}^{k}\frac{\beta_l^2}{\sigma^2_{Z_l}}\frac{\beta_{l'}^2}{\sigma^2_{Z_{l'}}}\kappa_l^2\\
&\quad -\frac{1}{\beta^2}\sum_{l=1}^{k}\sum_{l'\neq l}^{k}\frac{\beta_l^2}{\sigma^2_{Z_l}}\frac{\beta_{l'}^2}{\sigma^2_{Z_{l'}}}\left[\tfrac{1}{2} + \tfrac{1}{2}\kappa^2 + \tfrac{1}{4}\kappa^4 + \tfrac{1}{4}\sum_{l=1}^{k}\kappa_l^4\right],\\
n\mathrm{E}(\eta^*-1) &\doteq \tfrac{1}{4} + \bar{s}^2 + \tfrac{1}{4}\bar{s}^4 + \tfrac{1}{4}\frac{\bar{s}\phi(\bar{s})}{\Phi(\bar{s})}(3+\bar{s}^2),\\
n\mathrm{E}\zeta^* &\doteq 0, \qquad n\mathrm{E}\zeta^{*2} \doteq (\bar{s}^2+1)(1+\tfrac{1}{2}\kappa^2).
\end{aligned} \tag{3.10}$$

The mixed moments of $(\xi^* - 1)$, $(\eta^* - 1)$ and ζ^* are zero to the order considered.

Proof. See the Appendix. □

If we substitute the results of this lemma in (3.9), we can determine the correction term, which is

$$c = c_o + \frac{1}{n\beta^2}\sum_{l=1}^{k}\sum_{l'\neq l}^{k}\frac{\beta_l^2}{\sigma_{Z_l}^2}\frac{\beta_{l'}^2}{\sigma_{Z_{l'}}^2}\Big[\kappa_l^2(\tfrac{7}{4}-\kappa^2)+\tfrac{7}{4}-\tfrac{1}{2}\kappa^2-\tfrac{1}{8}\kappa^4+\tfrac{7}{8}\sum_{l=1}^{k}\kappa_l^4\Big]k(a_1)$$
$$-\frac{1}{4n\beta^2}\sum_{l=1}^{k}\sum_{l'\neq l}^{k}\frac{\beta_l^2}{\sigma_{Z_l}^2}\frac{\beta_{l'}^2}{\sigma_{Z_{l'}}^2}\Big[\kappa_l^2+1+\kappa^2+\tfrac{1}{2}\kappa^4+\tfrac{1}{2}\sum_{l=1}^{k}\kappa_l^4\Big]a_1k(a_1)[2k(a_1)-a_1], \tag{3.11}$$

where

$$c_o = \frac{1}{n}\Big\{\tfrac{1}{2}(1+\tfrac{1}{2}\kappa^2)k(a_1)+\tfrac{1}{4}(1+\kappa^2+\tfrac{1}{2}\kappa^4)[1+[2k(a_1)-a_1]a_1]k(a_1)$$
$$+\tfrac{1}{4}(1+4\bar{s}^2+\bar{s}^4+3\frac{\bar{s}\phi(\bar{s})}{\Phi(\bar{s})}+\bar{s}^2\frac{\bar{s}\phi(\bar{s})}{\Phi(\bar{s})})[k(a_1)-a_1] \tag{3.12}$$
$$+\tfrac{1}{2}k(a_1)(\bar{s}^2+1)(1+\tfrac{1}{2}\kappa^2)\Big\},$$

which is the same formula as the correction term if only one correlated measurement is used. This correction term c_o can also be found in Albers, Arts and Kallenberg (1995b).

Remark 3.1. If one wants to determine a test limit such that the consumer loss instead of the consumer risk is bounded, the following changes have to be made. The consumer loss if the test limit $t = \alpha + \beta s - a\sigma_Z$ is used is

$$CL = P(X > s, Y < t) = \int_a^\infty [\Phi[\bar{s}+\sigma(e-a)] - \Phi(\bar{s})]\phi(e)de.$$

A first order approximation of the consumer loss is equal to $\sigma\phi(\bar{s})g_1(a)$. If we set this approximation equal to the bound γ, we get a first order approximation of a, equal to

$$a_1 = g_1^{-1}\Big(\frac{\gamma}{\sigma\phi(\bar{s})}\Big),$$

instead of the one given in (2.7). The second order approximation of a becomes

$$a_2 = a_1 - \frac{1}{2}\sigma\bar{s}[a_1^2 + 1 - a_1k(a_1)],$$

instead of a_2 from (2.8), cf. Albers, Arts and Kallenberg (1995a). Due to the change of a_1 and a_2 also $m(\xi^*,\eta^*,\zeta^*) = g_1(\tilde{a}_1)$ changes, but only through η^*, which is now equal to

$$\eta^* = \frac{\hat{\beta}}{\beta^*}\frac{\hat{\sigma}_X}{\sigma_X}\frac{\phi(\bar{s})}{\phi[(s-\hat{\mu}_X)/\hat{\sigma}_X]},$$

instead of the one given in (3.7). In lemma 3.1, $n\mathrm{E}(\eta^* - 1)$ changes into

$$n\mathrm{E}(\eta^* - 1) \doteq \tfrac{1}{4}(1 + 4\bar{s}^2 + \bar{s}^4),$$

cf. Albers, Arts and Kallenberg (1995a). Finally the correction term in (3.11) changes only through a change in c_o. Instead of c_o from (3.12), we have

$$\begin{aligned} c_o &= \frac{1}{n}\Big\{\tfrac{1}{2}(1 + \tfrac{1}{2}\kappa^2)k(a_1) + \tfrac{1}{4}(1 + \kappa^2 + \tfrac{1}{2}\kappa^4)[1 + [2k(a_1) - a_1]a_1]k(a_1) \\ &\quad + \tfrac{1}{4}(1 + 4\bar{s}^2 + \bar{s}^4)[k(a_1) - a_1] + \tfrac{1}{2}k(a_1)(\bar{s}^2 + 1)(1 + \tfrac{1}{2}\kappa^2)\Big\}. \end{aligned} \qquad \square$$

For $k = 2$ and sample size $n = 100$, some simulations have been performed. The results are stated in Table I and Table II. First of all we see that the average consumer risk is close to γ and if two correlated measurements are used instead of one, the average consumer risk is even closer to γ and the standard deviation (between brackets) is smaller. Secondly, we see that the yield improves if two instead of one correlated measurements are used. The gain in yield depends on how large the error term U is relative to X and how large κ_1 and κ_2 are. The improvement varies from a few tenth of percents up to more than 10%. In both tables, we also added a column with consumer risk and yield using the second order test limit, if parameters are known. First of all it can be seen that the second order approximation works very well, since the consumer risk is very close to the prescribed bound γ. Secondly, having to estimate the parameters doesn't cost a lot of yield. The yield using the estimated optimal linear combination is close to the yield using the optimal linear combination if parameters are known. This and the small standard deviation of the yield is caused by the fact that the yield is not very sensitive to a change in parameters. In Albers, Arts and Kallenberg (1995a) an explanation is given why the yield is much less sensitive to parameter estimation than the consumer risk.

Table I. The average consumer risks (in ppm) and yields (in %), based on 10^4 simulations each. The first column corresponds to a test limit for one correlated measurement, the second to a test limit for two correlated measurements. The third column corresponds to the second order test limit using an optimal linear combination for two correlated measurements, with parameters known. The prescribed bound on the consumer risk γ has been set to 20 ppm and $\pi = P(X > s) = 0.15$. The parameters are estimated with a sample of size 100. Test limits are derived for different values of σ_U/σ_X, κ_1 and κ_2

σ_U/σ_X	κ_1	κ_2	$\mathrm{E}\widehat{CR}(1)$	$\mathrm{E}\widehat{CR}(2)$	$CR(2)$
0.01	0.5	0.5	20.4 (18.1)	20.3 (14.2)	20.0
		0.25	20.4 (18.1)	20.3 (15.8)	20.0
0.01	0.2	0.2	20.5 (19.8)	20.4 (14.7)	20.0
		0.1	20.5 (19.8)	20.5 (16.9)	20.0
0.05	0.5	0.5	21.1 (26.5)	20.9 (19.6)	20.0
		0.25	21.1 (26.5)	21.0 (22.6)	20.1
0.05	0.2	0.2	22.7 (29.8)	21.6 (20.3)	20.4
		0.1	22.7 (29.8)	22.3 (24.7)	20.9
0.10	0.5	0.5	22.3 (32.5)	21.7 (23.2)	20.2
		0.25	22.3 (32.5)	22.1 (27.5)	20.5
0.15	0.5	0.5	24.6 (38.7)	22.9 (26.6)	20.8
		0.25	24.6 (38.7)	24.0 (32.4)	21.6
σ_U/σ_X	κ_1	κ_2	$\mathrm{E}\widehat{YD}(1)$	$\mathrm{E}\widehat{YD}(2)$	$YD(2)$
0.01	0.5	0.5	83.9 (0.13)	84.2 (0.08)	84.3
		0.25	83.9 (0.13)	84.0 (0.11)	84.0
0.01	0.2	0.2	81.6 (0.36)	82.8 (0.19)	82.8
		0.1	81.6 (0.36)	82.0 (0.28)	82.1
0.05	0.5	0.5	76.9 (0.95)	79.8 (0.50)	80.0
		0.25	76.9 (0.95)	78.0 (0.71)	78.3
0.05	0.2	0.2	58.4 (2.93)	68.4 (1.48)	69.0
		0.1	58.4 (2.93)	62.2 (2.20)	63.1
0.10	0.5	0.5	65.0 (2.43)	72.5 (1.24)	73.0
		0.25	65.0 (2.43)	67.9 (1.80)	68.7
0.15	0.5	0.5	51.0 (4.01)	63.6 (2.15)	64.6
		0.25	51.0 (4.01)	55.8 (3.06)	57.2

Table II. The average consumer risks (in ppm) and yields (in %), based on 10^4 simulations each. The first column corresponds to a test limit for one correlated measurement, the second and third to a test limit for two correlated measurements. The third column corresponds to the second order test limit using an optimal linear combination for two correlated measurements, with parameters known. The prescribed bound on the consumer risk γ has been set to 20 ppm and $\pi = P(X > s) = 0.05$. The parameters are estimated with a sample of size 100. Test limits are derived for different values of σ_U/σ_X, κ_1 and κ_2

σ_U/σ_X	κ_1	κ_2	$\mathrm{E}\widehat{CR}(1)$	$\mathrm{E}\widehat{CR}(2)$	$CR(2)$
0.01	0.5	0.5	20.3 (17.8)	20.2 (15.2)	20.0
		0.25	20.3 (17.8)	20.2 (16.2)	20.0
0.01	0.2	0.2	20.4 (19.8)	20.3 (15.8)	20.0
		0.1	20.4 (19.8)	20.3 (17.6)	20.0
0.05	0.5	0.5	20.8 (25.8)	20.7 (20.6)	20.0
		0.25	20.8 (25.8)	20.8 (22.9)	20.0
0.05	0.2	0.2	21.3 (28.3)	21.0 (21.1)	20.0
		0.1	21.3 (28.3)	21.3 (24.5)	20.2
0.10	0.5	0.5	21.5 (31.0)	21.4 (24.2)	20.0
		0.25	21.5 (31.0)	21.5 (27.3)	20.1
0.15	0.5	0.5	22.5 (35.5)	22.2 (27.4)	20.1
		0.25	22.5 (35.5)	22.5 (31.2)	20.3
σ_U/σ_X	κ_1	κ_2	$\mathrm{E}\widehat{YD}(1)$	$\mathrm{E}\widehat{YD}(2)$	$YD(2)$
0.01	0.5	0.5	94.6 (0.07)	94.7 (0.04)	94.7
		0.25	94.6 (0.07)	94.6 (0.05)	94.6
0.01	0.2	0.2	93.6 (0.18)	94.1 (0.10)	94.1
		0.1	93.6 (0.18)	93.8 (0.15)	93.9
0.05	0.5	0.5	91.5 (0.51)	92.8 (0.28)	92.9
		0.25	91.5 (0.51)	92.0 (0.40)	92.2
0.05	0.2	0.2	80.9 (2.13)	87.0 (0.99)	87.5
		0.1	80.9 (2.13)	83.4 (1.57)	84.1
0.10	0.5	0.5	85.1 (1.59)	89.3 (0.78)	89.6
		0.25	85.1 (1.59)	86.8 (1.18)	87.3
0.15	0.5	0.5	75.7 (3.28)	84.2 (1.55)	85.0
		0.25	75.7 (3.28)	79.2 (2.41)	80.3

4 A numerical example

In this section we shall present an example from semiconductor industry. The data are made available by Philips Semiconductors Nijmegen. Of one of the products they make, a characteristic coded by #01645.2, has to satisfy a specification limit. The true value of this characteristic, denoted by X, has to fall below the specification limit $s = -45$. Beside measurements $\tilde{X}$ of this characteristic, there are also measurements available of two characteristics coded by #01645.0 and #01645.1, both highly correlated with the characteristic of interest. We will denote the measurements of these characteristics by Y_1 and Y_2. Since these measurements are available anyhow, because these characteristics have to be inspected too, we can use these to inspect #01645.2 as well. The dataset consists of measurements of 135 items. In case of characteristic #01645.2, each item is measured twice. The parameters can be estimated using (3.1) and (3.2). Using these results we can standardize the specification limit to $\hat{\bar{s}} = (s - \hat{\mu}_X)/\hat{\sigma}_X = 1.70$. Furthermore we have $\hat{\sigma}_U/\hat{\sigma}_X = 0.23$, $\hat{\sigma}_1 = \hat{\sigma}_{Z_1}/(\hat{\beta}_1\hat{\sigma}_X) = 0.47$ and $\hat{\sigma}_2 = \hat{\sigma}_{Z_2}/(\hat{\beta}_2\hat{\sigma}_X) = 0.43$. Since both $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are about twice as large as $\hat{\sigma}_U/\hat{\sigma}_X$, it is clear that using one correlated characteristic instead of the characteristic itself, causes a large reduction of the yield. If the characteristic itself is used we should accept an item if $\tilde{X} < -46.43$, which results in a yield of 84.3%. If #01645.1 is used, we should accept an item if $Y_2 < -37.08$. Indeed the yield goes down with more than 20%, to 63.1%. This loss can be reduced to 7.1% if also #01645.0 is used. The yield is 77.2% if we accept an item if the linear combination $2.09Y_1 + 2.19Y_2 < -161.77$. The question that remains for the producer is whether the loss of yield of 7.1% is sufficiently small to use the correlated measurements instead of measurements of the characteristic of interest. This will depend on how large the measurement costs are.

Appendix

Proof of lemma 2.1

Let $\mathbf{Y} = (Y_1, Y_2, ..., Y_k)$, $\mathbf{y} = (y_1, y_2, ..., y_k)$ and $f_{X,\mathbf{Y}}(x, \mathbf{y})$ be the joint density function of $(X, \mathbf{Y})$.

$$f_{X,\mathbf{Y}}(x,\mathbf{y}) = \frac{1}{(2\pi)^{\frac{1}{2}(k+1)}\sigma_X \prod_{l=1}^{k} \sigma_{Z_l}} \exp\Big\{-\frac{1}{2}\Big[\sum_{l=1}^{k} \frac{(y_l - \alpha_l - \beta_l x)^2}{\sigma_{Z_l}^2} + \Big(\frac{x-\mu_X}{\sigma_X}\Big)^2\Big]\Big\}.$$

This function can also be written as the product of the two density functions

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{(1-\rho^2)^{\frac{1}{2}}}{\prod_{l=1}^{k}\{\sqrt{2\pi}\sigma_{Z_l}\}} \exp\Big\{-\frac{1}{2}\Big[\sum_{l=1}^{k} \frac{(y_l - \mu_{Y_l})^2}{\sigma_{Z_l}^2} - (1-\rho^2)\Big(\sum_{l=1}^{k} \frac{\beta_l(y_l - \mu_{Y_l})\sigma_X}{\sigma_{Z_l}^2}\Big)^2\Big]\Big\}$$

and

$$f_{X|\mathbf{Y}}(x|\mathbf{y}) = \frac{(1-\rho^2)^{-\frac{1}{2}}}{\sqrt{2\pi}\sigma_X} \exp\Big\{-\frac{1}{2(1-\rho^2)}\Big[\frac{x-\mu_X}{\sigma_X} - (1-\rho^2)\sum_{l=1}^{k} \frac{\beta_l(y_l - \mu_{Y_l})\sigma_X}{\sigma_{Z_l}^2}\Big]^2\Big\},$$

where ρ is the correlation coefficient between $Y = \sum_{l=1}^{k} \frac{\beta_l}{\sigma^2_{Z_l}} Y_l$ and X. Note that $(1-\rho^2)^{-1} = 1 + \sum_{l=1}^{k} \frac{\beta_l^2 \sigma_X^2}{\sigma^2_{Z_l}}$. Define $V_{\mathbf{y}}$ and W_t as normally distributed variables:

$$\begin{aligned} V_{\mathbf{y}} &\sim N\Big(\mu_X + (1-\rho^2)\sum_{l=1}^{k} \frac{\beta_l (y_l - \mu_{Y_l})\sigma_X^2}{\sigma^2_{Z_l}}, (1-\rho^2)\sigma_X^2\Big), \\ W_t &\sim N\Big(\mu_X + (1-\rho^2)[t - \sum_{l=1}^{k} \frac{\beta_l \mu_{Y_l}}{\sigma^2_{Z_l}}]\sigma_X^2, (1-\rho^2)\sigma_X^2\Big). \end{aligned}$$

If $\mathbf{y} \in T^*$, i.e. $\sum_{l=1}^{k} \frac{\beta_l}{\sigma^2_{Z_l}} y_l < t$, we have $P(W_t > s) > P(V_{\mathbf{y}} > s)$ implying

$$\begin{aligned} P(W_t > s) &= \frac{\int_{T^*} f_{\mathbf{Y}}(\mathbf{y}) P(W_t > s) d\mathbf{y}}{P(\mathbf{Y} \in T^*)} > \frac{\int_{T^*} f_{\mathbf{Y}}(\mathbf{y}) P(V_{\mathbf{y}} > s) d\mathbf{y}}{P(\mathbf{Y} \in T^*)} \\ &= \frac{1}{P(\mathbf{Y} \in T^*)} \int_{T^*} \int_{s}^{\infty} f_{X|\mathbf{Y}}(x|\mathbf{y}) f_{\mathbf{Y}}(\mathbf{y}) dx d\mathbf{y} = \frac{P(X > s, \mathbf{Y} \in T^*)}{P(\mathbf{Y} \in T^*)} = \gamma, \end{aligned}$$

and thus $P(W_t < s) < 1 - \gamma$. So if $[1 - \gamma - P(W_t < s)](YD^* - YD) \geq 0$ then $YD^* - YD \geq 0$. Now, using $P(W_t < s) < P(V_{\mathbf{y}} < s)$ if $\mathbf{y} \in T^*$ and $P(W_t < s) \geq P(V_{\mathbf{y}} < s)$ if $\mathbf{y} \in \bar{T}^*$, where $\bar{T}^* = \mathbb{R}^k \setminus T^*$, we show that $[1 - \gamma - P(W_t < s)](YD^* - YD) \geq 0$.

$$[1 - \gamma - P(W_t < s)](YD^* - YD) =$$

$$\begin{aligned} &\int_{T^*} [1 - \gamma - P(W_t < s)] f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y} - \int_{T} [1 - \gamma - P(W_t < s)] f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y} = \\ &\int_{T^* \cap \bar{T}} [1 - \gamma - P(W_t < s)] f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y} - \int_{T \cap \bar{T}^*} [1 - \gamma - P(W_t < s)] f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y} \geq \\ &\int_{T^* \cap \bar{T}} [1 - \gamma - P(V_{\mathbf{y}} < s)] f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y} - \int_{T \cap \bar{T}^*} [1 - \gamma - P(V_{\mathbf{y}} < s)] f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y} = \\ &\int_{T^* \cap \bar{T}} [1 - \gamma - \int_{-\infty}^{s} f_{X|\mathbf{Y}}(x|\mathbf{y}) dx] f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y} - \int_{T \cap \bar{T}^*} [1 - \gamma - \int_{-\infty}^{s} f_{X|\mathbf{Y}}(x|\mathbf{y}) dx] f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y} = \\ &\int_{T^* \cap \bar{T}} [(1-\gamma) f_{\mathbf{Y}}(\mathbf{y}) - \int_{-\infty}^{s} f_{X,\mathbf{Y}}(x, \mathbf{y}) dx] d\mathbf{y} - \int_{T \cap \bar{T}^*} [(1-\gamma) f_{\mathbf{Y}}(\mathbf{y}) - \int_{-\infty}^{s} f_{X,\mathbf{Y}}(x, \mathbf{y}) dx] d\mathbf{y} = \\ &\int_{T^*} [(1-\gamma) f_{\mathbf{Y}}(\mathbf{y}) - \int_{-\infty}^{s} f_{X,\mathbf{Y}}(x, \mathbf{y}) dx] d\mathbf{y} - \int_{T} [(1-\gamma) f_{\mathbf{Y}}(\mathbf{y}) - \int_{-\infty}^{s} f_{X,\mathbf{Y}}(x, \mathbf{y}) dx] d\mathbf{y} = \end{aligned}$$

$$(1-\gamma)P(\mathbf{Y}\in T^*)-P(\mathbf{Y}\in T^*,X<s)-(1-\gamma)P(\mathbf{Y}\in T)+P(\mathbf{Y}\in T,X<s)=$$

$$(CR^*-\gamma)P(\mathbf{Y}\in T^*)-(CR-\gamma)P(\mathbf{Y}\in T)\geq 0 \qquad \square$$

Proof of lemma 3.1
Each term, (ξ^*-1), (η^*-1) and ζ^*, can be expanded in powers of

$$\begin{aligned} W_{Z_l} &= \frac{\hat{\sigma}^2_{Z_l}}{\sigma^2_{Z_l}}-1, \quad Q_l=\frac{\sigma_X}{\sigma_Z}(\hat{\beta}_l-\beta_l), \quad \zeta_l=-\frac{\hat{\alpha}_l-\alpha_l}{\sigma_Z}-\frac{\hat{\beta}_l-\beta_l}{\sigma_Z}s, \\ V_X &= \frac{\hat{\mu}_X-\mu_X}{\sigma_X}, \quad W_X=\frac{\hat{\sigma}^2_X}{\sigma^2_X}-1.\text{(A.1)} \end{aligned} \tag{A.1}$$

Using the moments of these variables we can approximate the moments of (ξ^*-1), (η^*-1) and ζ^*. Define $R_U=S_{U_\bullet U_\bullet}-\hat{\sigma}^2_U/2$, then from (A.1) together with (3.2) it follows that

$$\begin{aligned} \hat{\sigma}^2_{Z_l} &= \beta_l^2S_{XX}+2\beta_lS_{XZ_l}+S_{Z_lZ_l}-\frac{(\beta_lS_{XX}+\beta_lS_{XU_\bullet}+S_{XZ_l}+S_{U_\bullet Z_l})^2}{S_{XX}+2S_{XU_\bullet}+R_U}, \\ &\doteq \beta_l^2\Big(-\frac{S^2_{XU_\bullet}}{S_{XX}}+R_U\Big)+2\beta_l\Big(\frac{S_{XU_\bullet}S_{XZ_l}}{S_{XX}}-S_{U_\bullet Z_l}\Big)+\Big(S_{Z_lZ_l}-\frac{S^2_{XZ_l}}{S_{XX}}\Big) \\ Q_l &= \frac{\sigma_X}{\sigma_Z}\Big(\frac{S_{XZ_l}+S_{U_\bullet Z_l}-\beta_lS_{XU_\bullet}-\beta_lR_U}{S_{XX}+2S_{XU_\bullet}+R_U}\Big)\doteq\frac{\sigma_X}{\sigma_Z}\Big(\frac{S_{XZ_l}}{S_{XX}}-\beta_l\frac{S_{XU_\bullet}}{S_{XX}}\Big), \\ \zeta_l &= -\frac{Z_{l\bullet}}{\sigma_Z}+\frac{\beta_lU_{\bullet\bullet}}{\sigma_Z}-Q_l\frac{s-\hat{\mu}_X}{\sigma_X}. \end{aligned}$$

The moments of these variables and V_X and W_X, can be found using some general results for sample variances (see for example Fuller (1987), p88-89). For $l=1,...,k$, we have

$$\begin{aligned} n\mathrm{E}W_{Z_l} &\doteq -1-\tfrac{1}{2}\kappa_l^2, \quad \mathrm{E}Q_l\doteq 0, & \mathrm{E}\zeta_l &\doteq 0, \quad \mathrm{E}V_X=0, \quad \mathrm{E}W_X=0, \\ n\mathrm{E}W^2_{Z_l} &\doteq 2+2\kappa_l^2+\kappa_l^4, & n\ \mathrm{E}W_{Z_l}W_{Z_{l'}} &\doteq \kappa_l^2\kappa_{l'}^2,\ l'\neq l, \\ n\mathrm{E}Q_l^2 &\doteq \frac{\sigma^2_{Z_l}}{\sigma^2_Z}(1+\tfrac{1}{2}\kappa_l^2), & n\ \mathrm{E}Q_lQ_{l'} &\doteq \tfrac{1}{2}\frac{\sigma_{Z_l}\sigma_{Z_{l'}}}{\sigma^2_Z}\kappa_l\kappa_{l'},\ \ l'\neq l, \\ n\mathrm{E}\zeta_l^2 &\doteq \frac{\sigma^2_{Z_l}}{\sigma^2_Z}(1+\bar{s}^2)(1+\tfrac{1}{2}\kappa_l^2), & n\ \mathrm{E}\zeta_l\zeta_{l'} &\doteq \tfrac{1}{2}\frac{\sigma_{Z_l}\sigma_{Z_{l'}}}{\sigma^2_Z}\kappa_l\kappa_{l'}(1+\bar{s}^2),\ l'\neq l, \\ n\ \mathrm{E}Q_l\zeta_l &\doteq -\bar{s}\frac{\sigma^2_{Z_l}}{\sigma^2_Z}(1+\tfrac{1}{2}\kappa_l^2), & n\ \mathrm{E}Q_l\zeta_{l'} &\doteq -\tfrac{1}{2}\bar{s}\frac{\sigma_{Z_l}\sigma_{Z_{l'}}}{\sigma^2_Z}\kappa_l\kappa_{l'},\ \ l\neq l', \\ n\mathrm{E}V_X^2 &= 1, \quad n\mathrm{E}W_X^2\doteq 2. \end{aligned}$$

All other mixed moments of W_{Z_l}, $Q_{l'}$, $\zeta_{l''}$, V_X and W_X are negligible for all $l,l',l''=1,...,k$. Now we know the first and second order moments of the estimators, we can expand (ξ^*-1), (η^*-1) and ζ^* and take expectation. The first term we consider is

$$\xi^* = \frac{\hat\sigma_Z}{\sigma_{Z^*}} = \sqrt{\sum_{l=1}^k \frac{\hat\beta_l^2}{\hat\sigma_{Z_l}^2}} \Big/ \sqrt{\sum_{l=1}^k \frac{\hat\beta_l^2}{\hat\sigma_{Z_l}^4}\sigma_{Z_l}^2}.$$

Expanding this term in powers of W_{Z_l} and Q_l and taking expectation results in

$$\begin{aligned} \mathrm{E}(\xi^*-1) &\doteq \frac{1}{2\beta}\sum_{l=1}^k \frac{\beta_l^2}{\sigma_{Z_l}^2}\mathrm{E}W_{Z_l} + \frac{1}{2\beta}\sum_{l=1}^k \frac{\beta_l^2}{\sigma_{Z_l}^2}\Big(\frac{7}{4\beta}\frac{\beta_l^2}{\sigma_{Z_l}^2} - 2\Big)\mathrm{E}W_{Z_l}^2 \\ &\qquad + \frac{7}{8\beta^2}\sum_{l=1}^k\sum_{l'\neq l}\frac{\beta_l^2}{\sigma_{Z_l}^2}\frac{\beta_{l'}^2}{\sigma_{Z_{l'}}^2}\mathrm{E}W_{Z_l}W_{Z_{l'}} \\ &\doteq -\frac{1}{2n\beta}\sum_{l=1}^k\frac{\beta_l^2}{\sigma_{Z_l}^2} - \frac{1}{4n\beta}\sum_{l=1}^k\frac{\beta_l^2}{\sigma_{Z_l}^2}\kappa_l^2 + \frac{7}{4n\beta^2}\sum_{l=1}^k\frac{\beta_l^4}{\sigma_{Z_l}^4} + \frac{7}{4n\beta^2}\sum_{l=1}^k\frac{\beta_l^4}{\sigma_{Z_l}^4}\kappa_l^2 \\ &\quad + \frac{7}{8n\beta^2}\Big(\sum_{l=1}^k\frac{\beta_l^2}{\sigma_{Z_l}^2}\kappa_l^2\Big)^2 - \frac{2}{n\beta}\sum_{l=1}^k\frac{\beta_l^2}{\sigma_{Z_l}^2} - \frac{2}{n\beta}\sum_{l=1}^k\frac{\beta_l^2}{\sigma_{Z_l}^2}\kappa_l^2 - \frac{1}{n\beta}\sum_{l=1}^k\frac{\beta_l^2}{\sigma_{Z_l}^2}\kappa_l^4. \text{(A.2)} \end{aligned} \tag{A.2}$$

Using the definition of β from (2.4), we find that

$$\begin{aligned} &\sum_{l=1}^k\frac{\beta_l^2}{\sigma_{Z_l}^2} = \beta, \quad \sum_{l=1}^k\frac{\beta_l^4}{\sigma_{Z_l}^4} = \beta^2 - \sum_{l=1}^k\sum_{l'\neq l}\frac{\beta_l^2}{\sigma_{Z_l}^2}\frac{\beta_{l'}^2}{\sigma_{Z_{l'}}^2} \text{ and} \\ &\sum_{l=1}^k\frac{\beta_l^6}{\sigma_{Z_l}^6} = \beta^3 - \beta\sum_{l=1}^k\sum_{l'\neq l}\frac{\beta_l^2}{\sigma_{Z_l}^2}\frac{\beta_{l'}^2}{\sigma_{Z_{l'}}^2} - \sum_{l=1}^k\sum_{l'\neq l}\frac{\beta_l^4}{\sigma_{Z_l}^4}\frac{\beta_{l'}^2}{\sigma_{Z_{l'}}^2}. \text{(A.3)} \end{aligned} \tag{A.3}$$

So (A.2) together with (A.3) results in the first moment of (ξ^*-1) given in (3.10). The second moment of (ξ^*-1) is

$$\begin{aligned} \mathrm{E}(\xi^*-1)^2 &\doteq \frac{1}{4\beta^2}\sum_{l=1}^k\frac{\beta_l^4}{\sigma_{Z_l}^4}\mathrm{E}W_{Z_l}^2 + \frac{1}{4\beta^2}\sum_{l=1}^k\sum_{l'\neq l}\frac{\beta_l^2}{\sigma_{Z_l}^2}\frac{\beta_{l'}^2}{\sigma_{Z_{l'}}^2}\mathrm{E}W_{Z_l}W_{Z_{l'}} \\ &\doteq \frac{1}{2n\beta^2}\sum_{l=1}^k\frac{\beta_l^4}{\sigma_{Z_l}^4} + \frac{1}{2n\beta^2}\sum_{l=1}^k\frac{\beta_l^4}{\sigma_{Z_l}^4}\kappa_l^2 + \frac{1}{4n\beta^2}\Big(\sum_{l=1}^k\frac{\beta_l^2}{\sigma_{Z_l}^2}\kappa_l^2\Big)^2 \end{aligned}$$

Together with (A.3) this results in the second moment of (ξ^*-1) given in (3.10). Next, we take a look at

$$\begin{aligned} (\eta^*-1) &= \Big(\frac{\hat\beta}{\beta^*}-1\Big)\Big(\frac{\hat\sigma_X}{\sigma_X}\frac{\phi(\bar s)}{\Phi(\bar s)}\frac{\Phi[(s-\hat\mu_X)/\hat\sigma_X]}{\phi[(s-\hat\mu_X)/\hat\sigma_X]}-1\Big) \\ &\quad + \Big(\frac{\hat\beta}{\beta^*}-1\Big) + \Big(\frac{\hat\sigma_X}{\sigma_X}\frac{\phi(\bar s)}{\Phi(\bar s)}\frac{\Phi[(s-\hat\mu_X)/\hat\sigma_X]}{\phi[(s-\hat\mu_X)/\hat\sigma_X]}-1\Big) \text{(A.4)} \end{aligned} \tag{A.4}$$

We can expand $\frac{\hat\sigma_X}{\sigma_X}\frac{\phi(\bar s)}{\Phi(\bar s)}\frac{\Phi[(s-\hat\mu_X)/\hat\sigma_X]}{\phi[(s-\hat\mu_X)/\hat\sigma_X]}$ in powers of V_X and W_X and

$$\frac{\hat\beta}{\beta^*} = \Big(\sum_{l=1}^k\frac{\hat\beta_l^2}{\hat\sigma_{Z_l}^2}\Big)\Big/\Big(\sum_{l=1}^k\frac{\hat\beta_l}{\hat\sigma_{Z_l}^2}\beta_l\Big)$$

in powers of W_{Z_l} and Q_l. Since the mixed moments of V_X and W_X with W_{Z_l} and Q_l are zero up to the order considered, the first term in (A.4) has zero expectation up to the order considered. Expanding $(\hat{\beta}/\beta^* - 1)$ and taking expectation results in

$$\mathrm{E}\Big(\frac{\hat{\beta}}{\beta^*} - 1\Big) \doteq \sigma \sum_{l=1}^{k} \frac{\beta_l}{\sigma_{Z_l}^2} \mathrm{E}Q_l + \sigma^2 \sum_{l=1}^{k} \frac{1}{\sigma_{Z_l}^2}(\beta - \frac{\beta_l^2}{\sigma_{Z_l}^2})\mathrm{E}Q_l^2 - \sigma^2 \sum_{l=1}^{k}\sum_{l'\neq l}^{k} \frac{\beta_l}{\sigma_{Z_l}^2}\frac{\beta_{l'}}{\sigma_{Z_{l'}}^2}\mathrm{E}Q_l Q_{l'},$$

which is of order σ^2/n. So the expectation of $(\eta^* - 1)$ to the order considered is just the expectation of the last term in (A.4), which can be found in Albers, Arts and Kallenberg (1995b).

The last term to be considered is

$$\zeta^* = -\frac{\hat{\alpha} - \alpha^*}{\sigma_Z^*} - \frac{\hat{\beta} - \beta^*}{\sigma_Z^*} s = \frac{\sigma_Z}{\sigma_Z^*} \sum_{l=1}^{k} \frac{\hat{\beta}_l}{\hat{\sigma}_{Z_l}^2} \zeta_l.$$

Since the mixed moments of ζ_l with $W_{Z_{l'}}$ equal zero to the order considered for all $l, l' = 1, ..., k$, the expected value of ζ^* can be approximated by

$$\mathrm{E}\zeta^* \doteq \sum_{l=1}^{k} \frac{\beta_l}{\sigma_{Z_l}^2}\mathrm{E}\zeta_l + \sigma \sum_{l=1}^{k} \frac{\beta}{\sigma_{Z_l}^2}\mathrm{E}Q_l\zeta_l - \sigma \sum_{l=1}^{k} \frac{\beta_l^2}{\sigma_{Z_l}^4}\mathrm{E}Q_l\zeta_l - \sigma \sum_{l=1}^{k}\sum_{l'\neq l}^{k} \frac{\beta_l}{\sigma_{Z_l}^2}\frac{\beta_{l'}}{\sigma_{Z_{l'}}^2}\mathrm{E}\zeta_l Q_{l'},$$

which is of order σ/n and thus negligible. The second moment is approximated by

$$\begin{aligned}
\mathrm{E}\zeta^{*2} &\doteq \sum_{l=1}^{k} \frac{\beta_l^2}{\sigma_{Z_l}^4}\mathrm{E}\zeta_l^2 + \sum_{l=1}^{k}\sum_{l'\neq l}^{k} \frac{\beta_l}{\sigma_{Z_l}^2}\frac{\beta_{l'}}{\sigma_{Z_{l'}}^2}\mathrm{E}\zeta_l\zeta_{l'} \\
&= \frac{(1+\bar{s}^2)}{\sigma_Z^2 n}\Big[\sum_{l=1}^{k} \frac{\beta_l^2}{\sigma_{Z_l}^2}(1 + \tfrac{1}{2}\kappa_l^2) + \tfrac{1}{2}\sum_{l=1}^{k}\sum_{l'\neq l}^{k} \frac{\beta_l}{\sigma_{Z_l}}\frac{\beta_{l'}}{\sigma_{Z_{l'}}}\kappa_l\kappa_{l'}\Big] \\
&= \frac{1+\bar{s}^2}{\sigma_Z^2 n}\Big[\sum_{l=1}^{k} \frac{\beta_l^2}{\sigma_{Z_l}^2} + \tfrac{1}{2}\Big(\sum_{l=1}^{k} \frac{\beta_l}{\sigma_{Z_l}}\kappa_l\Big)^2\Big] = \frac{1}{n}(1+\bar{s}^2)(1 + \tfrac{1}{2}\kappa^2).
\end{aligned}$$

The mixed moments can be found by expanding each term up to first order and take expectation of their products. Since the mixed moments of W_{Z_l} with Q_l, V_X and W_X are all negligible, the mixed moment of $(\xi^* - 1)$ with $(\eta^* - 1)$ and ζ^* are negligible. The mixed moment of $(\eta^* - 1)$ and ζ^* is negligible since it is of order σ/n. □

References

Albers, W., Arts, G.R.J. and Kallenberg, W.C.M. (1995a). "Test limits using correlated measurements". Technical Report 1260, University Twente.

Albers, W., Arts, G.R.J. and Kallenberg, W.C.M. (1995b). "Test limits for correlated measurements under prescribed consumer risk". Technical Report 1282, University Twente.

Fuller, W.A. (1987). *"Measurement error models"*. Wiley, New York.