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Convexity Preservation of the Four-Point Interpolatory Subdivision Scheme

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Abstract

In this note we examine the convexity preserving properties of the (linear) four-point interpolatory subdivision scheme of Dyn, Gregory and Levin when applied to functional univariate strictly convex data. Conditions on the tension parameter guaranteeing preservation of convexity are derived. These conditions depend on the initial data. The resulting scheme is the four-point scheme with tension parameter bounded from above by a bound smaller than 1/16. Thus the scheme generates C^1 limit functions and has approximation order two.

Key words: stationary subdivision schemes, convexity preservation.

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We examine the four-point interpolatory subdivision scheme [3]:

$$\begin{cases} f_{2i}^{k+1} = f_i^k, \\ f_{2i+1}^{k+1} = -wf_{i-1}^k + \left(\frac{1}{2}+w\right) f_i^k + \left(\frac{1}{2}+w\right) f_{i+1}^k - wf_{i+2}^k. \end{cases} \tag{1}$$

This scheme is applied to a univariate data set $\{(x_i, f_i)\}_i$, with $x_i = ih$, where h is the mesh size of the initial data.

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The initial data are defined by $x_i^0 = x_i$ and $f_i^0 = f_i$, $\forall i$. Since the parameter values x_i^0 are equidistantly distributed, it is obtained that $x_i^k = 2^{-k}ih$. Application of scheme (1) to the data f_i^k defines a nested sequence of refined data sets $\{(x_i^k, f_i^k)\}_i$.

At any iteration, the function f^k is defined as the piecewise linear interpolant to the data $\{(x_i^k, f_i^k)\}_i$.

The parameter w in the scheme (1) is a tension parameter, and for w in the range $0 < w < 1/8$ the four-point scheme is known to converge to a continuously differentiable limit function, i.e., $f^\infty = \lim_{k\to\infty} f^k \in C^1$, see [3,4].

Subdivision scheme (1) does not have shape preserving properties in general. Only if w equals 0 the scheme preserves positivity, monotonicity and convexity, but the limit function is only continuous and not differentiable then.

In [1], conditions on w in terms of the initial data have been derived such that monotonicity is preserved. Although the tension parameter depends on the initial data in a nonlinear way, the construction generates a stationary interpolatory subdivision scheme that converges to C^1 limit functions which are monotone.

In [5], a stationary nonlinear subdivision scheme without a tension parameter has been constructed that preserves convexity and generates C^1 limit functions.

In this note data dependent conditions on w are derived, such that the four-point scheme (1) with w satisfying these conditions, is convexity preserving, when the initial data set $\{(ih, f_i^0)\}_i$ is strictly convex.

Theorem 1 *Given is a univariate equidistant data set $\{(ih, f_i^0)\}_i$, which is strictly convex.*

Define second order divided differences as $d_j^k = 2^{2k-1}(f_{j-1}^k - 2f_j^k + f_{j+1}^k)$, and q_i^k and q^0 as

$$q_i^k = \frac{1}{2}\frac{d_i^k}{d_{i-1}^k + 2d_i^k + d_{i+1}^k}, \quad q^0 = \min_i q_i^0.$$

Furthermore, let λ be an arbitrary real number with $1/2 < \lambda < 1$. Then, the four point scheme with

$$w = \min\{\lambda q^0, \frac{1}{4}\lambda(1-\lambda), \lambda - \frac{1}{2}\}, \tag{2}$$

preserves convexity and generates C^1 limit functions.

PROOF. The scheme for the second order divided differences d_j^k is given by [3]:

$$\begin{cases} d_{2i+1}^{k+1} = 8w(d_i^k + d_{i+1}^k), \\ d_{2i}^{k+1} = (2-8w)d_i^k - 4w(d_{i-1}^k + d_{i+1}^k). \end{cases}$$

It is necessary for preservation of strict convexity that $d_{2i+1}^{k+1} > 0$ and $d_{2i}^{k+1} > 0$, provided $d_i^k > 0$. Observe that the choice of w, $w > 0$, shows that $d_{2i+1}^{k+1} > 0$. Next, we prove by induction that

$$\lambda q_i^k \geq w, \quad \forall i, k, \tag{3}$$

which is sufficient for convexity preservation, as $\lambda < 1$. Indeed,

$$d_{2i}^{k+1} = 2d_i^k - 4w(d_{i-1}^k + 2d_i^k + d_{i+1}^k) = 2d_i^k\left(1 - \frac{w}{q_i^k}\right) > 0.$$

By (2), (3) holds for $k = 0$. The following estimates are obtained using the induction hypothesis and (2):

$$\begin{aligned} \lambda q_{2i}^{k+1} &= \lambda \frac{1}{2}\frac{d_{2i}^{k+1}}{d_{2i-1}^{k+1} + 2d_{2i}^{k+1} + d_{2i+1}^{k+1}} = \frac{1}{4}\lambda - w\frac{\lambda}{2}\frac{d_{i-1}^k + 2d_i^k + d_{i+1}^k}{d_i^k} \\ &= \frac{1}{4}\lambda - \frac{\lambda}{2}\frac{\lambda}{2}\frac{w}{\lambda q_i^k} \geq \frac{1}{4}\lambda - \frac{1}{4}\lambda^2 = \frac{1}{4}\lambda(1-\lambda) \geq w, \quad \text{and} \\ \lambda q_{2i+1}^{k+1} &= \lambda \frac{2w(d_i^k + d_{i+1}^k)}{(1+2w)(d_i^k + d_{i+1}^k) - 2w(d_{i-1}^k + d_{i+2}^k)} \\ &\geq \lambda \frac{2w(d_i^k + d_{i+1}^k)}{(1+2w)(d_i^k + d_{i+1}^k)} = \lambda \frac{2w}{1+2w} \geq (w + 1/2)\frac{2w}{1+2w} = w, \end{aligned}$$

which shows that convexity is preserved.

The tension parameter w is bounded from above by (2), hence

$$\frac{1}{4}\lambda^*(1-\lambda^*) = \lambda^* - \frac{1}{2} \quad \Longrightarrow \quad \lambda^* = \frac{1}{2}\sqrt{17} - \frac{3}{2},$$

i.e.,

$$0 < w \leq \lambda^* - \frac{1}{2} = \frac{1}{2}\sqrt{17} - 2 = 0.06155 < 0.0625 = \frac{1}{16},$$

which shows that the scheme has only approximation order two.

Remark 2 *Depending on the initial data, the actual value of* λ *can easily be optimised such that the tension parameter* w *is as large as possible. Then* w *is closer to the value* $1/16$ *for which the scheme is fourth order accurate and almost* C^2*, see [2].*

Furthermore, it is possible, using the above analysis, to construct non-stationary convexity preserving subdivision schemes by choosing w *in (1) depending on* k *(*w^k*). Indeed, it is proved in [6] that the four-point scheme is* C^1 *if* w^k *is chosen randomly in the interval* $[\epsilon, 1/8 - \epsilon]$ *for any* $\epsilon \in]0, 1/16[$.

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