Stability criteria for planar linear systems with state reset

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***Abstract*— In this work we perform a stability analysis for a class of switched linear systems, modeled as hybrid automata. We deal with a switched linear planar system, modeled by a hybrid automaton with one discrete state. We assume the guard on the transition is a line in the state space and the reset map is a linear projection onto the x-axis. We define necessary and sufficient conditions for stability of the switched linear system with fixed and arbitrary dynamics in the location.**

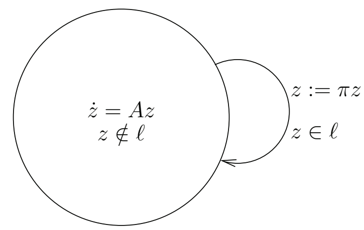

Fig. 1. Linear planar system with state reset

I. Introduction

In this paper we study a seemingly simple situation: a single linear planar system with a state reset. We derive a complete characterization and an algorithm to determine stability.

The paper is motivated by problems occurring in reset control. To overcome control limitations various nonlinear feedback controllers for linear time-invariant systems were proposed, particularly, reset control is one of such controllers. Basically it consists of a linear controller whose states is reset to zero when the input and output satisfy certain conditions. The first resetting element was introduced in 1958 by Clegg: the so-called Clegg integrator, which resets whenever the input is zero [7].

Furthermore, in a series of papers, [10], [11], reset control systems have been advanced by introducing the first-order reset element.

One of the main disadvantage of reset controllers is that the reset action may destabilize a stable feedback system. Recent work, [1], [3], [9], addressed the stability problem of this type of systems.

Reset control systems can be also considered as a special case of hybrid systems.

Stability analysis for hybrid systems is a much harder problem than it is for smooth systems. The reason appears to be the interplay between continuous time driven dynamics and discrete event driven dynamics. See [2], [5], [6], [4], [8], [12], [13] and the references therein.

The problems studied in this paper, simple as they might appear, form no exception to this observation.

II. Problem statement

The class of systems that we study can conveniently be modeled by a hybrid automaton, see Figure 1.

The dynamics in the location is described by a system of differential equations:

$$\dot{z} = Az, \tag{1}$$

which is asymptotically stable, i.e. $A \in \mathbb{R}^{2\times 2}$ is a Hurwitz matrix (every eigenvalue of A has strictly negative real part). The guard on the transition is a hyperplane in the state space, i.e. a line $\ell : y = kx$, for some $k \in \mathbb{R}$ and π is the orthogonal projection onto the x-axis.

The state is reset by orthogonal projection on the x-axis whenever the state trajectory crosses the switching line ℓ. Although A is Hurwitz, the state reset may lead to instability. The problem is particularly interesting for systems with oscillatory behavior, therefore we restrict our attention to matrices with complex conjugate eigenvalues:

$$\lambda = \alpha \pm \beta \mathrm{i}, \alpha < 0, \beta \neq 0. \tag{2}$$

For future reference we define:

$$\mathcal{A} = \{A \in \mathbb{R}^{2\times 2} | \mathrm{spec}(A) = \alpha \pm \beta \mathrm{i}, \alpha < 0, \beta \neq 0\}. \tag{3}$$

In the sequel, without loss of generality, we assume that all trajectories progress anti-clockwise in time. This corresponds to $a_{21} > 0$ for all matrices A that we consider. Indeed, all results holds, *mutatis mutandis*, for the cases that $a_{21} < 0$.

The following problems are treated:

1) Find a criterion that for a given pair (A, ℓ) determines its stability properties (Section 3).
2) For a given matrix A, find all switching lines ℓ for which the system is (asymptotically) stable (Section 4).
3) For a given switching line ℓ, find all matrices A for which the system is (asymptotically) stable (Section 5).

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III. Preliminaries

In this section we give definitions for hybrid automaton, hybrid trace, stability, asymptotic stability and instability of system described in Figure 1. For further details we refer to [12], [14].

Definition 3.1: A linear hybrid automaton is a tuple $H = (Z, Q, f, \mathrm{Init}, \mathrm{Inv}, E, \mathrm{Guard}, \mathrm{Re})$, where

- $Z = \mathbb{R}^2$ is the set of continuous states;
- $Q = \{q\}$ is the set of discrete states or locations;
- For $q \in Q$ $f(q, \cdot) : Z \to \mathbb{R}^2$ is a linear dynamics, given by $\dot{z} = f(z) = Az$, where $A \in \mathbb{R}^{2\times 2}$;
- $\mathrm{Init} \subseteq Q \times \mathbb{R}^2$ is a set of initial states;
- $\mathrm{Inv}(\cdot) : Q \to 2^{\mathbb{R}^2}$ is a set given by $\{z \in \mathbb{R}^2 | v_{\mathrm{e}}^T z \neq 0\}$ is an invariant for state z in the location q;
- $E = \{(q, q)\}$ is a set of transitions, called switches;
- $\mathrm{Guard}(\cdot, \cdot) : E \to 2^{\mathbb{R}^2}$ is a hyperplane l given by $\{z \in \mathbb{R}^2 | v_{\mathrm{e}}^T z = 0\}$
- $\mathrm{Re}(\cdot, \cdot, \cdot) : E \times \mathbb{R}^2 \to \mathbb{R}^2$ is a reset relation.

A hybrid automaton defines possible evolutions over time. Starting from initial condition $(q, z_0) \in \mathrm{Init}$, the continuous state z flows according to the differential equation $\dot{z} = Az$, as long as $z(t) \in \mathrm{Inv}(q)$. If at some point $z(t) \in \mathrm{Guard}(q, q)$ then a discrete transition takes place. During a discrete transition the continuous states can be switched to some value in $\mathrm{Re}(q, q, z)$.

Definition 3.2: A linear hybrid trace is an infinite sequence $\alpha = z_1 e_1 z_2 e_2 z_3 e_3 \ldots$ with associated infinite monotonically increasing time sequence of intervals $\tau = \{[\tau_i, \tau_{i+1})\}_{i=0}^{\infty}$, s.t. $\tau_i < \tau_{i+1}$ for all i, and $\tau_0 = 0$, which satisfies the following conditions:

- $(q, z_0(0))$ is the initial state;
- for all i, $z_i(\tau_{i+1}) \in \mathrm{Guard}(q)$ and $z_{i+1}(\tau_{i+1}) = \mathrm{Re}(q, q, z_i(\tau_{i+1}))$;
- for all i, as soon as $v_e^T z_i(\tau_i) = 0$, where $v_e \in \mathbb{R}^2$ then a transition takes place and $\mathrm{Re}(q, q, z_i) : E \times \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $\mathrm{Re}(q, q, x_i(\tau_{i+1}), y_i(\tau_{i+1})) = (x_{i+1}(\tau_{i+1}), 0)$;
- each z_i is the solution to the differential equation $\dot{z}_i = A_q z_i$ over time interval $[\tau_i, \tau_{i+1})$, starting at $z_i(\tau_i)$;
- for all $t \in [\tau_i, \tau_{i+1})$, $z_i(t) \in \mathrm{Inv}(q)$ for all i.

Denote by $||z||$ the Euclidean norm.

Definition 3.3: A linear hybrid automaton is stable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all hybrid traces $z_1 e_1 z_2 e_2 z_3 e_3 \ldots$, starting at (q, z_0),

$$||z_0|| < \delta \Rightarrow ||z_i(t)|| < \varepsilon, \quad \forall i \forall t \in [\tau_i, \tau_{i+1})$$

Definition 3.4: A linear hybrid automaton is asymptotically stable if it is stable and there exists δ such that for all hybrid traces $z_1 e_1 z_2 e_2 z_3 e_3 \ldots$, starting at (q, z_0),

$$||z_0|| < \delta \Rightarrow \lim_{i\to\infty} ||z_i(\tau_i)|| = 0, \quad \forall i$$

A linear hybrid automaton is called unstable if it is not stable.

IV. Stability conditions for a given pair (A, ℓ): Problem 1

In this section we formulate necessary and sufficient conditions for stability of the linear switched system depicted in Figure 1. We assume that the dynamics in the location and the switching line are given. Our objective is to find criteria that guarantee (asymptotic) stability of the system for the given pair (A, ℓ).
Consider any nonzero solution of $\dot{z} = Az$ and let $z_0 = (x_0, y_0)$ and $z_1 = (x_1, y_1)$ be two consecutive points on the intersection of the trajectory with the x-axis and the line $y = kx$ respectively. Define the projection gain $q = \frac{x_1}{x_0}$ (see Figure 2).

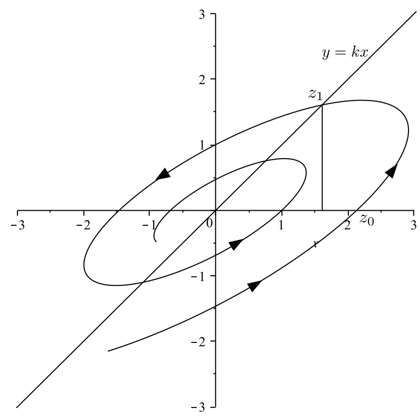

Fig. 2. Stable dynamics

Theorem 4.1: The system depicted in Figure 1 is asymptotically stable iff $q < 1$; is stable iff $q = 1$ and is unstable iff $q > 1$.
Proof
Suppose that a trajectory (a solution of the system (1)) crosses the x-axis at the time τ_0, define $z_0(\tau_0) = (x_0(\tau_0), 0)$ the intersection point and suppose the trajectory crosses the switching line $y = kx$ at time:

$$\tau_1 = \min\{\tau > \tau_0 |\ y(\tau) = kx(\tau)\}. \qquad (4)$$

Note that τ_0 and τ_1 are well-defined because the eigenvalues of A are $\alpha \pm \beta i$ with $\beta \neq 0$. Project the point $z_0(\tau_1)$ onto the x-axis, define $z_1(\tau_1) = (x_0(\tau_1), 0)$ the projection point. Consider the projection gain $q = \frac{|x_1(\tau_1)|}{|x_0(\tau_0)|}$. This ratio does not depend on the choice of τ_0, therefore for any trajectory we can choose any τ_0, τ_1, which satisfy (4).
Suppose that the projection gain $q < 1$. Assume that the trajectory crosses the switching line $y = kx$ $(k > 0)$ at time $\tau_1, \tau_2, \tau_3, \ldots$. Notice that $\tau_{i+1} - \tau_i$ is independent of i, say, $\tau_{i+1} - \tau_i = c$ (for all i). Obtain a hybrid trace (a solution) of the system, depicted in Figure 1 with initial condition $z_0(0) = z(\tau_0)$ as follows:

$$z_i(\tau) = \mathrm{e}^{A(\tau-\tau_i)} \cdot P \cdot \ldots \cdot P \cdot \mathrm{e}^{A(c)} P \cdot \mathrm{e}^{A(c)} \cdot P \cdot \mathrm{e}^{A(\tau_1)} z_0, \quad (5)$$

where $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is the orthogonal projection onto the x-axis.
Define:

$$z_i(\tau_i) = P\cdot e^{A(\tau_i-\tau_{i-1})}\cdot P\cdot\ldots\cdot P\cdot e^{A(\tau_2-\tau_1)}\cdot P\cdot e^{A(\tau_1)}z_0. \quad (6)$$

Since $||Pe^{A(\tau_i-\tau_{i-1})}|| = \frac{||z_{i+1}(\tau_{i+1})||}{||z_i(\tau_i)||} = \frac{||z_1(\tau_1)||}{||z_0(\tau_0)||} < 1$ for all i, where $||z_i(\tau_i)|| = ||(x_i(\tau_i), 0)|| = |x_i(\tau_i)|$, the sequence $\{|x_i(\tau_i)|\}$, converges to zero when i tends to infinity. Hence we have $\lim_{i\to\infty} ||z_i(\tau_i)|| = 0$. Moreover $\max_{\tau\in[0,c)} ||e^{A\tau}|| = M < \infty$. For all $\tau \in [0, c)$ we have $||z_i(\tau)|| \leqslant M||z_i(\tau_i)||$ and $\lim_{\tau\to\infty} ||z_i(\tau)|| = 0$. We conclude the system is asymptotically stable.
Suppose that the projection gain $q = 1$, we prove that the system is stable. As before $\frac{||z_{i+1}(\tau_{i+1})||}{||z_i(\tau_i)||} = \frac{||z_1(\tau_1)||}{||z_0(\tau_0)||} = 1$ for all i. Choose $\delta > 0$ s.t. $||z_0(\tau_0)|| < \delta$. Hence $||z_i(\tau_i)|| < \delta$ (for all i). Moreover $||z_i(\tau)|| \leqslant M||z_i(\tau_i)||$ for all $\tau \in [0, c)$ and therefore $||z_i(\tau)||$ is bounded. Hence the system is stable.
Suppose that the projection gain $q > 1$, we prove that the system is unstable. Since this projection gain does not depend on a choice of time we have $||z_{i+1}(\tau_{i+1})|| > q||z_i(\tau_i)||$ for all i. This means that the state $z(\tau)$ increases as τ tends to infinity and therefore the system is unstable.

□

V. Parameterization of the set $\mathcal{A}$

In this section we provide a complete parameterization of the class of matrices $\mathcal{A}$.
The parameterization is geometrically appealing and is particularly appropriate for our stability analysis.

Theorem 5.1: A matrix A belongs to the set $\mathcal{A}$ as defined in (3) iff there exist $r > 0$ and $\theta \in [0, 2\pi)$ such that:

$$A = S_\theta^{-1} S_r^{-1} \bar{A} S_r S_\theta, \quad (7)$$

where $\bar{A} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$, $S_r = \begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix}$ and

$$S_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \quad (8)$$

Proof

($\Leftarrow$) If A is given by (7) then the eigenvalues of A are equal to the eigenvalues of $\bar{A}$ since a basis transformation does not change the eigenvalues.

($\Rightarrow$) We show that for any A in the set $\mathcal{A}$ there exist r and θ such that (7) is satisfied.
Let A be an arbitrary matrix in $\mathcal{A}$. There exists $S \in \mathrm{GL}(2, \mathbb{R})$ such that:

$$A = S^{-1}\bar{A}S. \quad (9)$$

Using singular value decomposition (SVD) the matrix S can be written as follows:

$$S = UDV, \quad (10)$$

where U, V are orthonormal matrices and D is the diagonal matrix, consisting of the singular values of S: $D = \mathrm{diag}\{\sigma_1, \sigma_2\}$.

Substituting (10) into (9) we have:

$$S^{-1}\bar{A}S = V^T D^{-1} U^T \bar{A} U D V = V^T D^{-1} \bar{A} D V, \quad (11)$$

since U is a rotation matrix of the form (8), it is easy to verify that the matrices $\bar{A}$ and U commute.
From (11) it follows that we can take $S_r = \begin{bmatrix} \sqrt{\frac{\sigma_1}{\sigma_2}} & 0 \\ 0 & 1 \end{bmatrix}$ and $S_\theta = V$.

□

VI. Stability conditions for systems with given dynamics: Problem 2

In this section we formulate stability criteria for systems with given dynamics. The objective is to find all switching lines l that guarantee asymptotic stability of the system.
Let $A \in \mathcal{A}$ be given. Denote the upper left entry by a_{11}. Suppose, without loss of generality, that a trajectory (a solution of the system $\dot{z} = Az$) crosses the x-axis at the point $[1\ 0]^T$ at the time $\tau_0 = 0$.

Theorem 6.1: If $a_{11} \leqslant 0$ then the system is asymptotically stable for all lines $y = kx$ ($k > 0$), if $a_{11} > 0$ then there exists $k_1 > 0$, s.t. if $k > k_1$ then the system is asymptotically stable, if $k = k_1$ then the system is stable and if $0 < k < k_1$ then the system is unstable.
Proof
Suppose that $a_{11} \leqslant 0$. We prove that the system is asymptotically stable for all the lines $y = kx$ ($k > 0$). A solution $z(\tau) = (x(\tau), y(\tau))$ represents a parametric curve in the (x, y) plane, $z'(\tau) = (x'(\tau), y'(\tau))$ is its tangent vector. Since every matrix $A \in \mathcal{A}$ can be written in the form (7) it follows that the first component of $A\,[x\ y]^T$ is as follows:

$$\left[\alpha + (r\beta - \frac{\beta}{r})\sin\theta\cos\theta\right] x - \left[\frac{\beta}{r}\cos^2\theta + r\beta\sin^2\theta\right] y, \quad (12)$$

where $x \geqslant 0$, $y \geqslant 0$.
Obviously (12) is always negative for $\alpha < 0, \beta > 0, r > 0, 0 \leqslant \theta \leqslant 2\pi$. This means that $x(\tau)$ is decreasing in the first quadrant. Therefore the projection gain $\frac{|x(\tau_1)|}{|x(\tau_0)|} < 1$ for all $\tau_0 < \tau_1$. Asymptotic stability follows (see Figure 3).
Suppose that $a_{11} > 0$. We prove that there exists $k_1 > 0$, s.t. if $k > k_1$ then the system is asymptotically stable, if $k = k_1$ then the system is stable and if $0 < k < k_1$ then the system is unstable.
Since $a_{11} > 0$, the function $x(\tau)$ is increasing in a neighborhood of the point $[1\ 0]^T$. The fact that the matrix A

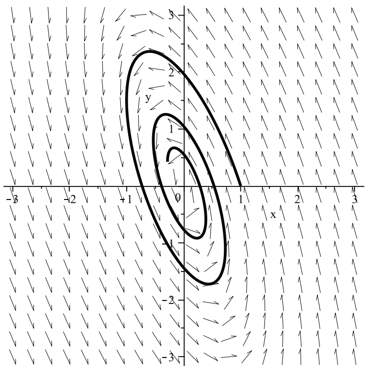

Fig. 3. Asymptotically stable dynamics for all switching lines $y = kx$ ($k > 0$)

is Hurwitz guarantees the function $x(\tau) \to 0$ as $\tau \to \infty$, therefore $\frac{|x(\tau)|}{|x(\tau_0)|} \to 0$ as $\tau \to \infty$. Since $x(\tau)$ is continuous there exists a time point $\tau_1 > \tau_0$ such that $\frac{|x(\tau_1)|}{|x(\tau_0)|} = 1$. This defines k_1.

The time point τ_1 is unique for $0 \leqslant \tau \leqslant \frac{\pi}{2}$ since there exist τ' such that $\dot{x}(\tau') = 0$ and $\tau_1 > \tau'$ therefore $\dot{x}(\tau) < 0$ for all $\tau_1 \leqslant \tau \leqslant \frac{\pi}{2}$ and $z(\tau)$ is asymptotically stable solution.

Hence there exists switching line $y = k_1 x$ ($k_1 = \frac{y(\tau_1)}{x(\tau_1)} > 0$), for which the system is stable, and for $k > k_1$ the system is asymptotically stable since the projection gain $\frac{|x(\tau_1)|}{|x(\tau_0)|} < 1$, and for $0 < k < k_1$ the system is unstable (see Figure 4). □

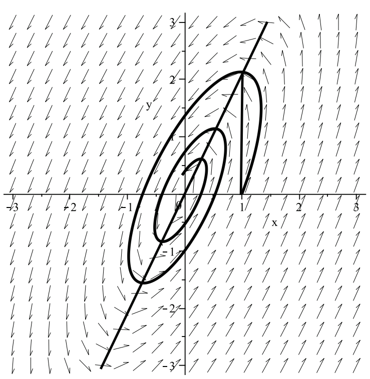

Fig. 4. Dynamics with splitting switching line

Corollary 6.2: The system, as depicted in Figure 1, with dynamics in the location of the form (7) is asymptotically stable for $\theta = 0$ and $\theta = \frac{\pi}{2}$.

Example 6.3: Let $\dot{z} = Az$ be the given dynamics in the location with

$$A = \begin{bmatrix} 1.775 & -2.125 \\ 2.125 & -1.975 \end{bmatrix}. \tag{13}$$

Matrix A is obtained from $\bar{A} = \begin{bmatrix} -0.1 & -1 \\ 1 & -0.1 \end{bmatrix}$ by taking $r = 4$, $\theta = \frac{\pi}{4}$, so that $A = S_\theta^{-1} S_r^{-1} \bar{A} S_r S_\theta$.

We find all switching lines $y = kx$, for which the system with above-mentioned dynamics is asymptotically stable, stable and unstable respectively.

Compute the solution of the system with initial conditions $z(0) = [1\ 0]^T$, i.e. $[x(t)\ y(t)]^T = \mathrm{e}^{At}[1\ 0]^T$, we have

$$x(t) = \mathrm{e}^{-0.1t}\left(\frac{15}{8}\sin t + \cos t\right), \tag{14}$$
$$y(t) = \frac{17}{8}\mathrm{e}^{-0.1t}\sin t.$$

Next we obtain that the trajectory (14) crosses the line $x = 1$ at time $t_1 = 2.036$, which satisfy (4). Compute $y(t_1) = 1.5493$. We can conclude that there exists $k_1 = 1.5493$ s.t. for $k > 1.5493$ the system (13) is asymptotically stable, for $k = 1.5493$ the system is stable and for $0 < k < 1.5493$ the system is unstable (see Figure 5).

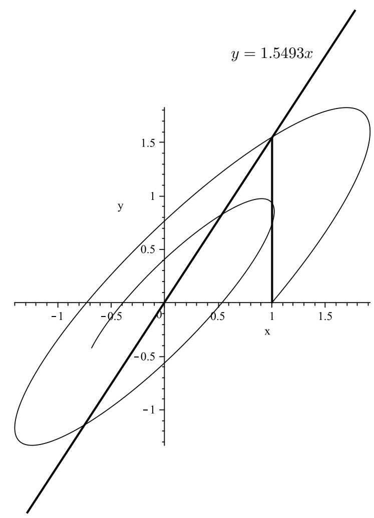

Fig. 5. Illustration of Example 6.3

VII. Stability conditions for systems with given switching line: Problem 3

In this section we formulate stability conditions for systems with arbitrary dynamics and given switching line. We assume that $k > 0$ is given and our objective to characterize all $r > 0$ and θ ($0 \leqslant \theta < 2\pi$) such that the system is (asymptotically) stable.

Let $z_0(\theta) = [x_0(\theta)\ y_0(\theta)]^T$ and $z_1(\theta) = [x_1(\theta)\ y_1(\theta)]^T$ be two consecutive points on the intersection of the trajectory with the x-axis and the line $y = kx$ respectively.

The projection gain can be expressed as a function of θ: $q(\theta) = \frac{|x_1(\theta)|}{|x_0(\theta)|}$.

We formulate and prove a lemma that we use for the proof of the next theorem.

Lemma 7.1: Suppose $f, g : [a, b] \to \mathbb{R}$ and f, g are C^1-functions. Assume that $g(a) < 0$, $g(b) > 0$, $g'(x) > 0$ for all $x \in [a, b]$. There exists $\varepsilon > 0$ s.t. for $|\alpha| < \varepsilon$ the function $h_\alpha(x) = \alpha f(x) + g(x)$ has a unique zero in the interval $[a, b]$.

PROOF

Consider $h_\alpha(a) = \alpha f(a) + g(a)$. Since $g(a) < 0$ there exists $\varepsilon_a > 0$ such that for $|\alpha| \leqslant \varepsilon_a$ $h_\alpha(a) = \alpha f(a) + g(a) < 0$.

Similarly, since $g(b) > 0$ there exists $\varepsilon_b > 0$ such that for $|\alpha| \leqslant \varepsilon_b$ $h_\alpha(b) = \alpha f(b) + g(b) < 0$.

Since $f(x)$ is a C^1–function it follows that $|f'(x)| \leqslant K$, for some $K > 0$, for all $x \in [a, b]$. Consider $h'_\alpha(x) = \alpha f'(x) + g'(x)$. Since $g'(x) > 0$ and $|f'(x)| \leqslant K$ there exists $\varepsilon_m > 0$ such that $h'_\alpha(x) > 0$ for $|\alpha| \leqslant \varepsilon_m$.

Finally we take $\varepsilon = \min\{\varepsilon_a, \varepsilon_b, \varepsilon_m\}$.

We conclude $h_\alpha(x)$ for all $|\alpha| < \varepsilon$ has a unique zero in the interval $[a, b]$.

□

Theorem 7.2: Consider the system (1). Let $k > 0$ be given. For all $r > 1$ there exists $\delta < 0$ such that for all $\alpha < \delta$ the system is asymptotically stable for all $\theta \in [0, 2\pi)$. There exists $\varepsilon > 0$ such that for $|\alpha| < \varepsilon$ there exist $\theta_1, \theta_2 \in [0, \frac{\pi}{2}]$ s.t. the system is:

1) Stable for $\theta = \theta_1$ and for $\theta = \theta_2$.
2) Unstable for $\theta_1 < \theta < \theta_2$.
3) Asymptotically stable for $0 \leqslant \theta < \theta_1$, $\theta_2 < \theta \leqslant \pi$.
4) Stable for $\theta = \theta_1 + \pi$ and for $\theta = \theta_2 + \pi$.
5) Unstable for $\theta_1 + \pi < \theta < \theta_2 + \pi$.
6) Asymptotically stable for $\pi \leqslant \theta < \theta_1 + \pi$, $\theta_2 + \pi < \theta \leqslant 2\pi$.

PROOF

For clarity of presentation, we equip the projection gain q with subscript α, to emphasize the dependency of q on α.

First we consider the case $\alpha = 0$. In this case the trajectories of $\dot{z} = Az$ are ellipsoids. We show that there exist at most 2 solutions such that $0 < \theta < \frac{\pi}{2}$, for which the projection gain $q_0(\theta) = 1$ see Figure 6.

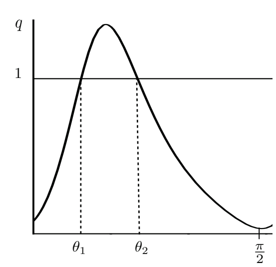

Fig. 6. Illustration of Theorem 7.2

It is easy to verify that

$$A^T S^T S + S^T S A = 0, \tag{15}$$

where $S = S_r S_\theta$ as defined in Theorem 5.1.

Define the Lyapunov function $V(z) = z^T P_\theta z$ with $P_\theta = S^T S$, $P_\theta = P_\theta^T > 0$. We use the subscript θ to emphasize the dependency of P on θ.

Next, we study the behavior of the projection gain $q_0(\theta)$ as a function of θ. For the particular case that $\alpha = 0$ $q_0(\theta)$ is characterized as follows.

Define the ellipse E as:

$$E = \left\{ z \in \mathbb{R}^2 \mid z^{\mathrm{T}} P_\theta z = 1 \right\}. \tag{16}$$

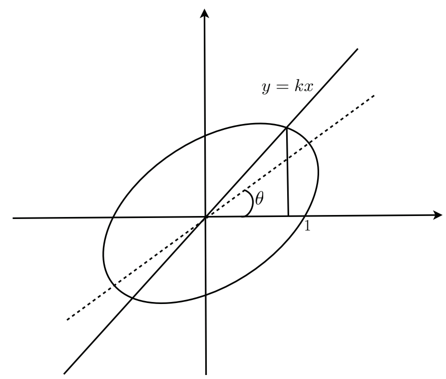

Fig. 7. Solution of the system (1) for $\alpha = 0$

Define the point z_0 as the intersection of E with the positive x-axis, and z_1 as the intersection in the first quadrant of E with the line $y = kx$. By the implicit function theorem it follows that both z_0 and z_1 depend C^1 on θ. In terms of $z_0(\theta)$ and $z_1(\theta)$, $q_0(\theta)$ may be expressed as:

$$q_0(\theta) = \frac{x_1(\theta)}{x_0(\theta)}, \tag{17}$$

and therefore also q_0 is a C^1 function on $[0, 2\pi]$. To determine the values for which the system is just stable we consider the equation in θ:

$$q_0(\theta) = 1. \tag{18}$$

This equation is equivalently characterized by:

$$\begin{bmatrix} x & 0 \end{bmatrix} P_\theta \begin{bmatrix} x \\ 0 \end{bmatrix} = 1 \text{ and } \begin{bmatrix} x & kx \end{bmatrix} P_\theta \begin{bmatrix} x \\ kx \end{bmatrix} = 1, \tag{19}$$

where

$$P_\theta = \begin{bmatrix} r^2 \cos^2(\theta) + \sin^2(\theta) & (1 - r^2)\sin(\theta)\cos(\theta) \\ (1 - r^2)\sin(\theta)\cos(\theta) & r^2 \sin^2(\theta) + \cos^2(\theta) \end{bmatrix}.$$

After some straightforward calculations this leads to:

$$(1 - r^2)\sin(2\theta) - k(r^2 - 1)\cos^2(\theta) + kr^2 = 0. \tag{20}$$

Using the substitution $w = \tan(\theta)$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ we find $\sin(2\theta) = \frac{2w}{1+w^2}$, $\cos^2(\theta) = \frac{1}{1+w^2}$ and we have a quadratic equation in variable w:

$$-kr^2 w^2 + 2(r^2 - 1)w - k = 0. \tag{21}$$

The equation (21) has at most 2 solutions and moreover the left hand-side has a unique maximum. Since the transformation $w = \tan(\theta)$ is monotonic, the same holds for (20): at most 2 solutions and the left hand-side of (20) has a unique maximum at θ^*, say. It is obvious from (21) that $\theta^* \in (0, \frac{\pi}{2})$.

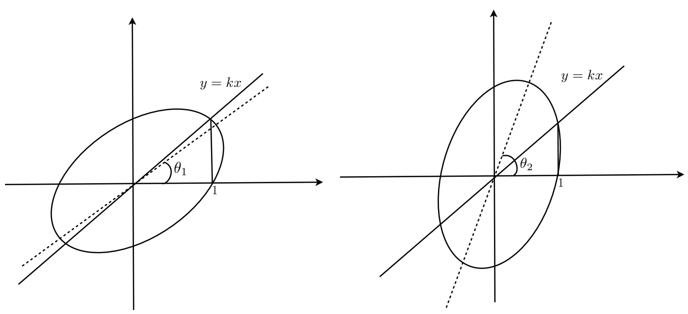

Fig. 8. Two solutions θ_1, θ_2 for $q_0(\theta) = 1$

For $\theta = \theta_1$ and $\theta = \theta_2$ the system is stable since the projection gain $q_0(\theta) = 1$. This establishes *1)*.
From Corollary 6.2 it follows that $q_0(0) < 1$ and $q_0(\frac{\pi}{2}) < 1$. Therefore $q_0(\theta) < 1$ for $0 \leqslant \theta < \theta_1$, $\theta_2 < \theta \leqslant \frac{\pi}{2}$. From Theorem 6.1 it follows that for $\theta \in [\frac{\pi}{2}, \pi]$ the system is asymptotically stable and therefore *3)* holds.
Suppose that $\theta_1 \neq \theta_2$. Since the left hand-side of (20) has a maximum at θ^* it follows that the projection gain $q_0(\theta) > 1$ for $\theta_1 < \theta < \theta_2$, this means the system is unstable in this interval and *2)* holds.
Since $\tan\theta$ is a periodic function with period π we conclude that *4)* − *6)* hold.

We determine the 'traveling time' from the x-axis to the line $y = kx$ as follows.

$$z(\tau) = \mathrm{e}^{A_\theta \tau} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{22}$$

The traveling time τ_1 as a function of θ is now defined as:

$$\tau_1(\theta) = \min\left\{\tau > 0 \mid y(\tau) = kx(\tau)\right\}. \tag{23}$$

A routine calculation yields:

$$\tau_1(\theta) = \frac{1}{\beta}\arctan\left(\frac{kr}{\frac{k}{2}(1-r^2)\sin 2\theta + \sin^2\theta + r^2\cos^2\theta}\right), \tag{24}$$

where $\arctan$ is taken in $[0, \pi]$ rather than in $(-\pi/2, \pi/2)$. It follows that τ_1 is a strictly positive C^1 function on $[0, 2\pi]$.

Next we consider the case $\alpha < 0$.
The projection gain is given by:

$$q(\theta) = \mathrm{e}^{\alpha\tau_1(\theta)} q_0(\theta). \tag{25}$$

There are two possibilities: if $q_0(\theta) \leqslant 1$ for all $\theta \in [0, 2\pi]$, then obviously $q_\alpha(\theta) < 1$ for all θ and the system is asymptotically stable for all θ.
If, however, $q_0(\theta) > 1$ for some θ, then we know from the case $\alpha = 0$ that there are exactly two values θ_1 and θ_2 for which $q_0(\theta) = 1$.

First we show that there exists $\delta < 0$ s.t. for all $\alpha < \delta$ the system depicted in Figure 1 is asymptotically stable for all $\theta \in [0, 2\pi]$.
Since the functions $\tau_1(\theta)$ and $q_0(\theta)$ are C^1 we can define $\bar{\tau}_1 = \min_{0 \leqslant \theta \leqslant 2\pi} \tau_1(\theta)$ and $\bar{q}_0 = \max_{0 \leqslant \theta \leqslant 2\pi} q_0(\theta)$.
It follows that:

$$\begin{aligned} q_\alpha(\theta) &= \mathrm{e}^{\alpha\tau_1(\theta)} q_0(\theta), \\ &\leqslant \mathrm{e}^{\alpha\bar{\tau}_1}\bar{q}_0. \end{aligned} \tag{26}$$

Choose $\delta = -\frac{1}{\bar{\tau}_1}\log \bar{q}_0(\theta)$, and it follows that for $\alpha < \delta$ the projection gain $q_\alpha(\theta) < 1$ for all $\theta \in [0, 2\pi]$.

Finally we show that there exists $\varepsilon > 0$ s.t. for $|\alpha| < \varepsilon$ the equation

$$q_\alpha(\theta) = 1. \tag{27}$$

Or, equivalently, the equation

$$\alpha\tau_1(\theta) + \log q_0(\theta) = 0 \tag{28}$$

has at most 2 solutions θ', θ'' in the interval $[0, \frac{\pi}{2}]$.
Since the left hand-side of (20) has a unique maximum at θ^* it follows that the function $\log q_0(\theta)$ has a unique maximum at θ^* as well.
Take $a = 0$ and $b \in (\theta_1, \theta^*)$. With $f(\theta) = \tau_1(\theta)$ and $g(\theta) = \log q_0(\theta)$ the conditions of Lemma 7.1 are satisfied. It follows that there exists $\varepsilon_1 > 0$ s.t. for $|\alpha| \leqslant \varepsilon_1$ there exists a unique zero of $\alpha\tau_1(\theta) + \log q_0(\theta)$ in the interval $[0, b]$.
Likewise, with $c \in (\theta^*, \theta_2)$, there exists $\varepsilon_2 > 0$ s.t. for $|\alpha| \leqslant \varepsilon_2$ there exists a unique zero of $\alpha\tau_1(\theta) + \log q_0(\theta)$ in the interval $[c, \frac{\pi}{2}]$.
Moreover, there exists $\varepsilon_3 > 0$ s.t. for $|\alpha| \leqslant \varepsilon_3$ and $\theta \in [b, c]$

$$\alpha\tau_1(\theta) + \log q_0(\theta) > 0. \tag{29}$$

Finally we take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$.

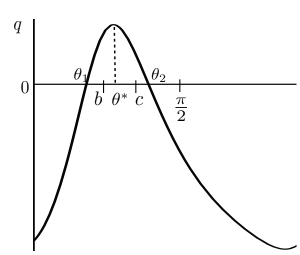

Fig. 9. The function $\log q_0(\theta)$

□

Remark 7.3: A similar result holds for $0 < r < 1$. In that case $\theta_i \in [\pi, \frac{3\pi}{2}]$, $i = 1, 2$.

Remark 7.4: In Theorem 7.2 we presented the stability conditions for $\alpha < \delta$ and $|\alpha| < \varepsilon$. For intermediate values of α a proof along the lines of the proof of Theorem 7.2 has not yet been established. However, extensive simulations for various values of α strongly suggest that the statements 1-6 of Theorem 7.2 remain true. See Figure 10 for an account of this claim.

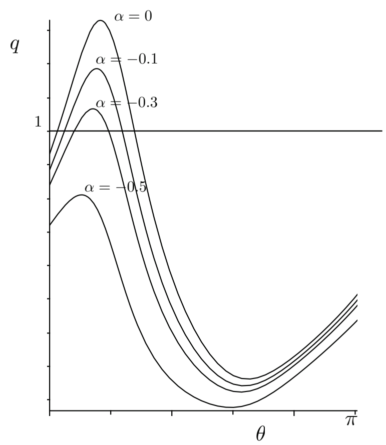

Fig. 10. Projection gain $q_\alpha(\theta)$ for different values of α

Example 7.5: Consider the linear system of differential equations $\dot{z} = Az$. Suppose that $A \in \mathcal{A}$. Find θ $(0 \leqslant \theta \leqslant 2\pi)$ s.t. the system, as depicted in Figure 1, is (asymptotically) stable.

Let be $r = 3, \alpha = -0.1, \beta = 1$ and $y = 2x$ the switching line are given.

We plot the graph to demonstrate the relation between θ and q. It is easily to seen that there exist $\theta_1 = \frac{\pi}{3}$ and $\theta_2 = \frac{3\pi}{7}$ belong to the interval $[0, \frac{\pi}{2}]$, for which $q(\theta) = 1$ (see Figure 11).

Therefore for $\theta = \{\frac{\pi}{3}, \frac{\pi}{3} + \pi, \frac{3\pi}{7}, \frac{3\pi}{7} + \pi\}$ the system is stable; for $\frac{\pi}{3} < \theta < \frac{3\pi}{7}$ and $\frac{\pi}{3} + \pi < \theta < \frac{3\pi}{7} + \pi$ the system is unstable and asymptotically stable for all other values of $\theta \in [0, 2\pi]$.

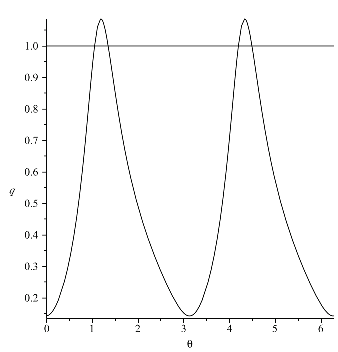

Fig. 11. Illustration of Example 7.5

VIII. Conclusions

In this paper, we have derived stability criteria for switched linear planar systems, modeled by hybrid automaton with one discrete state. We have formulated necessary and sufficient conditions for stability of the switched linear system with fixed and arbitrary dynamics in the location.

A geometry driven parameterization of all systems with given spectrum enabled an analysis on the flow of the system rather than on the equations making the approach to a large extent behavioral in nature.

The ideas sketched here can be extended to higher dimensions at the cost of a considerable more involved analysis.

We have provided the numerical examples and descriptive graphs to illustrate a better understanding of our results.

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