Spherical Hamiltonian Isentropic Two-Layer Model for Atmospheric Dynamics

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Introduction Recently, it has been shown that the numerical Hamiltonian particle mesh method of Frank, Gottwald and Reich [4] arises from a parcel Eulerian-Lagrangian (EL) formulation [2,3]. A parcel Eulerian-Lagrangian Hamiltonian formulation consists of a non-autonomous Hamiltonian description of a particular fluid parcel with as single parcel Hamiltonian function the sum of its kinetic energy (the velocity magnitude squared) and an Eulerian potential evaluated at the parcel's position. However, the fluid is a continuum collection of such particular fluid parcels. The Eulerian potential depends on an Eulerian (pseudo)density and, furthermore, this (pseudo)density is related to all fluid parcel by an integral relation which thus establishes the continuum nature of the fluid. Bokhove and Oliver [2,3] show that several geophysical fluid systems have a parcel EL Hamiltonian formulations, which are readily related to corresponding Eulerian Hamiltonian formulations. The advantages of the new parcel formulation are that it provides new insights into Hamiltonian systems and sometimes simplifies the use of mathematical techniques. Bokhove [1] first used these EL formulations to study balanced models with asymptotic theory and to revisit classic parcel instabilities. As an example, a two-layer model with an isentropic tropospheric and an isentropic stratospheric layer will be derived here on the sphere within the parcel EL framework. Subsequently, it can be shown that the Eulerian Hamiltonian formulation of the two-layer equations readily follows from the parcel EL formulation. Further questions under investigation are whether numerical Hamiltonian particle mesh methods remain worthwhile in the presence of weak forcing and dissipation.

Parcel Eulerian-Lagrangian Hamiltonian formulation Consider an isentropic two-layer atmosphere on the sphere. In the troposphere the potential temperature (or entropy) is taken constant, $\theta = \theta_2$, and, similarly, in the stratosphere the potential temperature is constant $\theta = \theta_1$ with $\theta_1 > \theta_2$. The variables in these layers and at their interfaces are sketched in Fig. 1. The pressure at the bottom topography at $r = r_2(\lambda, \phi)$ is $p_2(\lambda, \phi, t)$ with r the radial coordinate, λ and ϕ the latitude-longitude coordinates, and t the time. The pressure at the tropopause, the interface between troposphere and stratosphere at $r = r_1(\lambda, \phi)$, is $p_1(\lambda, \phi, t)$. The pressure at the top $r = r_0(\lambda, \phi, t)$ of the stratosphere is $p_0 \approx 0$. In these layers the horizontal velocity depends on the latitude-longitude coordinates and time. When the horizontal length and velocity scales along the sphere are much larger than the vertical ones this is a leading order approximation. Furthermore, hydrostatic balance holds to leading order in the radial or vertical direction.

We use subscripts $\alpha = 1, 2$ for the variables in and associated with the tropospheric and stratospheric layers. These are dropped when no confusion arises. Consider a fluid parcel in one of the layers, on a rotating Earth. It has coordinates $X^{\mathbf{1}} = \lambda$ and $X^{\mathbf{2}} = \phi$. The corresponding velocities follow from the spherical geometry as $U = R\cos\phi\,\dot{\lambda}$ and $V = R\,\dot{\phi}$ with $\dot{\lambda} = \mathrm{d}\lambda/\mathrm{d}t$ and $\dot{\phi} = \mathrm{d}\phi/\mathrm{d}t$, using

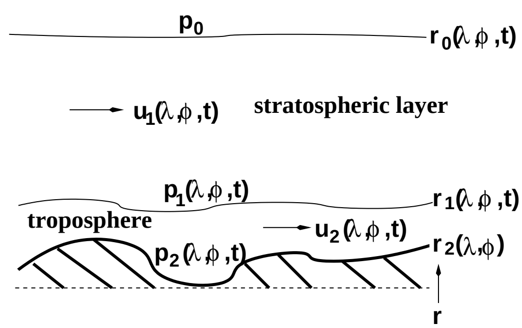

FIGURE 1. Sketch of an atmosphere consisting of an isentropic tropospheric and isentropic stratospheric layer.

$r \approx R$. Related to these velocities are the following covariant (bold lower indices) and contravariant (bold upper indices) vectors

$$U_{\mathbf{1}} = U^{\mathbf{1}}\, R^2\, \cos^2\phi = U\, R\, \cos\phi \quad \text{and} \quad U_{\mathbf{2}} = U^{\mathbf{2}}\, R^2 = V\, R. \tag{1}$$

The kinetic energy on a sphere rotating with a speed Ω is

$$E_{\text{kin}} = \frac{1}{2}\left(R^2 \cos^2\phi\,(\dot\lambda + \Omega)^2 + (R\,\dot\phi)^2\right) = \frac{1}{2}\,(\tilde U^2 + \tilde V^2) \tag{2}$$

for each layer. The effective potential for each layer α is

$$V = M(\lambda, \phi) + \frac{1}{2}\,\Omega^2\, R^2\, \cos^2\phi \tag{3}$$

with Eulerian Montgomery potential $M = M(X^{\mathbf{1}}, X^{\mathbf{2}})$ and a centrifugal force contribution. The Euler-Lagrange equations for one fluid parcel then follow from the variational principle $0 = \delta \int_{t_0}^{t_1} E_{\text{kin}} - V \mathrm{d}t$. The associated canonical Hamilton's equations subsequently emerge via a Legendre transformation. Instead, we prefer the following non-canonical formulation. The non-canonical Hamiltonian equations for each layer parcel in this spherical shell geometry then become

$$\frac{\mathrm{d}X^{\mathbf{1}}}{\mathrm{d}t} = \frac{U_{\mathbf{1}}}{R^2\,\cos^2\,X^{\mathbf{2}}} = \frac{\partial H}{\partial U_{\mathbf{1}}} \quad \text{and} \quad \frac{\mathrm{d}X^{\mathbf{2}}}{\mathrm{d}t} = \frac{U_{\mathbf{2}}}{R^2} = \frac{\partial H}{\partial U_{\mathbf{2}}} \tag{4}$$

$$\frac{\mathrm{d}U_{\mathbf{1}}}{\mathrm{d}t} = 2\,\Omega\,R^2\cos X^{\mathbf{2}}\sin X^{\mathbf{2}}\,\frac{\partial H}{\partial U_{\mathbf{1}}} - \frac{\partial H}{\partial X^{\mathbf{1}}} \tag{5}$$

$$\frac{\mathrm{d}U_{\mathbf{2}}}{\mathrm{d}t} = -2\,\Omega\,R^2\cos X^{\mathbf{2}}\sin X^{\mathbf{2}}\,\frac{\partial H}{\partial U_{\mathbf{2}}} - \frac{\partial H}{\partial X^{\mathbf{2}}} \tag{6}$$

with in each layer the Hamiltonian

$$H = \frac{1}{2}\,(U_{\mathbf{1}}\,U^{\mathbf{1}} + U_{\mathbf{2}}\,U^{\mathbf{2}}) + M(X^{\mathbf{1}}, X^{\mathbf{2}}, t) = \frac{1}{2}\,(U^2 + V^2) + M(\lambda, \phi, t). \tag{7}$$

Montgomery potentials The next step is to derive the Montgomery potentials M_α for $\alpha = 1, 2$. In the troposphere the potential temperature is constant $\theta = \theta_2$ and the ideal gas law $p = \rho\,R\,T$ is used with temperature $T = \theta\,\eta^\kappa$, density ρ and gas constant R. We can therefore rewrite the following expression

$$(\nabla p)/\rho + \nabla(g\,r) = \nabla(\theta\,\Pi + g\,r) = \nabla M, \tag{8}$$

where $\Pi = c_p\,\theta\,\eta^\kappa$ is the Exner function, $\eta = p/p_r$ with reference pressure p_r, and $\kappa = R/c_p$ the ratio of R over the specific heat c_p at constant pressure. We have assumed that hydrostatic balance holds in the radial direction, which thus obtains the form $\partial M/\partial r = 0$ after using (8) in the respective layers. After integration of this hydrostatic balance relation from the Earth's surface at $r = r_2$ to $r < r_1$ one obtains

$$M = c_p\,\theta_2\,\eta^\kappa + g\,r = M_2 = c_p\,\theta\,\eta_2^\kappa + g\,r_2. \tag{9}$$

Integration of hydrostatic balance from $r > r_1$ to $r = r_0$ in the stratosphere gives likewise

$$M = c_p\,\theta_1\,\eta^\kappa + g\,(r - R_0) = g\,(r_0 - R_0) \tag{10}$$

with $p_0 \approx 0$ and a constant reference height R_0. Using continuity of pressure $p_1 = p_2$ at interface r_1, one finds

$$M_1 = c_p\,\theta_1\,\eta_1^\kappa + c_p\,\theta_2\,(\eta_2^\kappa - \eta_1^\kappa) + g\,(r_2 - R_0). \tag{11}$$

In addition, pseudo-densities $\sigma_1 = p_1/g$ and $\sigma_2 = (p_2 - p_1)/g$ are defined such that $M_\alpha = M_\alpha(\sigma_1, \sigma_2, z_2)$.

To derive the Eulerian Hamiltonian formulation on the sphere from the parcel Hamiltonian formulation (4)–(7), relations between functional and function variational and time derivatives are required as in [2,3]. The mass of a layer column is $\mathrm{d}M = \mathrm{d}a\,\mathrm{d}b = \sigma\,R^2\,\cos x^{\mathbf{2}}\,\mathrm{d}x^{\mathbf{1}}\,\mathrm{d}x^{\mathbf{2}}$. By definition

$$\sigma_\alpha(x^{\mathbf{1}}, x^{\mathbf{2}}, t) = \iint \sigma_\alpha(\tilde{x}^{\mathbf{1}}, \tilde{x}^{\mathbf{2}}, t)\,\delta\big(x^{\mathbf{1}} - \tilde{x}^{\mathbf{1}}\big)\,\delta\big(x^{\mathbf{2}} - \tilde{x}^{\mathbf{2}}\big)\,\mathrm{d}\tilde{x}^{\mathbf{1}}\,\mathrm{d}\tilde{x}^{\mathbf{2}} \tag{12}$$

$$= \iint \frac{\delta\big(x^{\mathbf{1}} - \boldsymbol{\chi}_\alpha^{\mathbf{1}}\big)\,\delta\big(x^{\mathbf{2}} - \boldsymbol{\chi}_\alpha^{\mathbf{2}}\big)}{\sqrt{\mu}}\,\mathrm{d}a\,\mathrm{d}b \tag{13}$$

with parcel coordinates $\tilde{x}^{\mathbf{1}} = \boldsymbol{\chi}_\alpha^{\mathbf{1}}(a, b, t), \tilde{x}^{\mathbf{2}} = \boldsymbol{\chi}_\alpha^{\mathbf{2}}(a, b, t)$ and parcel labels $(a, b)^T$, and $\sqrt{\mu} = R^2\,\cos\phi_\alpha$. For a particular label $(A, B)^T$: $(X^{\mathbf{1}}(t), X^{\mathbf{2}}(t))^T = \boldsymbol{\chi}(A, B, t)$. These integral relations reveal the continuum nature of the parcel EL Hamiltonian formulation, (4)-(7) and (12), since the M_α's depend on these pseudo-densities.

Conclusion The parcel EL Hamiltonian formulation has been constructed for an isentropic two-layer model of atmospheric dynamics on a sphere. From this EL formulation one can derive the corresponding Eulerian Hamiltonian formulation in a relatively easy way. Details of similar derivations can be found in Bokhove and Oliver [2,3].

References

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