**Proceedings of the 45th IEEE Conference on Decision & Control**
**Manchester Grand Hyatt Hotel**
**San Diego, CA, USA, December 13-15, 2006**



# A Finite Dimensional Approximation of the shallow water Equations: The port-Hamiltonian Approach

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***Abstract*— We look into the problem of approximating a distributed parameter port-Hamiltonian system which is represented by a *non-constant* Stokes-Dirac structure. We here employ the idea where we use different finite elements for the approximation of geometric variables (forms) describing a infinite-dimensional system, to spatially discretize the system and obtain a finite-dimensional port-Hamiltonian system. In particular we take the example of a special case of the shallow water equations.**

## I. INTRODUCTION

In recent publications, see for e.g. [6], [7], the Hamiltonian formulation of distributed parameter systems has been successfully extended to incorporate boundary conditions corresponding to non-zero energy flow, by defining a Dirac structure on certain spaces of differential forms on the spatial domain and its boundary, based on the use of Stokes' theorem. This is essential from a control and interconnection point of view, since in many applications interaction of system with its environment takes place through the boundary of the system. This framework has been applied to model various kinds of systems from different domains, like telegraphers equations, fluid dynamical systems, Maxwell equations, flexible beams and so on.

Consider a mixed finite and infinite-dimensional port-Hamiltonian system, where we interconnect finite-dimensional systems to infinite-dimensional systems. It has been shown in [3] that such an interconnection again defines a port-Hamiltonian system. A typical example of such a system is a power-drive consisting of a power converter, transmission line and electrical machine. From the control and simulation point of view of such systems, it may be crucial to approximate the infinite-dimensional subsystem with a finite-dimensional one. The finite-dimensional approximation should be such that it is again a port-Hamiltonian system which retains all the properties of the infinite-dimensional model, like energy balance and other conserved quantities. Furthermore, the port-variables of the approximated system should be such that it can easily be replaced in the original system, in other words the original interconnection constraints should be retained. It has been shown in [1] how the intrinsic Hamiltonian formulation suggests finite element methods which result in finite-dimensional approximations which are again port-Hamiltonian systems. Given the port-Hamiltonian formulation of distributed parameter systems it is natural to use *different* finite-elements for the approximation of functions and forms. In [1] this method was used for discretization of the ideal transmission line and the two dimensional wave equation. In this paper we extend this method to a special case of shallow water equations, which is a 1-D port-Hamiltonian system defined with respect to a non-constant Stokes-Dirac structure.

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## II. NOTATIONS

We apply the differential geometric framework of differential forms on the spatial domain $Z$ of the system. The shallow water equations are a case of a distributed parameter system with a one-dimensional spatial domain and in this context it means that we distinguish between zero-forms (functions) and one forms defined on the interval representing the spatial domain of the canal. One forms are objects which can be integrated over every sub-interval of the interval where as zero-forms or functions can be evaluated at any points of the interval. If we consider a spatial coordinate $z$ for the interval $Z$, then a function is simply given by the values $f(z) \in \mathbb{R}$ for every coordinate value in $z$ in the interval, while a one-form $g$ is given as $\tilde{g}(z)dz$ for a certain density function $g$. We denote the set of zero forms and one-forms on $Z$ by $\Omega^0(Z)$ and $\Omega^1(Z)$ respectively. Given a coordinate $z$ for the spatial domain we obtain by spatial differentiation of a function $f(z)$ the one-form $\omega := \frac{df}{dz}(z)dz$. In coordinate free language this is denoted as $\omega = \mathrm{d}f$, where $\mathrm{d}$ is called the exterior derivative mapping zero forms to one forms. We denote by $*$, the Hodge star operator mapping one forms to zero-forms, meaning that given a one-form $g$ on $Z$, the star operator converts the one form g to a function $g$, mathematically given as $*g(z) = \tilde{g}(z)$. Also denote by $\wedge$, the wedge product of two differential forms. Given a $k$-form $\omega_1$ and an $l$-form $\omega_2$, the wedge product $\omega_1 \wedge \omega_2$ is a $k + l$-form.

## III. PORT-HAMILTONIAN FORMULATION OF THE SHALLOW WATER EQUATIONS

Consider flow of water through a canal as shown in Figure 1, where $h(z, t)$ is the height of the water level in the canal $u(z, t)$ and $v(z, t)$ are the two velocity components. Here we restrict ourselves to the case where the height and the velocity components depend on only one spatial coordinate and hence we can model the system as an infinite-dimensional system with a 1-D spatial domain. The dynamics of the system are described by the following set of equations [4]

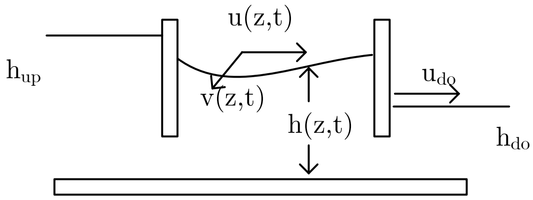


Fig. 1. Flow of water through a canal: The $h, u, v$ formulation

$$\begin{aligned}
\partial_t \tilde{h} &= -\partial_z(\tilde{h}\tilde{u}) \\
\partial_t(\tilde{h}\tilde{u}) &= -\partial_z(\tilde{h}\tilde{u}^2 + \frac{1}{2}g\tilde{h}^2) \\
\partial_t(\tilde{h}\tilde{v}) &= -\partial_z(\tilde{h}\tilde{u}\tilde{v}),
\end{aligned} \tag{1}$$

with $\tilde{h}(z,t)$ the height of the water level, $\tilde{u}(z,t)$ and $\tilde{v}(z,t)$ the water velocity components, with $g$ the acceleration due to gravity. The first equation again corresponds to mass balance, while the second and third equations correspond to the momentum balance. The above set of equations can alternatively be written as

$$\begin{aligned}
\partial_t \tilde{h} &= -\partial_z(\tilde{h}\tilde{u}) \\
\partial_t \tilde{u} &= -\partial_z(\frac{1}{2}\tilde{u}^2 + g\tilde{h}) \\
\partial_t \tilde{v} &= -\tilde{u}\partial_z\tilde{v}.
\end{aligned} \tag{2}$$

In the port-Hamiltonian framework this is modeled as follows. The energy variables now are $h(z,t)$, $u(z,t)$ and $v(z,t)$, the Hamiltonian of the system is given by

$$\mathcal{H} = \int_Z \frac{1}{2}\left(\tilde{h}(\tilde{u}^2 + \tilde{v}^2) + g\tilde{h}^2\right) dz, \tag{3}$$

and the variational derivatives are given by $\delta\mathcal{H} = [\frac{1}{2}(\tilde{u}^2+\tilde{v}^2) \;\; \tilde{h}\tilde{u} \;\; \tilde{h}\tilde{v}]^T$. As before the interaction of the system with the environment takes place through the boundary of the system $\{0, l\}$. The Stokes-Dirac structure corresponding to the shallow water equations (2), and modeled as a 1-D fluid flow, is defined as follows: The spatial domain $Z \subset \mathbb{R}$ as before is represented by a 1-D manifold with point boundaries. The height of the water flow through the canal $h(z,t)$ is identified with a 1-form on $Z$ and again assuming the existence of a *Riemannian metric* $<,>$ on $W$, we can identify (by index raising w.r.t this Riemannian metric) the Eulerian vector fields $u$ and $v$ on $Z$ with a 1-form. This leads to the consideration of the (linear) space of energy variables,.

$$X := \Omega^1(Z) \times \Omega^1(Z) \times \Omega^1(Z).$$

To identify the boundary variables we consider space of 0-forms, i.e., the space of functions on $\partial Z$, to represent the boundary height ,the dynamic pressure and the additional velocity component at the boundary. We thus consider the space of boundary variables

$$\Omega^0(\partial Z) \times \Omega^0(\partial Z) \times \Omega^0(\partial Z).$$

We will now define the Stokes-Dirac structure on $X \times \Omega^0(\partial Z)$, (i.e., the space of energy variables and part of the space of the boundary variables) in the following way

*Proposition 1:* (non-constant Stokes-Dirac structure) Let $Z \subset \mathbb{R}$ be a 1-dimensional manifold with boundary $\partial Z$. Consider $V = X \times \Omega^0(\partial Z) = \Omega^1(Z) \times \Omega^1(Z) \times \Omega^1(Z) \times \Omega^0(\partial Z)$, together with the bilinear form

$$\begin{aligned}
&<< (f_h^1, f_u^1, f_v^1, f_b^1, e_h^1, e_u^1, e_v^1, e_b^1), (f_h^2, f_u^2, f_v^2, f_b^2, e_h^2, e_u^2, e_v^2, e_b^2) >> \\
&:= \int_Z (e_h^1 \wedge f_h^2 + e_h^2 \wedge f_h^1 + e_u^1 \wedge f_u^2 + e_u^2 \wedge f_u^1 + e_v^1 \wedge f_v^2 + e_v^2 \wedge f_v^1) \\
&+ \int_{\partial Z} (e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1),
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
&f_h^i \in \Omega^1(Z), f_u^i \in \Omega^1(Z), f_v^i \in \Omega^1(Z), f_b^i \in \Omega^0(\partial Z) \\
&e_h^i \in \Omega^0(Z), e_u^i \in \Omega^0(Z), e_v^i \in \Omega^0(Z), e_b^i \in \Omega^0(\partial Z).
\end{aligned}$$

Then $\mathcal{D} \subset V \times V^*$ defined as

$$\begin{aligned}
&\mathcal{D} = \{(f_h, f_u, f_v, f_b, e_h, e_u, e_v, e_b) \in V \times V^* \mid \\
&\begin{bmatrix} f_h \\ f_u \\ f_v \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{d} & 0 \\ \mathrm{d} & 0 & -\frac{1}{*h}\mathrm{d}(*v) \\ 0 & \frac{1}{*h}\mathrm{d}(*v) & 0 \end{bmatrix} \begin{bmatrix} e_h \\ e_u \\ e_v \end{bmatrix}; \\
&\begin{bmatrix} f_b \\ e_b \\ e'_v \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \frac{1}{*h} \end{bmatrix} \begin{bmatrix} e_u \mid_{\partial Z} \\ e_h \mid_{\partial Z} \\ e_v \mid_{\partial Z} \end{bmatrix}.
\end{aligned} \tag{5}$$

is a Dirac structure, that is $\mathcal{D} = \mathcal{D}^\perp$, where $\perp$ is with respect to (4).

In terms of shallow water equations with an additional velocity component the above terms would correspond to

$$\begin{aligned}
f_h &= -\frac{\partial}{\partial t}h(z,t), \\
e_h &= \delta_h\mathcal{H} = (\frac{1}{2}((*u)(*u) + (*v)(*v)) + g(*h)) \\
f_u &= -\frac{\partial}{\partial t}u(z,t), e_u = \delta_u\mathcal{H} = (*h)(*u) \\
f_v &= -\frac{\partial}{\partial t}v(z,t), e_v = \delta_v\mathcal{H} = (*h)(*v) \\
f_b &= \delta_u\mathcal{H} \mid_{\partial W}, e_b = -\delta_h\mathcal{H} \mid_{\partial W}, \\
e'_v &= \frac{1}{*h}\delta_v\mathcal{H} \mid_{\partial W} .
\end{aligned} \tag{6}$$

with the Hamiltonian given as

$$\mathcal{H} = \int_Z \frac{1}{2}\left((*u)h(*u) + (*u)h(*u)\right) + \frac{1}{2}g(*h)h.$$

Substituting (6) into (5), we obtain the equations (2).

*Proof:* The proof is based on the skew symmetric term in the $3 \times 3$ matrix and also that the boundary variable $e'_v$ in (5)does not contribute to the bilinear form (4) and also follows a procedure as in [6]. ■

*Remark 2:* The Dirac structure above is no more a constant Dirac structure as it depends on the energy variables $h, u$ and $v$. Moreover, of the three boundary variables $f_b, e_b$ and $e'_v$, only $f_b$ and $e_b$ play a role in the power exchange through the boundary as will be seen in the expression for energy balance.

*1) Properties of the port-Hamiltonian model:*

- It follows from the power-conserving property of a Dirac structure that the modified Stokes-Dirac structure defined above has the property

$$\int_W (e_h \wedge f_h + e_u \wedge f_u + e_v \wedge f_v) + \int_{\partial W} e_b \wedge f_b = 0,$$

and hence we can get the energy balance

$$\frac{d}{dt}\mathcal{H} = \int_{\partial W} e_b \wedge f_b, \quad (7)$$

which can also be seen by the following

$$\begin{aligned}\frac{d}{dt}\mathcal{H} &= \int_Z [\delta_h\mathcal{H} \wedge \frac{\partial h}{\partial t} + \delta_u\mathcal{H} \wedge \frac{\partial u}{\partial t} + \delta_v\mathcal{H} \wedge \frac{\partial v}{\partial t}] \\ &= -\int_Z d[\delta_h\mathcal{H} \wedge \delta_u\mathcal{H}] = \int_{\partial Z} \delta_h\mathcal{H} \wedge \delta_u\mathcal{H} \\ &= \tilde{h}\tilde{u}(\frac{1}{2}\tilde{u}^2 + g\tilde{h}) \mid_0^L \\ &= (\tilde{u}(\frac{1}{2}\tilde{h}\tilde{u}^2 + \frac{1}{2}g\tilde{h}^2)) \mid_0^L + (\tilde{u}(\frac{1}{2}g\tilde{h}^2)) \mid_0^L .\end{aligned}$$

The first term in last line of the above expression for energy balance corresponds to the energy flux (the total energy times the velocity) through the boundary and the second term is the work done by the hydrostatic pressure given by pressure times the velocity. It is also seen that the boundary variables which contribute to the power at the boundary are $f_b$ and $e_b$ and the third boundary variable $e'_v$ does not contribute to it.

- Conservation laws or Casimirs are obtained by applying the theory of Casimirs for infinite-dimensional systems [3]. It has been shown in [2] that the Casimirs are all the functionals $\mathcal{C}$ which satisfy

$$\begin{aligned}\delta_u\mathcal{C} &= 0 \\ \mathrm{d}\delta_h\mathcal{C} &= \frac{1}{h}\mathrm{d}(*v)\delta_v\mathcal{C}. \quad (8)\end{aligned}$$

The solution to the above PDE is of the form, see [5]

$$\mathcal{C} = \int_W h.\phi(\frac{1}{*h}\mathrm{d}(*v)) \quad (9)$$

for any function $\phi$. We discuss here a few specific examples of Casimir functions:
Case 1: where $\phi(\frac{1}{h}\mathrm{d}(*v)) = 1$, we have $\mathcal{C} = \int_W h$ which corresponds to mass conservation as in the above case
Case 2: $\phi(\frac{1}{*h}\mathrm{d}(*v)) = \frac{1}{*h}\mathrm{d}(*v)$, in which case $\mathcal{C} = \int_W \mathrm{d}(*v)$ which is called *vorticity*
Case 3: $\phi(\frac{1}{*h}\mathrm{d}(*v)) = (\frac{1}{*h}\mathrm{d}(*v))^2$, and this corresponds to $\mathcal{C} = \int_W \frac{1}{*h}(\mathrm{d}(*v))^2$, which is called *mass weighted potential enstrophy*.

## IV. Spatial discretization of the shallow water equations

Consider a part of the canal between two points $a$ and $b$ ($0 \leq a < b \leq L$). The spatial manifold corresponding to this part of the canal is $Z_{ab} = [a, b]$. The mass flow through point $a$ is denoted by $e_a^B$ and the Bernoulli function by $f_a^B$, similarly for the point $b$ with $e_b^B$ and $f_b^B$ respectively.

**Approximation of $f_h$, $f_u$ and $f_v$:** As in the case of a *constant* Dirac structure, we approximate the infinitesimal height $f_h$, the velocities $f_u$, $f_v$ on $Z_{ab}$ as

$$f_{h,u,v}(t, z) = f_{ab}^{h,u,v}(t)\omega_{ab}^{h,u,v}(z) \quad (10)$$

where again the one-forms $\omega_{ab}^h$, $\omega_{ab}^u$ and $\omega_{ab}^v$satisfy

$$\int_{Z_{ab}} \omega_{ab}^h = 1, \quad \int_{Z_{ab}} \omega_{ab}^u(z) = 1, \quad \int_{Z_{ab}} \omega_{ab}^v(z) = 1. \quad (11)$$

**Approximation of $e_h$ and $e_u$:** The co-energy variables $e_h(z, t)$ and $e_u(z, t)$ are approximated as

$$e_{h,u,v}(t, z) = e_a^{h,u,v}(t)\omega_a^{h,u,v}(z) + e_b^{h,u,v}(t)\omega_{h,u,v}^b(z) \quad (12)$$

where the zero-forms $\omega_a^h, \omega_b^h, \omega_a^u, \omega_b^u, \omega_a^v, \omega_b^v \in \Omega^0(Z_{ab})$ satisfy

$$\begin{array}{llll}\omega_a^h(a) = 1, & \omega_a^h(b) = 0, & \omega_b^h(a) = 0, & \omega_b^h(b) = 1, \\ \omega_a^u(a) = 1, & \omega_a^u(b) = 0, & \omega_b^u(a) = 0, & \omega_b^u(z) = 1, \\ \omega_a^v(a) = 1, & \omega_a^v(b) = 0, & \omega_b^v(a) = 0, & \omega_b^v(b) = 1.\end{array}$$

Furthermore we also approximate $*v(t, z)$ in (5) with a zero form instead of approximating $v(t, z)$ with a one form) as,

$$*v(t, z) = v_a(t)\omega_a(t) + v_b(t)\omega_b(t), \quad (13)$$

where the zero forms $\omega_a(z)$ and $\omega_a(z)$ satisfies

$$\begin{array}{ll}\omega_a(a) = 1, & \omega_a(b) = 0, \\ \omega_b(a) = 0, & \omega_b(b) = 1.\end{array}$$

this gives

$$f_{ab}^h(t)\omega_{ab}^h(z) = e_a^u(t)\mathrm{d}\omega_a^u(z) + e_b^u(t)\mathrm{d}\omega_b^u(z) \quad (14)$$

$$f_{ab}^u(t)\omega_{ab}^u(z) = e_a^h(t)\mathrm{d}\omega_a^h(z) + e_b^h(t)\mathrm{d}\omega_b^h(z) - \frac{1}{h_{ab}(t) * \omega_{ab}^h(z)}(v_a(t)\mathrm{d}\omega_a(t) + v_b(t)\mathrm{d}\omega_b(t))(e_a^v(t)\omega_a^v(z) + e_b^v(t)\omega_b^v(z)) \quad (15)$$

$$f_{ab}^v(t)\omega_{ab}^v(z) = \frac{1}{h_{ab}(t) * \omega_{ab}^h(z)}(v_a(t)\mathrm{d}\omega_a(t) + v_b(t)\mathrm{d}\omega_b(t))(e_a^u(t)\omega_a^u(z) + e_b^u(t)\omega_b^u(z)), \quad (16)$$

where the height $h(z, t)$ is approximated as

$$h(z, t) = h_{ab}(t)\omega_{ab}^h(z), \quad \text{where,} \int_{Z_{ab}} \omega_{ab}^h(z) = 1.$$

**Compatibility of forms:** In the first line of the above equation the one form $\omega_{ab}^h$ and the functions $\omega_a^u(z)$ and $\omega_u^b(z)$ should be chosen in such a way that for every $e_a^u$ and $e_b^u$, we can find $f_{ab}^h$ such that (14) is satisfied. The satisfaction of such conditions leads to the following equations

$$f_{ab}^h(t) = e_a^u(t) - e_b^u(t). \quad (17)$$

The above expression can also be obtained by integrating (14) over $Z_{ab}$ and substituting the conditions on the zero and one forms (11,12). Similar satisfaction of compatibility conditions for (14) gives us the following equations

$$f_{ab}^u(t) = e_a^h(t) - e_b^h(t) - \frac{1}{h_{ab}(t)} (c_1 v_a(t)e_a^v(t) + c_2 v_a(t)e_v^b(t) + c_3 v_b(t)e_a^v(t) + c_4 v_b(t)e_b^v(t)), \quad (18)$$

where the constants are given by (again this is obtained by integrating over $Z_{ab}$)

$$c_1 = \int_{Z_{ab}} \frac{\mathrm{d}\omega_a}{*\omega^h_{ab}} \omega^a_v, \qquad c_2 = \int_{Z_{ab}} \frac{\mathrm{d}\omega_a}{*\omega^h_{ab}} \omega^v_b,$$

$$c_3 = \int_{Z_{ab}} \frac{\mathrm{d}\omega_b}{*\omega^h_{ab}} \omega^a_v, \qquad c_4 = \int_{Z_{ab}} \frac{\mathrm{d}\omega_b}{*\omega^h_{ab}} \omega^v_b.$$

Similar satisfaction of compatibility conditions for (16) and integrating it over $Z_{ab}$ yields

$$f^{ab}_v(t) = \frac{1}{h_{ab}(t)}(c'_1 v_a(t)e^u_a(t) + c'_2 v_a(t)e^u_b(t) + c'_3 v_b(t)e^u_a(t) + c'_4 v_b(t)e^u_b(t)), \tag{19}$$

where

$$c'_1 = \int_{Z_{ab}} \frac{\mathrm{d}\omega_a}{*\omega^h_{ab}} \omega^a_u, \qquad c'_2 = \int_{Z_{ab}} \frac{\mathrm{d}\omega_a}{*\omega^h_{ab}} \omega^u_b,$$

$$c'_3 = \int_{Z_{ab}} \frac{\mathrm{d}\omega_b}{*\omega^h_{ab}} \omega^a_u, \qquad c'_4 = \int_{Z_{ab}} \frac{\mathrm{d}\omega_b}{*\omega^h_{ab}} \omega^u_b.$$

For the sake of clarity the argument $t$ is omitted in the rest of the section. The relations describing the spatially discretized interconnection structure of the part of the canal are given by

$$\begin{bmatrix} e^B_a \\ e^B_b \\ f^B_a \\ f^B_b \\ e^B_{va} \\ e^B_{vb} \\ f^h_{ab} \\ f^u_{ab} \\ f^v_{ab} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -k_1 & -k_2 \\ 0 & 0 & k_1 & k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^h_a \\ e^h_b \\ e^u_a \\ e^u_b \\ e^v_a \\ e^v_b \end{bmatrix}.$$

where $k_1 = \frac{1}{h_{ab}}(c_1 v_a + c_3 v_b)$ and $k_2 = \frac{1}{h_{ab}}(c_2 v_a + c_4 v_b)$.The net power in the considered part of the canal is

$$\int_{Z_{ab}} [e_h f_h + e_u f_u + e_v f_v] - e^B_a f^B_a + e^B_b f^B_b.$$

We then get

$$\begin{aligned} P^{net}_{ab} &= [\alpha_{ab} e^h_a + (1-\alpha_{ab})e^h_b] f^h_{ab} \\ &+ [(1-\alpha_{ab})e^u_a + \alpha_{ab} e^u_b] f^u_{ab} + [\beta_1 e^v_a + \beta_2 e^v_b] f^v_{ab}, \end{aligned} \tag{20}$$

where $\alpha_{ab} := \int_{Z_{ab}} \omega^h_a(z)\omega^h_{ab}(z), \alpha_{ba} := \int_{Z_{ab}} \omega^h_b(z)\omega^h_{ab}(z)$, $\beta_1 := \int_{Z_{ab}} \omega^v_a(z)\omega^v_{ab}(z)$, $\beta_2 = \int_{Z_{ab}} \omega^v_b(z)\omega^v_{ab}(z)$. We use the above expression for identifying the port variables in the discretized interconnection structure. The flow variable corresponding to the mass density is $f^h_{ab}$ and the effort variable is $\alpha_{ab} e^h_a + (1-\alpha_{ab})e^h_b$. Thus we define

$$\begin{aligned} e^h_{ab} &:= [\alpha_{ab} e^h_a + (1-\alpha_{ab})e^h_b] \\ e^u_{ab} &:= [(1-\alpha_{ab})e^u_a + \alpha_{ab} e^u_b] \\ e^u_{ab} &:= [\beta_1 e^v_a + \beta_2 e^v_b]. \end{aligned} \tag{21}$$

We also have the following properties of the corresponding zero and one forms which are the same as case for the transmission line derived in [1].

$$\begin{aligned} \omega^h_a(z) + \omega^h_a(z) &= 1 \\ \omega^u_a(z) + \omega^u_b(z) &= 1 \\ \int_{Z_{ab}} \omega^h_a(z)\omega^h_{ab}(z) + \int_{Z_{ab}} \omega^h_b(z)\omega^h_{ab}(z) &= 1 \\ \int_{Z_{ab}} \omega^u_a(z)\omega^u_{ab}(z) + \int_{Z_{ab}} \omega^u_b(z)\omega^u_{ab}(z) &= 1 \\ \int_{Z_{ab}} \omega^h_a(z)\omega^h_{ab}(z) + \int_{Z_{ab}} \omega^u_b(z)\omega^u_{ab}(z) &= 1 \end{aligned} \tag{22}$$

*Proposition 3:* Under the assumption that $\omega^v_{ab} = \omega^u_{ab}$, the constants $\alpha_{ab}, \alpha_{ba}, \beta_1, \beta_2, c_1, c_2, c_3, c_4, c'_1, c'_2, c'_3, c'_4$ satisfy

$$\begin{aligned} &\alpha_{ba} c_1 = \beta_1 c'_1 \quad \alpha_{ba} c_3 = \beta_1 c'_3 \quad \alpha_{ba} c_2 = \beta_2 c'_1 \quad \alpha_{ba} c_4 = \beta_2 c'_3, \\ &\alpha_{ab} c_1 = \beta_1 c'_2 \quad \alpha_{ab} c_3 = \beta_1 c'_4 \quad \alpha_{ab} c_2 = \beta_2 c'_2 \quad \alpha_{ab} c_4 = \beta_2 c'_4. \end{aligned} \tag{23}$$

*Proof:* We know from (22 that

$$\omega^u_a(z) + \omega^u_b(z) = 1.$$

Hence, by satisfying the compatibility conditions of (14,15,16) we have the following

$$(c_1 + c_2)\omega^u_{ab} = \frac{d(\omega_a + \omega_b)}{*\omega^h_{ab}}(\omega^v_a + \omega^v_b)$$

$$(c_3 + c_4)\omega^u_{ab} = \frac{d(\omega_a + \omega_b)}{*\omega^h_{ab}}(\omega^v_a + \omega^v_b)$$

$$(c'_1 + c'_2)\omega^v_{ab} = \frac{d(\omega_a + \omega_b)}{*\omega^h_{ab}}; \; (c'_3 + c'_4)\omega^v_{ab} = \frac{d(\omega_a + \omega_b)}{*\omega^h_{ab}}.$$

using the above equalities the relations (23) can be easily proved. ■

Then, the net expression for power becomes

$$P^{net}_{ab} := f^h_{ab} e^h_{ab} + f^u_{ab} e^u_{ab} + f^v_{ab} e^v_{ab} - e^B_a f^B_a + e^B_b f^B_b. \tag{24}$$

*Remark 4:* We see that the additional port variables arising due to the velocity component $v$ does not play any role in the expression for energy balance. This property was also observed in the infinite-dimensional case in (7).

Now by substituting

$$e^h_a = e^B_a, e^h_b = e^B_b, e^u_a = f^B_a, e^u_b = f^B_b, e^v_a = f^B_v, e^v_b = e^v_b,$$

yields

$$\begin{bmatrix} -1 & 0 & 0 & \alpha_{ab} & \alpha_{ba} & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\beta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & \frac{c_2 v^a + c_4 v^b}{h_{ab}} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^h_{ab} \\ e^u_{ab} \\ e^v_{ab} \\ e^B_a \\ e^B_b \\ e^v_b \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{ba} & \alpha_{ab} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta_1 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{c_1 v^a + c_3 v^b}{h_{ab}} \\ 0 & 0 & 1 & -\frac{c'_1 v^a + c'_3 v^b}{h_{ab}} & -\frac{c'_2 v^a + c'_4 v^b}{h_{ab}} & 0 \end{bmatrix} \begin{bmatrix} f^h_{ab} \\ f^u_{ab} \\ f^v_{ab} \\ f^B_a \\ f^B_b \\ f^v_b \end{bmatrix} \tag{25}$$

The above equation represents the spatially discretized interconnection structure, abbreviated as

$$D_{ab} = \{(f^{ab}, e^{ab}) \in \mathbb{R}^{12} : E_{ab}e_{ab} + F_{ab}f_{ab} = 0\}.$$

It can easily be shown that the above subspace $D_{ab}$ is a Dirac structure with respect to the bilinear form

$$<< (f_1^{ab}, e_1^{ab}), (f_2^{ab}, e_2^{ab}) >>:=< e_1^{ab}, f_2^{ab} > + < e_2^{ab}, f_1^{ab} > . \quad (26)$$

*A. Approximation of the energy part*

For the discretization of the energy part we proceed as follows: The flow variables $f_h$, $f_u$ and $f_v$ and the energy variables $h$, $u$ and $v$ are one-forms. Since $f_h$, $f_u$ and $f_v$ are approximated by (10) and are related to $h$, $u$ and $v$ by (5), it is consistent to approximate $h$,$u$ and $v$ on $Z_{ab}$ in the same way by

$$\begin{aligned} h(t,z) &= h_{ab}(t)\omega_{ab}^h(z) \\ u(t,z) &= u_{ab}(t)\omega_{ab}^u(z) \\ v(t,z) &= v_{ab}(t)\omega_{ab}^u(z), \end{aligned} \quad (27)$$

where

$$-\frac{dh_{ab}(t)}{dt} = f_{ab}^h(t), -\frac{du_{ab}(t)}{dt} = f_{ab}^u(t), -\frac{dv_{ab}(t)}{dt} = f_{ab}^v(t). \quad (28)$$

Here $h_{ab}$ represents the total amount of water in the considered part of the canal and $u_{ab}$, $v_{ab}$ the average velocities of the same part of the canal. The kinetic energy as a function of the energy variables $u$ and $v$ is given by

$$\int_{Z_{ab}} \frac{1}{2}[(*u(t,z))h(t,z)(*u(t,z)) + (*v(t,z))h(t,z)(*v(t,z))].$$

Approximation of the infinite-dimensional energy variables $u$ and $v$ by (28) means that we restrict the infinite-dimensional space of one-forms $\Omega^1(Z_{ab})$ to its one-dimensional subspace spanned by $\omega_{ab}^h, \omega_{ab}^u, \omega_{ab}^v$. This leads to the approximation of the kinetic energy of the considered part of the canal by

$$H_{ab}^{u,v}(h_{ab}, u_{ab}, v_{ab}) = \frac{1}{2}(C_1 h_{ab} u_{ab}^2 + C_2 h_{ab} v_{ab}^2),$$

where

$$\begin{aligned} C_1 &= \int_{Z_{ab}} (*\omega_{ab}^u(z))\omega_{ab}^h(z) * \omega_{ab}^u(z) \\ C_2 &= \int_{Z_{ab}} (*\omega_{ab}^v(z))\omega_{ab}^h(z) * \omega_{ab}^v(z). \end{aligned}$$

Note that this is nothing else than the restriction of the kinetic energy function to the one dimensional subspace of $\Omega^1(Z_{ab})$. Similarly the potential energy is approximated by

$$H_{ab}^h(h_{ab}) = \frac{C_3}{2} g h_{ab}^2,$$

where

$$C_3 = \int_{Z_{ab}} (*\omega_{ab}^h(z))\omega_{ab}^h(z).$$

Therefore, the total energy in the considered part of the canal is approximated by

$$\begin{aligned} H_{ab}(h_{ab}, u_{ab}, v_{ab}) &= H_{ab}^{u,v}(h_{ab}, u_{ab}, v_{ab}) + H_{ab}^h(h_{ab}) \\ &= \frac{1}{2}\left(C_1 h_{ab} u_{ab}^2 + C_2 h_{ab} v_{ab}^2 + C_3 g h_{ab}^2\right). \end{aligned}$$

Next, in order to describe the discretized dynamics, we equate the discretized effort variables $e_{ab}^h, e_{ab}^u, e_{ab}^v$ of the discretized interconnection structure defined in (21) with co-energy variables corresponding to the total approximated energy $H_{ab}$ of the considered part of the canal

$$\begin{aligned} e_{ab}^h &= \tfrac{\partial H(h_{ab},u_{ab},v_{ab})}{\partial h_{ab}}(t) = \tfrac{1}{2}(C_1 u_{ab}^2 + C_2 v_{ab}^2) + C_3 g h_{ab} \\ e_{ab}^u &= \tfrac{\partial H(h_{ab},u_{ab},v_{ab})}{\partial u_{ab}}(t) = C_1 h_{ab} u_{ab} \\ e_{ab}^v &= \tfrac{\partial H(h_{ab},u_{ab},v_{ab})}{\partial v_{ab}}(t) = C_2 h_{ab} v_{ab}. \end{aligned} \quad (29)$$

The equations (25) (the interconnected structure) together with (28),(29) represent a finite-dimensional model of the shallow water equations with a non-constant Stokes-Dirac structure. To sum up we have the following set of equations for a single lump of the finite-dimensional model

$$\begin{aligned} &-\frac{dh_{ab}}{dt} = hu\mid_a - hu\mid_b \\ &-\frac{du_{ab}}{dt} = \frac{1}{2}(u^2+v^2) + gh\mid_a - \frac{1}{2}(u^2+v^2) + gh\mid_a \\ &-\left(\frac{c_2 v_a + c_4 v_b}{h_{ab}} hv\mid_a + \frac{c_1 v_a + c_3 v_b}{h_{ab}} hv\mid_b\right) \\ &-\frac{dv_{ab}}{dt} = \frac{c_2' v_a + c_4' v_b}{h_{ab}} hu\mid_a + \frac{c_1' v_a + c_3' v_b}{h_{ab}} hu\mid_b \\ &\frac{1}{2}(C_1 u_{ab}^2 + C_2 v_{ab}^2) + C_3 g h_{ab} = \alpha_{ab}(\frac{1}{2}(u^2+v^2) + gh\mid_a) \\ &\qquad + \alpha_{ba}(\frac{1}{2}(u^2+v^2) + gh\mid_b) \\ &C_1 h_{ab} u_{ab} = \alpha_{ba}(hu\mid_a) + \alpha_{ab}(hu\mid_b) \\ &C_2 h_{ab} v_{ab} = \beta_1(hv\mid_a) + \beta_2(hv\mid_b). \end{aligned} \quad (30)$$

*1) Spatial discretization of the entire system:* The canal is split into $n$ parts. The $i$th part $(S_{i-1}, S_i)$ is discretized as explained in the previous subsections, where $a = S_{i-1}$ and $b = S_i$. The resulting model consists of $n$ submodels each of them representing a port-Hamiltonian system. Since a power conserving interconnection of a number of port-Hamiltonian systems is again a port-Hamiltonian system, the total discretized system is also a port-Hamiltonian system, whose interconnection structure is given by the composition of the $n$ Dirac structures on $(S_{i-1}, S_i)$, while the total Hamiltonian is the sum of individual Hamiltonians as

$$H(h,u) = \sum_{i=1}^{n} [C_{1i} h_l u_{S_{i-1},S_i} + C_{2i} h_l v_{S_{i-1},S_i} + C_3 g h_{S_{i-1},S_i}^2].$$

Here $h = (h_{S_0,S_1}, h_{S_1,S_2}, ..., h_{S_{n-1},S_n})^T$ are the discretized heights and $u = (u_{S_0,S_1}, u_{S_1,S_2}, ..., u_{S_{n-1}}, u_{s_n})$ and $v = (v_{S_0,S_1}, v_{S_1,S_2}, ..., v_{S_{n-1}}, v_{s_n})$ are the discretized velocities. The total discretized model still has two ports. The port $(f_{S_0}^B, e_{S_0}^B) = (f_0^B, e_0^B)$ is the incoming port and the port

$(f^B_{S_n}, e^B_{Sn}) = (f^B_S, e^B_S)$ is the outgoing port, resulting in the energy balance of the discretized model

$$\frac{dH(h(t), u(t), v(t))}{dt} - e^B_0 f^B_0 + e^B_S f^B_S = 0.$$

Equation (17) for the $i$th part becomes $f^h_{S_{i-1},S_i}(t) = e^u_{S_{i-1}}(t) - e^u_{S_i}(t)$. Taking into account (28) and $e^u_{S_0} = f^B_0$, $e_{S_n} = f^B_S$, we have $\frac{dh(t)}{dt} = f^B_0 - f^B_S$, where $h := \sum_{i=1}^{n} h_{S_{i-1},S_i}$ is the total mass (amount of water) in the canal, this represents mass conservation.

*2) The input-state-output model:* In this section we write the discretized system in the input state output model, which could help us further analyze the properties of the finite-dimensional model and compare it with the infinite-dimensional model. To simplify the model we use the following choices for the approximating zero and one forms. The zero-forms are approximated as constant density functions, i.e.

$$\omega^{h,u,v}_{ab} = \frac{1}{b-a},$$

and the zero-forms as linear splines, i.e.

$$\omega^{h,u,v}_{a} = \frac{b-z}{b-a}, \quad \omega^{h,u,v}_{b} = \frac{z-a}{b-a}.$$

Computing the values for the constants in (23), we have

$$\begin{bmatrix} f^h_{ab} \\ f^u_{ab} \\ f^v_{ab} \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 2K \\ 0 & -2K & 0 \end{bmatrix} \begin{bmatrix} e^h_{ab} \\ e^u_{ab} \\ e^v_{ab} \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} f^B_a \\ -e^B_b \end{bmatrix}$$
$$\begin{bmatrix} e^B_a \\ f^B_b \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} e^h_{ab} \\ e^u_{ab} \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f^B_a \\ -e^B_b \end{bmatrix},$$

where

$$K = \frac{(v_a - v_b)}{h_{ab}}.$$

If we now apply the theory of Casimirs for an *autonomous* port-Hamiltonian system [3] for a single lump, we see that the Casimirs are all functions $C(h, u, v)$ which satisfy

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 2K \\ 0 & -2K & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial C}{\partial h_{ab}} \\ \frac{\partial C}{\partial u_{ab}} \\ \frac{\partial C}{\partial v_{ab}} \end{bmatrix},$$

from the above equation we have

$$\frac{\partial C}{\partial u_{ab}} = 0$$
$$\frac{\partial C}{\partial h_{ab}} - \frac{(v_a - v_b)}{h_{ab}} \frac{\partial C}{\partial v} = 0. \qquad (31)$$

This means that the Casimirs are independent of the $u$ component of velocity which is consistent with the continuous case. Equation (31) could be seen as an analogue of (8), the solution of which would result in a class of functions which would be conserved quantities for the finite-dimensional model.

## V. CONCLUSIONS AND FUTURE WORKS

In this paper we have extended the general methodology for spatial discretization of boundary control systems modelled as port-Hamiltonian systems which are now defined with respect to a *non-constant* Stokes-Dirac structure. It is observed that a key feature of this methodology is that the discretized system is again a port-Hamiltonian system. The advantages of it are that the physical properties of the infinite-dimensional model can be translated to the finite-dimensional approximation. The finite-dimensional model can be interconnected to other systems in the same was as that for the infinite dimensional model.

Here we have treated the spatial discretization of a special case of the shallow water equations and we have seen that the energy and the mass conservation laws also hold for the finite-dimensional model. However, what is a matter of further investigation is to see how solutions of equation (31) relate to the class of conserved quantities as in the infinite-dimensional model (9). The next step would also be to use this finite-dimensional model for actual numerical simulations and also to obtain bounds on error between the infinite-dimensional model and its finite-dimensional approximation.

## VI. ACKNOWLEDGMENTS

This work has been done in context of the European sponsored project GeoPleX IST-2001-34166. For more information see http://www.geoplex.cc.

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