On Interconnections of Infinite-dimensional Port-Hamiltonian Systems

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Abstract

Network modeling of complex physical systems leads to a class of nonlinear systems called port-Hamiltonian systems, which are defined with respect to a Dirac structure (a geometric structure which formalizes the power-conserving interconnection structure of the system). A power conserving interconnection of Dirac structures is again a Dirac structure. In this paper we study interconnection properties of *mixed finite and infinite dimensional* port-Hamiltonian systems and show that this interconnection again defines a port-Hamiltonian system. We also investigate which closed-loop port-Hamiltonian systems can be achieved by power conserving interconnections of finite and infinite dimensional port-Hamiltonian systems. Finally we study these results with particular reference to the transmission line.

Keywords : port-Hamiltonian systems, Dirac structures, Casimir functions

1 Introduction

The framework of port-Hamiltonian systems has emerged as a powerful tool for modeling and control of complex physical systems [2]. Port-Hamiltonian systems are defined with respect to a Dirac structure (formalizing the power conserving interconnection property of the system). The key property of a Dirac structure is that a power conserving interconnection of a number of Dirac structures again leads to a Dirac structure, implying that a power conserving interconnection of a number of port-Hamiltonian systems is again a port-Hamiltonian system, with Dirac structure the interconnection of the individual Dirac structures and Hamiltonian the sum of the individual Hamiltonians. These properties have been studied in [3] for the finite dimensional case.

This framework has recently been extended to deal with infinite dimensional systems, see [4] and various systems have been encorporated within this framework. In this paper we extend the results in [3] to the *mixed finite and infinite dimensional* case. First we define interconnections in this case and show that the interconnection is indeed a port-Hamiltonian system. Next we derive conditions for the achievable closed loop Dirac structures, analogous to the finite dimensional case and then charactarize the set of achievable Casimirs. Finally we apply these results for the case of the transmission line.

2 Port Hamiltonian systems and Dirac structures.

It is well known [2, 4] that the notion of power conserving interconnections can be formulated by a geometric structure called Dirac structure, which is a subspace of the space of efforts and flows. We briefly discuss these concepts here both for finite dimensional systems as well as infinite dimensional systems with scalar spatial variable. Refer [2, 4] for details.

2.1 Finite Dimensional systems

To define the notion of Dirac structures for finite dimensional systems, we start with a space of power variables $\mathcal{F} \times \mathcal{F}^*$, for some linear space $\mathcal{F}$, with power defined by

$$P = < e \mid f >, \ (f, e) \in \mathcal{F} \times \mathcal{F}^*,$$

where $< e \mid f >$ denotes the duality product, that is, the linear functional $e \in \mathcal{F}^*$ acting on $f \in \mathcal{F}$. $\mathcal{F}$ is called the space of flows and $\mathcal{F}^*$ the space of efforts, with the power of a signal $(f, e) \in \mathcal{F} \times \mathcal{F}^*$ denoted as $< e \mid f >$.

There exists on $\mathcal{F} \times \mathcal{F}^*$ a canonically defined *bilinear form* $\ll, \gg$, defined as

$$\begin{aligned} \ll (f^a, e^a), (f^b, e^b) \gg \ &:= < e^a \mid f^b > + < e^b \mid f^a >, \\ &\qquad (f^a, e^a), (f^b, e^b) \in \mathcal{F} \times \mathcal{F}^* \end{aligned} \tag{1}$$

Definition 2.1 *[2] A constant Dirac structure on $\mathcal{F} \times \mathcal{F}^*$ is a subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ such that $\mathcal{D} = \mathcal{D}^\perp$ with respect to the bilinear form (1).*

As an immediate corollary of the definition we see that for all $(f, e) \in \mathcal{D}$ we have that $< e \mid f > = 0$. Hence a Dirac structure defines a power conserving relation.

Consider a lumped-parameter physical system given by power-conserving interconnection defined by a constant Dirac structure $\mathcal{D}$ and energy storing elements with energy variables x. For simplicity we assume that the energy variables are living in a linear space $\mathcal{X}$ although everything can be generalized to the case of manifolds. The constitutive relations of the energy storing elements are specified by their stored energy functions $H(x)$.

The space of flows is naturally partitioned as $\mathcal{X} \times \mathcal{F}$ with $f_x \in \mathcal{X}$, the flows corresponding to the energy storing elements, and $f \in \mathcal{F}$ denoting the remaining flows (corresponding to dissipative elements and ports/sources). Correspondingly, the space of effort variables is split as $\mathcal{X}^* \times \mathcal{F}^*$, with $e_x \in \mathcal{X}^*$ the efforts corresponding to the energy-storing elements and $e \in \mathcal{F}^*$ the remaining efforts.The bilinear form now takes the form:

$$\begin{aligned} &\ll (f_x^a, e_x^a, f^a, e^a), (f_x^b, e_x^b, f^b, e^b) \gg := \\ &< e_x^a \mid f_x^b > + < e_x^b \mid f_x^a > + < e^a \mid f^b > + < e^b \mid f^a > \end{aligned} \tag{2}$$

with $f_x^a, f_x^b \in \mathcal{X}, \ f^a, f^b \in \mathcal{F}, \ e_x^a, e_x^b \in \mathcal{F}^*, \ e^a, f^b \in \mathcal{F}^*$. The Dirac structure $\mathcal{D}$ can then be given in matrix kernel representation [2, 3] as

$$\begin{aligned} \mathcal{D} = \{(f_x, e_x, f, e) \in \mathcal{X} \times \mathcal{X}^* \times \mathcal{F} \times \mathcal{F}^* \mid F_x f_x + E_x e_x + F f + E e = 0\}, \\ E_x F_x^T + F_x E_x^T + E F^T + F E^T = 0 \\ \text{with } rank[F_x \vdots E_x \vdots F \vdots E] = \dim(\mathcal{X} \times \mathcal{F}) \end{aligned} \tag{3}$$

Now the flows of the energy storing elements are given by $\dot{x}$, and equated with $-f_x$(the negative sign is included to have a consistent energy flow direction). The efforts e_x corresponding to the energy storing elements are given as $\frac{\partial H}{\partial x} = e_x$. Substituting these into (3) leads to the description of the physical system by the set of DAE's

$$F_x \dot{x}(t) = E_x \frac{\partial H}{\partial x}(x(t)) + Ff(t) + Ee(t) \tag{4}$$

with f, e the port power variables (some of which may be terminated by resistive elements). The system of equations (4) is called a port-Hamiltonian system.

Because of (3) we obtain the power balance

$$\frac{dH}{dt} = \left(\frac{\partial H}{\partial x}(x)\right)^T \dot{x} = e^T f \tag{5}$$

which means that the increase in internal energy of the port-Hamiltonian system is equal to the externally supplied power. (In case the ports are terminated by resistive elements then the increase in energy will be the supplied energy minus the power dissipated in the energy-dissipating elements).

2.2 Infinite Dimensional systems

The key concept in order to define an infinite-dimensional port-Hamiltonian system on a bounded spatial domain, with non-zero energy flow through the boundary, is the introduction of a special type of Dirac structure on suitable spaces of differential forms on the spatial domain and its boundary, making use of Stokes' theorem (see [4]). Let Z be an n-dimensional manifold with a smooth $(n-1)$ dimensional boundary ∂Z, representing the space of spatial variables. Define now the linear space

$$\mathcal{F}_{p,q} := \Omega^p(Z) \times \Omega^q(Z) \times \Omega^{n-p}(\partial Z)$$

for any pair p, q of positive integers satisfying $p + q = n + 1$, and correspondingly define

$$\mathcal{F}^*_{p,q} := \Omega^{n-p} \times \Omega^{n-q} \times \Omega^{n-q}(\partial Z)$$

Here $\Omega^k(Z)$, $k = 0, 1, ..., n$, is the space of exterior k-forms on Z, and $\Omega^k(\partial Z)$, $k = 0, 1, ..., n-1$, the space of k-forms on ∂Z.

There is a natural pairing between $\Omega^k(Z)$ and $\Omega^{n-k}(Z)$ (similarly between $\Omega^k(\partial Z)$ and $\Omega^{n-k}(\partial Z)$) given by

$$< \beta \mid \alpha > := \int_Z \beta \wedge \alpha, \quad (\in \mathbb{R}) \tag{6}$$

with $\alpha \in \Omega^k(Z)$, $\beta \in \Omega^{n-k}(Z)$, with $\wedge$ the usual wedge product of differential forms yielding the n-form $\beta \wedge \alpha$.

Then the pairing (6) yields a pairing between $\mathcal{F}_{p,q}$ and $\mathcal{F}^*_{p,q}$, and symmetrization of this pairing leads to the following bilinear form on $\mathcal{F}_{p,q} \times \mathcal{F}^*_{p,q}$ with values in $\mathbb{R}$:

$$\begin{aligned} &\ll (f_p^1, f_q^1, f_b^1, e_p^1, e_q^1, e_b^1), (f_p^2, f_q^2, f_b^2, e_p^2, e_q^2, e_b^2) \gg \\ &:= \int_Z [e_p^1 \wedge f_p^2 + e_q^1 \wedge f_q^2 + e_p^2 \wedge f_p^1 + e_q^2 \wedge f_q^1] + \int_{\partial Z} [e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1] \end{aligned} \tag{7}$$

where for $i = 1, 2$

$$\begin{array}{ll} f_p^i \in \Omega^p(Z), & f_q^i \in \Omega^q(Z) \\ e_p^i \in \Omega^{n-p}(Z), & e_p^i \in \Omega^{n-q}(Z) \\ f_b^i \in \Omega^{n-p}(\partial Z), & e_b^i \in \Omega^{n-q}(\partial Z) \end{array}$$

The spaces of differential forms $\Omega^p(Z)$ and $\Omega^q(Z)$ represent the energy variables of two different physical energy domains interacting with each other, while $\Omega^{n-p}(\partial Z)$ and $\Omega^{n-q}(\partial Z)$ denotes the boundary variables whose (wedge) product represents the boundary energy flow. It has thus been shown in [4] that the following system defines a port-Hamiltonian system

$$\begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & (-1)^r d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}; \quad \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} e_p \mid_{\partial Z} \\ e_q \mid_{\partial Z} \end{bmatrix} \tag{8}$$

with $|_{\partial Z}$ denoting the restriction to the boundary ∂Z and $r := pq + 1$. The space of all admissible flows and efforts satisfying (8) represents a Dirac structure called *Stokes'* Dirac structure.

Example 2.2 *Consider an ideal lossless transmission line with $Z = [0,\ 1] \in \mathbb{R}$. The energy variables are the charge density one-form $Q = Q(t,z)dz \in \Omega^1([0,1])$, and the flux density one-form $\phi = \phi(t,z)dz \in \Omega^1([0,1])$ $(p = q = n = 1)$. The total energy stored at time t in the transmission line is given as*

$$H(Q,\phi) = \int_0^1 \frac{1}{2}\left(\frac{Q^2(t,z)}{C(z)} + \frac{\phi^2(t,z)}{L(z)}\right) dz \tag{9}$$

with co-energy variables

$$\begin{aligned} \delta_Q H &= \frac{Q(t,z)}{C(z)} = V(t,z), \quad \textit{(voltage)} \\ \delta_\phi H &= \frac{\phi(t,z)}{L(z)} = I(t,z), \quad \textit{(current)} \end{aligned} \tag{10}$$

where $C(z)$, $L(z)$ are respectively, the distributed capacitance and distributed inductance of the line.

The resulting port-Hamiltonian system is given by the telegrapher's equations

$$\begin{aligned} \frac{\partial Q}{\partial t} &= -\frac{\partial I}{\partial z} \\ \frac{\partial \phi}{\partial t} &= -\frac{\partial V}{\partial z} \end{aligned} \tag{11}$$

together with the boundary variables

$$\begin{array}{ll} f_b^0(t) = V(t,0), & f_b^l(t) = V(t,l) \\ e_b^0(t) = -I(t,0), & e_b^l(t) = -I(t,l) \end{array} \tag{12}$$

with the resulting energy balance as

$$\frac{dH}{dt} = \int_{\partial([0,l])} e_b f_b = -I(t,l)V(t,l) + I(t,0)V(t,0) \tag{13}$$

3 Interconnection of Dirac structures (The mixed finite and infinite dimensional case)

We consider here composition of two Dirac structures (denoted $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively) interconnected to each other via a Stokes' Dirac structure (denoted $\mathcal{D}_\infty$). For simplicity we consider the case $p = q = n = 1$ throughout, for the Stokes' Dirac structure. (An immediate example of this case is that of a transmission line.)

First we consider the composition of the two Dirac structures $\mathcal{D}_1$ and $\mathcal{D}_\infty$. Consider $\mathcal{D}_1$ on the product space $\mathcal{F}_1 \times \mathcal{F}_0$ of two linear spaces $\mathcal{F}_1$ and $\mathcal{F}_0$, and the Stokes' Dirac structure $\mathcal{D}_\infty$ on the product space $\mathcal{F}_0 \times \mathcal{F}_{p,q} \times \mathcal{F}_l$, with $\mathcal{F}_0$ and $\mathcal{F}_l$ being linear spaces (representing the space of boundary variables of the Stokes' Dirac structure) and $\mathcal{F}_{p,q}$ an infinite dimensional function space with p, q representing the two different physical energy domains interacting with each other. The linear space $\mathcal{F}_0$ is the space of shared flow variables and its dual $\mathcal{F}_0^*$, the space of shared effort variables between $\mathcal{D}_1$ and $\mathcal{D}_\infty$. Next consider the composition of $\mathcal{D}_\infty$ and $\mathcal{D}_2$. Considering $\mathcal{D}_2$ as defined on the product space $\mathcal{F}_l \times \mathcal{F}_2$ of two linear spaces, we have the linear space $\mathcal{F}_l$ the space of shared flow variables and its dual $\mathcal{F}_l^*$, the space of shared effort variables between $\mathcal{D}_2$ and $\mathcal{D}_\infty$.

We define the two interconnections as follows:

The interconnection of the two Dirac structures $\mathcal{D}_1$ and $\mathcal{D}_\infty$ is defined as

$$\begin{aligned} \mathcal{D}_1 \| \mathcal{D}_\infty := \; & \{ f_1, e_1, f_p, f_q, e_p, e_q, f_l, e_l) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_l \times \mathcal{F}_l^* \mid \\ & \exists (f_0, e_0) \in \mathcal{F}_0 \times \mathcal{F}_0^* \; s.t \\ & (f_1, e_1, f_0.e_0) \in \mathcal{D}_1 \text{ and } (-f_0, e_0, f_p, f_q.e_p, e_q, f_l, e_l) \in \mathcal{D}_\infty \} \end{aligned}$$

Similarly, the interconnection of $\mathcal{D}_\infty$ and $\mathcal{D}_2$ is defined as

$$\begin{aligned} \mathcal{D}_\infty \| \mathcal{D}_2 := \; & \{ -f_0, e_0, f_p, f_q, e_p, e_q, f_2, e_2) \in \mathcal{F}_0 \times \mathcal{F}_0^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_2 \times \mathcal{F}_2^* \mid \\ & \exists (f_l, e_l) \in \mathcal{F}_l \times \mathcal{F}_l^* \; s.t \\ & (-f_0, e_0, f_p, f_q.e_p, e_q, f_l, e_l) \in \mathcal{D}_\infty \text{ and } (-f_l, e_l, f_2, e_2) \in \mathcal{D}_2 \} \end{aligned}$$

Hence we can define the total interconnection of $\mathcal{D}_1$,$\mathcal{D}_\infty$ and $\mathcal{D}_2$ as (also see figure below)

$$\begin{aligned} \mathcal{D}_1 \| \mathcal{D}_\infty \| \, \mathcal{D}_2 := \; & \{ (f_1, e_1, f_p, f_q, e_p, e_q, f_2, e_2) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_2 \times \mathcal{F}_2^* \mid \\ & \exists (f_0, e_0) \in \mathcal{F}_0 \times \mathcal{F}_0^* \; s.t \; (f_1, e_1, f_0.e_0) \in \mathcal{D}_1 \text{ and } (-f_0, e_0, f_p, f_q.e_p, e_q, f_l, e_l) \in \mathcal{D}_\infty \\ \text{and} \quad & \exists (f_l, e_l) \in \mathcal{F}_l \times \mathcal{F}_l^* \; s.t \; (-f_0, e_0, f_p, f_q.e_p, e_q, f_l, e_l) \in \mathcal{D}_\infty \text{ and } (-f_l, e_l, f_2.e_2) \in \mathcal{D}_2 \} \end{aligned} \tag{14}$$

This yields the following bilinear form on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$:

$$\begin{aligned} & \ll (f_1^a, f_p^a, f_q^a, f_2^a, e_1^a, e_p^a, e_q^a, e_2^a), (f_1^b, f_p^b, f_q^b, f_2^b, e_1^b, e_p^b, e_q^b, e_2^b) \gg \\ & := < e_1^b | f_1^a > + < e_1^a | f_1^b > + < e_2^a | f_2^b > + < e_2^b | f_2^a > \\ & + \int_z \left[e_p^a \wedge f_p^b + e_p^b \wedge f_p^a + e_q^b \wedge f_q^a + e_q^a \wedge f_q^b \right] \end{aligned} \tag{15}$$

Theorem 3.1 *Let $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_\infty$ be Dirac structures as said above (defined respectively with respect to $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_0 \times \mathcal{F}_0^*$, $\mathcal{F}_l \times \mathcal{F}_l^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$ and $\mathcal{F}_0 \times \mathcal{F}_0^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_l \times \mathcal{F}_l^*$). Then $D = \mathcal{D}_1 \| \mathcal{D}_\infty \| \, \mathcal{D}_2$ is a Dirac structure defined with respect to the bilinear form on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$ given by (15).*

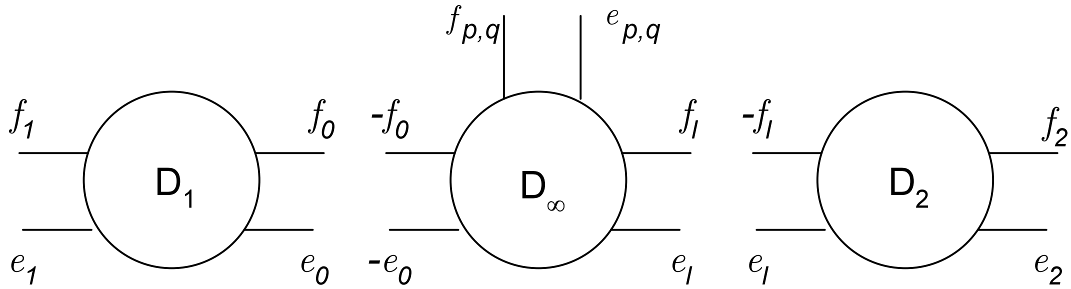

Figure 1: $\mathcal{D}_1 \parallel \mathcal{D}_\infty \parallel \mathcal{D}_2$

We use the following facts for the proof (as we know that $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_\infty$ individually are Dirac structures).
On $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_0 \times \mathcal{F}_0^*$ the bilinear form is defined as

$$\begin{aligned} &\ll (f_1^a, f_0^a, e_1^a, e_0^a), (f_1^b, f_0^b, e_1^b, e_0^b) \gg \\ &:= < e_1^b | f_1^a > + < e_1^a | f_1^b > + < e_0^b | f_0^a > + < e_0^a | f_0^b > \end{aligned} \tag{16}$$

and $\mathcal{D}_1 = \mathcal{D}_1^\perp$ with respect to the bilinear form as in (16).
Similarly on $\mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_l \times \mathcal{F}_l^*$ the bilinear form is defined as

$$\begin{aligned} &\ll (-f_l^a, e_l^a, f_2^a, e_2^a), (-f_l^b, e_l^b, f_2^b, e_2^b) \gg \\ &:= < e_2^b | f_2^a > + < e_2^a | f_2^b > - < e_l^b | f_l^a > - < e_l^a | f_l^b > \end{aligned} \tag{17}$$

and $\mathcal{D}_2 = \mathcal{D}_2^\perp$ with respect to the bilinear form as in (17).
On $\mathcal{F}_0 \times \mathcal{F}_0^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_l \times \mathcal{F}_l^*$ the bilinear form takes the following form

$$\begin{aligned} &\ll (f_p^a, f_q^a, f_b^a, e_p^a, e_q^a, e_b^a), (f_p^b, f_q^b, f_b^b, e_p^b, e_q^b, e_b^b) \gg \\ &:= \int_z \Big[e_p^a \wedge f_p^b + e_p^b \wedge f_p^a + e_q^b \wedge f_q^a + e_q^a \wedge f_q^b \Big] + \Big[< e_l^a | f_l^b > + < e_l^b | f_l^a > - < e_0^a | f_0^b > + < e_0^b | f_0^a > \Big] \end{aligned} \tag{18}$$

and $\mathcal{D}_\infty = \mathcal{D}_\infty^\perp$ with respect to the bilinear form as in (18).

Proof. (i) $\mathcal{D} \subset \mathcal{D}^\perp$: Let $(f_1^a, f_p^a, f_q^a, f_2^a, e_1^a, e_p^a, e_q^a, e_2^a) \in \mathcal{D}$ and consider any other $(f_1^b, f_p^b, f_q^b, f_2^b, e_1^b, e_p^b, e_q^b, e_2^b) \in \mathcal{D}$ and the bilinear form on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$ as in (15).

Then $\exists$ (f_0^a, e_0^a), (f_l^a, e_l^a) $s.t$ $(f_1^a, e_1^a, f_0^a, e_0^a) \in \mathcal{D}_1$, $(-f_0^a, e_0^a, f_p^a, f_q^a, e_p^a, e_q^a, f_l^a, e_l^a) \in \mathcal{D}_\infty$ and $(-f_l^a, e_l^a, f_2^a, e_2^a) \in \mathcal{D}_2$,

and $\exists$ (f_0^b, e_0^b), (f_l^b, e_l^b) $s.t$ $(f_1^b, e_1^b, f_0^b, e_0^b) \in \mathcal{D}_1$, $(-f_0^b, e_0^b, f_p^b, f_q^b, e_p^b, e_q^b, f_l^b, e_l^b) \in \mathcal{D}_\infty$ and $(-f_l^b, e_l^b, f_2^b, e_2^b) \in \mathcal{D}_2$.

Since $\mathcal{D}_\infty$ is a Dirac structure with respect to (18)

$$\int_z \Big[e_p^a \wedge f_p^b + e_p^b \wedge f_p^a + e_q^b \wedge f_q^a + e_q^a \wedge f_q^b \Big] = - < e_l^a | f_l^b > - < e_l^b | f_l^a > + < e_0^a | f_0^b > + < e_0^b | f_0^a > \tag{19}$$

Substituting (19) in (15) and using the fact that the bilinear from (16) is zero on $\mathcal{D}_1$ and (17) is zero on $\mathcal{D}_2$, we get

$$< e_1^b|f_1^a > + < e_1^a|f_1^b > + < e_2^b|f_2^a > + < e_2^a|f_2^b >$$
$$+ \int_z \left[e_p^a \wedge f_p^b + e_p^b \wedge f_p^a + e_q^b \wedge f_q^a + e_q^a \wedge f_q^b \right] = 0$$

and hence $\mathcal{D} \subset \mathcal{D}^\perp$

(ii) $\mathcal{D}^\perp \subset \mathcal{D}$: We know that the flow and effort variables of $\mathcal{D}_\infty$ are related as

$$D_\infty \triangleq \left\{ (f,e) \in \mathcal{F} \times \mathcal{F}^* \mid \begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e_p|\partial z \\ e_q|\partial z \end{bmatrix} \right\} \tag{20}$$

Let $(f_1^a, f_p^a, f_q^a, f_2^a, e_1^a, e_p^a, e_q^a, e_2^a) \in \mathcal{D}^\perp$, then for all $(f_1^b, f_p^b, f_q^b, f_2^b, e_1^b, e_p^b, e_q^b, e_2^b) \in \mathcal{D}$ the right side of equation (15) is zero. Since we already know [3] that in the absence of $\mathcal{D}_\infty$, the interconnection $\mathcal{D}_1 \parallel \mathcal{D}_2$ defines a Dirac structure, it would be sufficient to show that $(f_1^a, f_p^a, f_q^a, f_2^a, e_1^a, e_p^a, e_q^a, e_2^a) \in \mathcal{D}^\perp$ satisfies (20).

Now consider the vectors $(f_1^b, f_p^b, f_q^b, f_2^b, e_1^b, e_p^b, e_q^b, e_2^b) \in \mathcal{D}$ with $f_1^b = f_2^b = e_1^b = e_2^b = 0$. Then from (20) and (15) we have

$$\int_z \left[e_p^a \wedge de_q^b + e_p^b \wedge f_p^a + e_q^b \wedge f_q^a + e_q^a \wedge de_p^b \right] = 0 \tag{21}$$

This implies (see the proof of theorem 2.1 in [4])

$$f_p^a = de_q^a \text{ and } f_q^a = de_p^a \tag{22}$$

Substituting (22) in (15) we have

$$< e_1^b|f_1^a > + < e_1^a|f_1^b > + < e_2^b|f_2^a > + < e_2^a|f_2^b > + \int_z \left[e_p^a \wedge de_q^b + e_p^b \wedge de_q^a + e_q^b \wedge de_p^a + e_q^a \wedge de_p^b \right] = 0$$

This yields by Stokes' theorem

$$< e_1^b|f_1^a > + < e_1^a|f_1^b > + < e_2^b|f_2^a > + < e_2^a|f_2^b > + \left[< e_p^a|e_q^b > + < e_p^b|e_q^a > \right] \Big|_0^l = 0$$

Since we already know that $\mathcal{D}_1$ and $\mathcal{D}_2$ are Dirac structures, with respect to the bilinear forms (16) and (17) the above equation can be written as

$$< e_0^a|f_0^b > + < e_0^b|f_0^a > - < e_l^a|f_l^b > - < e_l^b|f_l^a >$$
$$- < e_{pl}^a \mid f_l^b > + < e_l^b \mid e_{ql}^a > + < e_{p0}^a \mid f_0^b > - < e_0^b \mid e_{q0}^a >= 0$$

from the above equation we get the following:

$$\begin{aligned} f_0^a &= -e_q^a|_0, \quad e_0^a = e_p^a|_0 \\ f_l^a &= -e_q^a|_l, \quad e_l^a = e_p^a|_l \end{aligned} \tag{23}$$

and hence $\mathcal{D}^\perp \subset \mathcal{D}$, completing the proof. ■

4 Achievable Dirac Structures:

We investigate which closed-loop port-Hamiltonian systems can be achieved by interconnecting a plant port-Hamiltonian system P (which in our case is an infinite dimensional port-Hamiltonian system) with a controller port-Hamiltonian system(s), (which here are port-Hamiltonian systems connected at the boundaries of P), in particular we investigate what closed-loop Dirac structures can be achieved. That is given a $\mathcal{D}_\infty$ (which is considered as the plant Dirac structure $\mathcal{D}_p$) and to be designed $\mathcal{D}_1$ and $\mathcal{D}_2$ (which comprise the controller Dirac structure $\mathcal{D}_c$), what are the achievable $\mathcal{D}_1\| \mathcal{D}_\infty\| \mathcal{D}_2$ (or $\mathcal{D}_p\| \mathcal{D}_c$). See figure (2).

In the figure, $f_b = \begin{bmatrix}-f_0 & f_l\end{bmatrix}^T, e_b = \begin{bmatrix}e_0 & e_l\end{bmatrix}^T$, $f = \begin{bmatrix}f_1 & f_2\end{bmatrix}^T, e = \begin{bmatrix}e_1 & e_2\end{bmatrix}^T$, $f_b' = \begin{bmatrix}f_0 & -f_l\end{bmatrix}^T, e_b = \begin{bmatrix}e_0 & e_l\end{bmatrix}$, such that
$(f_1, e_1, f_0, e_0) \in \mathcal{D}_1$ and $(-f_l, e_l, f_2, e_2) \in \mathcal{D}_2$

Theorem 4.1 *Given any plant Dirac structure $\mathcal{D}_p$, a certain interconnected $\mathcal{D} = \mathcal{D}_p\| \mathcal{D}_c$ can be achieved by a proper choice of the controller Dirac structure D_c if and only if the following two conditions are satisfied*

$$\begin{aligned} \mathcal{D}_p^0 &\subset \mathcal{D}^0 \\ \mathcal{D}^\pi &\subset \mathcal{D}_p^\pi \end{aligned} \tag{24}$$

$$\textit{where} \quad \begin{cases} \mathcal{D}_p^0 := \{(f_p, f_q, e_p, e_q) \mid (f_p, f_q, e_p, e_q, 0, 0) \in \mathcal{D}_p\} \\ \mathcal{D}_p^\pi := \{(f_p, f_q, e_p, e_q) \mid \exists (f_b, e_b) : (f_p, f_q, e_p, e_q, f_b, e_b) \in \mathcal{D}_p\} \\ \mathcal{D}^0 := \{(f_p, f_q, e_p, e_q) \mid (f_p, f_q, e_p, e_q, 0, 0) \in \mathcal{D}\} \\ \mathcal{D}^\pi := \{(f_p, f_q, e_p, e_q) \mid \exists (f, e) : (f_p, f_q, e_p, e_q, f, e) \in \mathcal{D}\} \end{cases} \tag{25}$$

Proof. As in [3] the proof is based on the "copy" of $\mathcal{D}_p$ (see figure (3)) defined as follows

$$\mathcal{D}_p^* := \{(f_p, f_q, e_p, e_q, f_b, e_b) \mid (-f_p, -f_q, e_p, e_q - f_b, e_b) \in \mathcal{D}_p\}$$

The necessity of (24) and (25) is obvious and proof of sufficienfy follows the same procedure as in [3] by first proving $\mathcal{D} \subset \mathcal{D}_p \parallel \mathcal{D}_c$, and then $\mathcal{D}_p \parallel \mathcal{D}_c \subset \mathcal{D}$, hence we omit the proof here. ■

Remark 4.2 *We can also consider other mixed cases where we can take $\mathcal{D}_p$ as the interconnection of the Stokes' Dirac structure and a Dirac structure connected to one of its boundary, and $\mathcal{D}_c$ would then be a Dirac structure interconneted to the other end of the Stokes' Dirac structure. This is the case if we want to control a plant which is interconnected to a controller through a infinite dimensional system, which is also one of the cases we consider in the next section.*

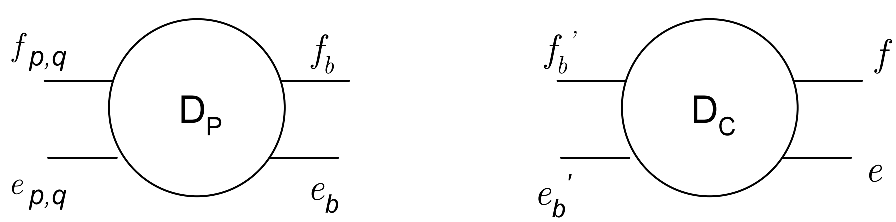

Figure 2: $\mathcal{D}_p \parallel \mathcal{D}_c$

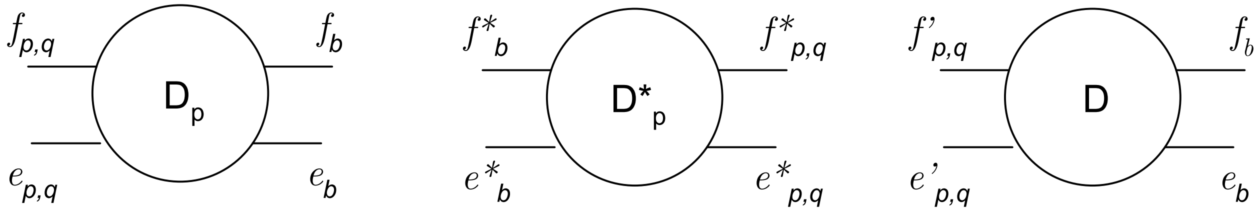

Figure 3: $\mathcal{D} = \mathcal{D}_p \parallel \mathcal{D}_p^* \parallel \mathcal{D}$

5 Achievable Casimirs

This section answers the question of which are the *Casimir functions* that can be achieved for the closed loop system by interconnecting a given plant port-Hamiltonian system with associated Dirac structure $\mathcal{D}_p$ with a controller port-Hamiltonian system with associated Dirac structure $\mathcal{D}_c$.

A *Casimir function* $C : \mathcal{X} \longrightarrow \mathbb{R}$ of a port-Hamiltonian system is defined to be a function which is constant along all trajectiories of the port-Hamiltonian system, irrespective of the Hamiltonian H.

We consider here the question of characterizing the set of achievable Casimirs (again for the mixed finite and infinite dimensional case) for the closed-loop system $\mathcal{D}_p \| \mathcal{D}_c$, where $\mathcal{D}_p$ is the given Dirac structure of the plant port-Hamiltonian system with Hamiltonian H, and $\mathcal{D}_c$ is the controller Dirac stucture. We here consider the case where $\mathcal{D}_p$ is a Stokes' Dirac structure (although we can also consider other mixed cases as stated in the previous section) and investigate as to what are the achievable Casimirs. Again for simplicity we consider the case $p = q = n = 1$.

Consider the notation in figure(2), and assume ports in $(f_{p,q}, e_{p,q})$[1] are connected to the (given) energy storing elements of the plant port-Hamiltonian system that is

$$f_p = -\frac{\partial \alpha_p}{\partial t}, \; f_q = -\frac{\partial \alpha_q}{\partial t}$$
$$e_p = \delta_p H, \quad e_q = \delta_q H$$

while (f, e) are connected to the (to be designed) energy storing elements of the controller port-Hamiltonian system(s). We know that from the power variable description of an infinite-dimensional port-Hamiltonian system, that the Casimir functions are determined by the subspace $\{e_p, q \in \mathcal{F}_p, q^* \mid (0, e_p, q) \in \mathcal{D}_\infty\}$. In this situation the achievable Casimir functions are functions $C(x, \xi)$ such that $\frac{\partial^T C(x)}{\partial x}$ belongs to the space

$$P_{Cas} = \{e_{p,q} \mid \exists \mathcal{D}_c \; s.t \; \exists \; e : (0, e_{p,q}, 0, e) \in \mathcal{D}_p \| \mathcal{D}_c\} \tag{26}$$

where again as in the previous section the controller Dirac structure $\mathcal{D}_c$ comprises the Dirac structures $\mathcal{D}_1$ and $\mathcal{D}_2$ and $e = \begin{bmatrix} e_1 & e_2 \end{bmatrix}^T$. (See figure (2)). Similar to the finite dimensional case [3] the following theorem addresses the question of characterizing the achievable Casimirs of the closed-loop system, regarded as functions of the plant state x , by finding a characterization of the space P_Cas.

[1]For brevity we use the notation $(f_{p,q}, e_{p,q})$ for (f_p, f_q, e_p, e_q).

Theorem 5.1 *The space P_{Cas} defined in (26) is equal to the linear space*

$$\tilde{P} = \{e_{p,q} \mid \exists (f_b, e_b) : (0, e_{p,q}, f_b, e_b) \in \mathcal{D}_p\}$$

Proof. The inclusion $P_{Cas} \subset \tilde{P}$ is obvious, and taking the controller Dirac structure $\mathcal{D}_c = \mathcal{D}_p^*$, the second inclusion $\tilde{P} \subset P_{Cas}$ is obtained.
Since for all $e_{p,q} \in \tilde{P}$ we have $f_{p,q} = 0$ and $(0, e_{p,q}, -f_b, e_b) \in \mathcal{D}_p^*$. And with respect to (20) this would mean that the space $\tilde{P}$ is such that e_p and e_q are constants as functions of the spatial variable, which in addition would mean $f_0 = f_l$ and $e_0 = e_l$, thus resulting in finite dimensional controllers. ■

Example 5.2 *(Continued)*
We see that in case of the transmission line

$$\begin{aligned} &P_{Cas} = \{(V(t,z), I(t,z) \mid \exists (f_b, e_b)\ s.t \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} V(t,z) \\ I(t,z) \end{bmatrix}; \\ &V(t,z) = V(t,0) = V(t,l);\ I(t,z) = I(t,0) = I(t,l). \end{aligned}$$

which means that the set of achievable Casimir functions is such that $V(t,z)$ and $I(t,z)$ are constant as a function of z, or in other words every Casimir function should be linear with respect to the spatial variables

Next we consider a case where $\mathcal{D}_p$ is the interconnection of a Stokes' Dirac structure with a Dirac structure interconnected to one of its boundary. (In terms of figure (2) this would mean $\mathcal{D}_p := \mathcal{D}_1 \parallel \mathcal{D}_\infty$) and then characterise the set of achievable Casimirs for the closed-loop system $\mathcal{D}_p \parallel \mathcal{D}_c$ (also see figure(4)). We assume that the ports in (f_1, e_1) are connected to the energy storing elements of $\mathcal{D}_1$ (meaning $f_1 = -\dot{x}$, $e_1 = \frac{\partial^T H}{\partial x}$), the ports in $(f_{p,q}, e_{p,q})$ are connected to the (given) energy storing elements of $\mathcal{D}_\infty$ that is

$$\begin{aligned} &f_p = -\frac{\partial \alpha_p}{\partial t},\ f_q = -\frac{\partial \alpha_q}{\partial t} \\ &e_p = \delta_p H, \quad e_q = \delta_q H \end{aligned}$$

and (f_2, e_2) are connected to the energy storing elements of the (to be designed) controller port-Hamiltonian system $\mathcal{D}_2$ (that is $f_2 = -\dot{\xi}$, $e_2 = \frac{\partial^T H}{\partial \xi}$). Then the achievable Casimirs are functions $C(x, \xi)$ such that $\frac{\partial^T C}{\partial x}(x, \xi)$ belongs to the space

$$P_{Cas} = \{(e_1, e_{p,q}) \mid \exists \mathcal{D}_c\ s.t. \exists e_2 : (0, e_1, 0, e_{p,q}, 0, e_2) \in \mathcal{D}_p \parallel \mathcal{D}_c\} \tag{27}$$

Again as above the following theorem characterizes the set of achievable Casimir's of the closed-loop system, regarded as functions of the plant state x, by finding a characterization of the space P_{Cas}.

Theorem 5.3 *The space P_{Cas} defined in (27) is equal to the linear space*

$$\tilde{P} = \{(e_1, e_{p,q}) \mid \exists (f_l, e_l) : (0, e_1, 0, e_{p,q}, f_l, e_l) \in \mathcal{D}_p\}$$

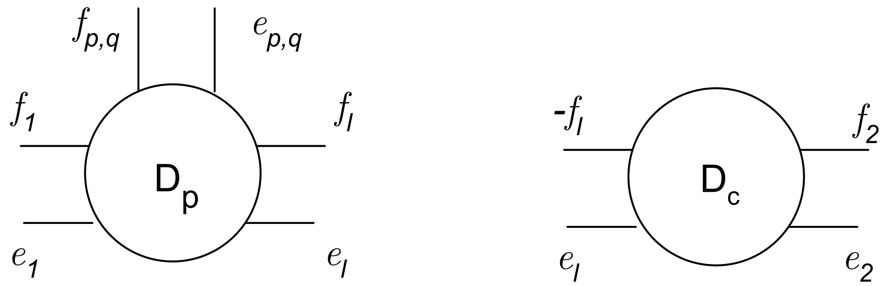

Figure 4: $\mathcal{D}_p \parallel \mathcal{D}_c$

Proof. The inclusion $P_{Cas} \subset \tilde{P}$ is obvious, and by taking the controller Dirac structure $\mathcal{D}_c = \mathcal{D}_1^*$, the inclusion $\tilde{P} \subset P_{Cas}$ is obtained immediately. D_1^* is defined as

$$\mathcal{D}_1^* = \{(f_1, e_1, f_0, e_0) \mid (-f_1, e_1, -f_0, e_0) \in \mathcal{D}_1\}$$

This is also equivalent to considering $\mathcal{D}_c = (\mathcal{D}_p^* = \mathcal{D}_1^* \parallel \mathcal{D}_\infty^*)$ as in the previous theorem as $f_{p,q} = 0$ would again mean that the space $\tilde{P}$ is such that e_p and e_q are constants as functions of the spatial variable, which in addition would mean $f_0 = f_l$ and $e_0 = e_l$ hence we can take $\mathcal{D}_c = \mathcal{D}_1^*$ ■

Example 5.4 *Consider the case as in figure(4), where the Dirac structure of the plant is given by*

$$\begin{bmatrix} f_1 \\ f_p \\ f_q \end{bmatrix} = \begin{bmatrix} -J(x) & 0 & 0 \\ 0 & 0 & d \\ 0 & d & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_p \\ e_q \end{bmatrix} + \begin{bmatrix} -g(x) \\ 0 \\ 0 \end{bmatrix} e_{p0}; \; \begin{bmatrix} f_l \\ e_l \end{bmatrix} = \begin{bmatrix} -e_{ql} \\ e_{pl} \end{bmatrix}$$

$$e_0 = g^T(x) e_1$$

In this case

$$P_{Cas} = \{(e_1, e_{p,q}) \mid \exists \; (f_l, e_l) \; s.t \;\; \begin{array}{l} 0 = J(x)e_1 + g(x)e_{p0} \; and \\ 0 = \frac{\partial}{\partial z} e_p; \; 0 = \frac{\partial}{\partial z} e_q \end{array}$$

which again shows that the set of achievable Casimirs is such that e_p and e_q are constant with respect to the spatial variable z and the x dependency of the Casimir functions are the Hamiltonian functions corresponding to the input vector fields given by the columns of $g(x)$.

Concluding Remarks

In this paper we have presented a first stage extension of some of the fundamental results on interconnections of Dirac structures to the infinite dimensional case, or the mixed finite and infinite dimensional case in particular. We have shown how the composition of finite and infinite-dimensional Dirac structures again defines a Dirac structure and have presented extensions of results concerning the achievable closed-loop Dirac structures and achievable Casimir functions, to the mixed case.

Future work would be to consider systems with dissipation and generalize this framework to the more general case of $Z \subset \mathbb{R}^n$ (or $p, q > 1$) and apply these results to stabilization of infinite-dimensional port-Hamiltonian systems.

Acknowledgements

This work has been done in context of the European sponsored project GeoPleX IST-2001-34166. For more information see http://www.geoplex.cc

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