

Implicit Lagrangian equations and the mathematical modeling of physical systems

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Abstract— We introduce a class of optimal control problems on manifolds which gives rise (via the Pontryagin maximum principle) to a class of *implicit Lagrangian systems* (a notion which is introduced in the paper). We apply this to the mathematical modeling of interconnected mechanical systems and mechanical systems with singularities.

Keywords— Lagrangian and Hamiltonian dynamics, Dirac structures, modeling, physical systems.

I. INTRODUCTION

Lagrangian and Hamiltonian dynamics continue to attract a large amount of attention in the engineering literature, partially because many mechanical and electromechanical systems may be modeled within this framework and also because a number of successful control algorithms are available for these classes of systems; see, e.g., [13], [19], [14], [5], [4].

During the past ten years, a far-reaching generalization of the Hamiltonian framework has been developed in a series of papers, including [11], [20], [17], [6], [8], [18], [3]. This generalization, which is based on *generalized Dirac structures*, explicitly takes into account the modular structure characteristic of many engineering systems, and gives rise to *implicit Hamiltonian systems*. (See [10] for applications in the study of electrical power converters and [19], [15] for a more general control perspective.)

The aim of the present research is to contribute to a complementary theory of *implicit Lagrangian systems*, following the line of research initiated in [12]. In that paper it is shown that, for the particular class of electrical inductor-capacitor circuits, the implicit Hamiltonian models from [11], [20], [6] have a Lagrangian counterpart and a corresponding variational interpretation. The variational interpretation is based on the maximum principle [16] of optimal control theory, which may be seen as a generalization of the classical theory of the calculus of variations. The approach in [12] is limited to constant Dirac structures on vector spaces.

In the present paper, we extend the approach of [12] from vector spaces towards manifolds. We show how a class of optimal control problems on manifolds gives rise to a class of *implicit Lagrangian systems* (the notion of implicit Lagrangian system is introduced in the paper) with respect

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to generalized Dirac structures on manifolds. The variational interpretation is important, as it provides a systematic approach for deriving the equations of motion. The obtained implicit Lagrangian models may, under appropriate invertibility conditions (concerning a Legendre transformation), be converted into implicit Hamiltonian models. The present research thus provides an alternative approach for deriving (some classes of) implicit Hamiltonian models. For some applications, it may be desirable to work directly with a Lagrangian formulation instead of an implicit Hamiltonian description. This may for example be the case when the Legendre transformation gives rise to tedious manipulations, or when the Legendre transformation is not invertible. The present research provides the theoretical background for such a Lagrangian description. The theory is illustrated by means of two applications: interconnected mechanical systems and mechanical systems with singularities.

We conclude this introduction with some hints for future research. The fact that the implicit Lagrangian systems of the present paper are characterized by a Dirac structure is very important. It suggests a modularity whereby interconnections of implicit (or classical) Lagrangian systems might again result in implicit Lagrangian system. This could be a subject of further research. The ultimate goal of the present research is to provide models that are useful for control design purposes. However, none of the Lagrangian control algorithms reported in [14], [5], [4] applies directly to the present class of implicit Lagrangian systems. The generalization of these control algorithms therefore remains an important topic for further research.

The paper is organized as follows. In Section II we introduce an optimal control problem and in Section III we discuss the Lagrangian and Hamiltonian nature of equations associated to this optimal control problem. In Sections IV and V we present two applications.

II. AN OPTIMAL CONTROL PROBLEM

We introduce the optimal control problem that (under appropriate assumptions) will give rise to implicit Lagrangian systems. Before we state this optimal control problem, we introduce some notation.

Let X be a smooth n_x -dimensional manifold ($n_x \in \mathbb{N}$). We consider a control system on X

$$\dot{x} = E_a(x)u_a \quad (1)$$

where E_1, \dots, E_{n_u} are smooth vector fields on X ($n_u \in \mathbb{N}$). (Throughout the paper we adhere to the summation convention according to which indices that appear twice in the same term are automatically summed over. In equation (1), for example, $E_a(x)u_a$ is a short-hand notation for $\sum_a E_a(x)u_a$.)

Let \mathcal{V} be a vector bundle over X with n_v -dimensional fibers ($n_v \in \mathbb{N}$). We associate with every state-input pair (x, u) an element ν of \mathcal{V} according to

$$\nu = F_a(x)u_a \quad (2)$$

where F_1, \dots, F_{n_u} are smooth sections of \mathcal{V} .

Let L be a smooth real-valued function on \mathcal{V} . We are interested in the following optimal control problem: among all piecewise continuous control inputs $u : [t_i, t_f] \rightarrow \mathbb{R}^{n_u}$ that steer x from x_i at t_i to x_f at t_f (x_i, t_i, x_f, t_f are prescribed) according to the dynamics (1), find that control input that minimizes $\int_{t_i}^{t_f} L(\nu) dt$ where ν is given by (2).

We apply the Pontryagin maximum principle to this problem and we study the Lagrangian and Hamiltonian structure of equations that determine its *normal* Pontryagin extremals. (Throughout the paper we restrict attention to the normal Pontryagin extremals.) In order to carry out this plan, it is convenient to introduce local coordinates $(x, v) = (x_1, \dots, x_{n_x}, v_1, \dots, v_{n_v})$ on \mathcal{V} where (x_1, \dots, x_{n_x}) are local coordinates on X and (v_1, \dots, v_{n_v}) are linear coordinates on the fibers. Formulated in terms of these local coordinates, the optimal control problem becomes: among all piecewise continuous control inputs $u : [t_i, t_f] \rightarrow \mathbb{R}^{n_u}$ that steer x from x_i at t_i to x_f at t_f according to the dynamics $\dot{x}_i = E_{ai}(x)u_a$, find that control that minimizes $\int_{t_i}^{t_f} L(x, v) dt$ where $v_m = F_{am}(x)u_a$. An application of the Pontryagin maximum principle yields the set of equations

$$\dot{x}_i = E_{ai}(x)u_a, \quad (3)$$

$$\dot{\xi}_i = \frac{\partial L}{\partial x_i}(x, v) + \frac{\partial L}{\partial v_m}(x, v) \frac{\partial F_{am}}{\partial x_i}(x)u_a - \xi_j \frac{\partial E_{aj}}{\partial x_i}(x)u_a, \quad (4)$$

$$\frac{\partial L}{\partial v_m}(x, v) F_{am}(x) = \xi_i E_{ai}(x), \quad (5)$$

$$v_m = F_{am}(x)u_a. \quad (6)$$

Here $(x_1, \dots, x_{n_x}, \xi_1, \dots, \xi_{n_x})$ may be interpreted as local coordinates on T^*X , the cotangent bundle of X , that are induced by the local coordinates (x_1, \dots, x_{n_x}) on X .

We take the derivative of equation (5) with respect to time and eliminate $\dot{\xi}$ by means of equation (4). This leads to the set of equations

$$\dot{x}_i = E_{ai}(x)u_a, \quad (7)$$

$$F_{am}(x) \frac{d}{dt} \left(\frac{\partial L}{\partial v_m}(x, v) \right) - E_{ai}(x) \frac{\partial L}{\partial x_i}(x, v) = \xi_i \left\{ E_{bj}(x) \frac{\partial E_{ai}}{\partial x_j}(x) - E_{aj}(x) \frac{\partial E_{bi}}{\partial x_j}(x) \right\} u_b + \frac{\partial L}{\partial v_m}(x, v) \left\{ E_{ai}(x) \frac{\partial F_{bm}}{\partial x_i}(x) - E_{bi}(x) \frac{\partial F_{am}}{\partial x_i}(x) \right\} u_b, \quad (8)$$

$$\frac{\partial L}{\partial v_m}(x, v) F_{am}(x) = \xi_i E_{ai}(x), \quad (9)$$

$$v_m = F_{am}(x)u_a. \quad (10)$$

The factor between brackets in the first term of the right hand side of (8) corresponds to the coordinate expression of the Lie bracket of the vector fields E_b and E_a . If the vector fields E_1, \dots, E_{n_u} form an involutive distribution, that is (in coordinates), if there exist scalar functions $c_{bad}(x)$ such that

$$E_{bj}(x) \frac{\partial E_{ai}}{\partial x_j}(x) - E_{aj}(x) \frac{\partial E_{bi}}{\partial x_j}(x) = c_{bad}(x) E_{di}(x), \quad (11)$$

then ξ may be eliminated from equations (7)–(10), yielding a closed system of differential and algebraic equations in x ,

v and u :

$$\dot{x}_i = E_{ai}(x)u_a, \quad (12)$$

$$F_{am}(x) \frac{d}{dt} \left(\frac{\partial L}{\partial v_m}(x, v) \right) - E_{ai}(x) \frac{\partial L}{\partial x_i}(x, v) = \frac{\partial L}{\partial v_m}(x, v) \left\{ F_{dm}(x) c_{bad}(x) + E_{ai}(x) \frac{\partial F_{bm}}{\partial x_i}(x) - E_{bi}(x) \frac{\partial F_{am}}{\partial x_i}(x) \right\} u_b, \quad (13)$$

$$v_m = F_{am}(x)u_a. \quad (14)$$

Finally we introduce a Legendre transformation. Let the variables p_m ($m = 1, \dots, n_v$) be defined by

$$p_m = \frac{\partial L}{\partial v_m}(x, v). \quad (15)$$

If equation (15) is solvable for $v = v(x, p)$, we may eliminate v from equations (12)–(14), leading to a closed system of differential and algebraic equations in x, p and u :

$$\dot{x}_i = E_{ai}(x)u_a, \quad (16)$$

$$F_{am}(x) \dot{p}_m + E_{ai}(x) \frac{\partial H}{\partial x_i}(x, p) = p_m \left\{ F_{dm}(x) c_{bad}(x) + E_{ai}(x) \frac{\partial F_{bm}}{\partial x_i}(x) - E_{bi}(x) \frac{\partial F_{am}}{\partial x_i}(x) \right\} u_b, \quad (17)$$

$$\frac{\partial H}{\partial p_m}(x, p) = F_{am}(x)u_a. \quad (18)$$

Here we have introduced the Hamiltonian $H(x, p) = p_m v_m(x, p) - L(x, v(x, p))$.

III. LAGRANGIAN AND HAMILTONIAN INTERPRETATION

In order to characterize the Lagrangian (respectively Hamiltonian) nature of equations (12)–(14) (respectively equations (16)–(18)), we would like to give a coordinate-free interpretation to these equations. We are aware of at least two important cases where this is possible: in the case of \mathcal{V} being a trivial bundle and in the case that there exists a bundle homomorphism $\sigma : TX \rightarrow \mathcal{V}$ such that $F_a = \sigma \circ E_a$ ($a = 1, \dots, n_u$). These two cases are not mutually exclusive. Rigid body dynamics modeled by Euler-Poincaré equations, for example, fall into both classes. Classical Euler-Lagrange equations on manifolds, on the other hand, falls outside the scope of the first case, but fits the framework of the second case. We focus exclusively on the second case in the remainder of this paper.

We assume the existence of a smooth bundle homomorphism $\sigma : TX \rightarrow \mathcal{V}$ such that $F_a = \sigma \circ E_a$ ($a = 1, \dots, n_u$) or, equivalently (see equations (1) and (2)), such that

$$\nu = \sigma(x)\dot{x}. \quad (19)$$

In coordinates, the assumed relation between E_a and F_a may be expressed as

$$F_{ai}(x) = \sigma_{ij}(x) E_{aj}(x). \quad (20)$$

With equation (20), equations (12)–(14) reduce to

$$\dot{x}_i = E_{a_i}(x)u_a, \quad (21)$$

$$E_{a_i}(x)\sigma_{m_i}(x)\frac{d}{dt}\left(\frac{\partial L}{\partial v_m}(x, v)\right) - E_{a_i}(x)\frac{\partial L}{\partial x_i}(x, v) \quad (22)$$

$$= E_{a_i}(x)\frac{\partial L}{\partial v_m}(x, v)\left\{\frac{\partial \sigma_{m_j}}{\partial x_i}(x) - \frac{\partial \sigma_{m_i}}{\partial x_j}(x)\right\}\dot{x}_j, \quad (23)$$

$$v_m = \sigma_{m_i}(x)\dot{x}_i$$

and equations (16)–(18) reduce to

$$\dot{x}_i = E_{a_i}(x)u_a, \quad (24)$$

$$E_{a_i}(x)\sigma_{m_i}(x)\dot{p}_m + E_{a_i}(x)\frac{\partial H}{\partial x_i}(x, p) \quad (25)$$

$$= E_{a_i}(x)p_m\left\{\frac{\partial \sigma_{m_j}}{\partial x_i}(x) - \frac{\partial \sigma_{m_i}}{\partial x_j}(x)\right\}\dot{x}_j,$$

$$\frac{\partial H}{\partial p_m}(x, p) = \sigma_{m_i}(x)\dot{x}_i. \quad (26)$$

A. Hamiltonian interpretation

First, we introduce a generalized Dirac structure \mathcal{D} on \mathcal{V}^* , the dual of the vector bundle \mathcal{V} , as follows. (See the Appendix for terminology and notation.) The bundle homomorphism σ gives rise to a dual bundle homomorphism σ^* from \mathcal{V}^* to T^*X . The pull-back of the natural symplectic structure on T^*X by means of σ^* defines a presymplectic structure ω on \mathcal{V}^* . We define \mathcal{D} to be the generalized Dirac structure on \mathcal{V}^* that is induced by the presymplectic structure ω on \mathcal{V}^* and the vector fields E_1, \dots, E_{n_u} on X , in the sense of Lemma 1 in the Appendix.

Next, we introduce a Hamiltonian function H on \mathcal{V}^* , as follows. The function L gives rise to a Legendre transformation $\mathbb{F}L$ from \mathcal{V} to \mathcal{V}^* , defined (in coordinates) as follows:

$$\mathbb{F}L : \mathcal{V} \rightarrow \mathcal{V}^* : (x, v) \mapsto \mathbb{F}L(x, v) = \left(x, \frac{\partial L}{\partial v}(x, v)\right). \quad (27)$$

This leads to the definition of a Hamiltonian function on \mathcal{V} , which we denote by \mathcal{E} :

$$\mathcal{E} : \mathcal{V} \rightarrow \mathbb{R} : \nu \mapsto \mathcal{E}(\nu) = \langle \mathbb{F}L(\nu), \nu \rangle - L(\nu). \quad (28)$$

Assuming that the Lagrangian L is *hyperregular*, that is, that $\mathbb{F}L$ is a smooth global diffeomorphism, we define H as the push-forward of \mathcal{E} by means of $\mathbb{F}L$.

The following theorem asserts the Hamiltonian nature of equations (24)–(26).

Theorem 1. *Assume that the Lagrangian L is hyperregular. Then the set of equations (24)–(26) is a coordinate expression of the Hamiltonian system on \mathcal{V}^* with respect to the generalized Dirac structure \mathcal{D} and with Hamiltonian H :*

$$(\dot{\pi}, dH(\pi)) \in \mathcal{D}_\pi, \quad \pi \in \mathcal{V}^*. \quad (29)$$

B. Lagrangian interpretation

In contrast with the previous subsection, we do not assume hyperregularity of the Lagrangian in the present subsection.

We introduce a generalized Dirac structure \mathcal{D}^L on \mathcal{V} as follows. The pull-back of the presymplectic structure ω on \mathcal{V}^* by means of $\mathbb{F}L$ defines a presymplectic structure ω^L on \mathcal{V} . We define \mathcal{D}^L to be the generalized Dirac structure on \mathcal{V}

that is induced by the presymplectic structure ω^L on \mathcal{V} and the vector fields E_1, \dots, E_{n_u} on X , in the sense of Lemma 1 in the Appendix.

The following theorem asserts the Lagrangian nature of equations (21)–(23).

Theorem 2. *The set of equations (21)–(23) is a coordinate expression of the dynamics*

$$(\dot{\nu}, d\mathcal{E}(\nu)) \in \mathcal{D}_\nu^L, \quad \nu \in \mathcal{V}, \quad (30)$$

$$\nu = \sigma(x)\dot{x}, \quad x \in X. \quad (31)$$

(Here x is the natural projection of ν onto X .) If the Lagrangian L is regular, that is, if $\mathbb{F}L$ is a smooth local diffeomorphism, then equation (31) is redundant and the implicit Lagrangian system (30)–(31) is actually an implicit Hamiltonian system on \mathcal{V} with respect to the generalized Dirac structure \mathcal{D}^L and with Hamiltonian \mathcal{E} .

We refer to equations (30)–(31) as an *implicit Lagrangian system*. This terminology is motivated by the fact that the coordinate-free description (30)–(31) resembles the coordinate-free interpretation of the classical Euler-Lagrange equation; see, e.g., [1, 3.5.17 Theorem].

C. Discussion of Theorems 1 and 2

The Lagrangian description is more generally applicable than the Hamiltonian description. The interpretation of equations (21)–(23) as an implicit Lagrangian system on \mathcal{V} does not assume regularity of the Lagrangian, whereas the derivation of equations (24)–(26) and hence its interpretation as an implicit Hamiltonian system on \mathcal{V}^* assumes hyperregularity of the Lagrangian. This is because we take the Lagrangian L (or more generally, the associated optimal control problem) as the starting point for our derivations.

It is clear (e.g. from equations (1), (21) or (24)) that x is constrained to move in a direction spanned by the vector fields E_1, \dots, E_{n_u} . The vector fields E_1, \dots, E_{n_u} may be thought of as providing an image representation of this constraint. This is in contrast with many studies in the literature (e.g. in the context of nonholonomic mechanical systems) where constraints are given a kernel representation. The difference between kernel and image representations of constraints is illustrated in Section IV. (See the last paragraph of that section.)

The manipulations that have brought us from the original maximum principle equations (3)–(6) to the final equations (21)–(23) and (24)–(26) are not standard in the optimal control literature. A more standard approach is to eliminate (if possible) the u variable, yielding a Hamiltonian system on T^*X . In the present paper we eliminate ξ (assuming that the vector fields E_1, \dots, E_{n_u} form an involutive distribution), leading to an implicit Lagrangian system on \mathcal{V} and (if the Legendre transformation is invertible) an implicit Hamiltonian system on \mathcal{V}^* .

The variable u in equations (21)–(23) and (24)–(26) characterizes the generalized Dirac structure, it is not a state variable; that is, its initial value need not be specified in initial value problems.

If the vector fields E_1, \dots, E_{n_u} span the complete tangent space $T_x X$ at each $x \in X$, then the generalized Dirac structure \mathcal{D} (and \mathcal{D}^L) is presymplectic. This case is complementary to the case discussed in [21]. That paper considers variational problems on a vector bundle (a *Lie algebroid*) \mathcal{V}

over X that is equipped, among others, with a bundle homomorphism $\rho: \mathcal{V} \rightarrow TX$, giving rise to a *Poisson structure* on \mathcal{V}^* .

IV. APPLICATION 1: INTERCONNECTED MECHANICAL SYSTEMS

When two mechanical systems switch from a non-interconnected configuration to an interconnected configuration (characterized by a holonomic constraint), the total number of degrees of freedom decreases. This may be taken into account by introducing new generalized coordinates for the interconnected configuration. In that approach there is no obvious relationship between the interconnected and the non-interconnected model.

Alternatively, if a Hamiltonian description of the individual mechanical systems is available, then the interconnected system may be described in terms of the original configuration and momentum variables as an implicit Hamiltonian system with respect to a generalized Dirac structure that takes into account the interconnection constraint.

This section illustrates that a complementary Lagrangian description of the interconnected configuration is possible in terms of the original configuration and velocity variables and that this description derives from a variational principle.

We illustrate the approach by means of an academic example. We consider two particles in a plane, whose unconstrained dynamics are determined by their joint Lagrangian. As a constraint, we fix the distance between both particles, e.g., by putting a rigid massless link between them. This situation bears some similarity with grasping maneuvers where two robot arms manipulate a common rigid light object.

We identify the plane with \mathbb{R}^2 and denote the natural coordinates on the configuration space $X = \mathbb{R}^2 \times \mathbb{R}^2$ by (x_1, x_2, x_3, x_4) . Here x_1 and x_2 (respectively x_3 and x_4) represent the position of the first particle (respectively of the second particle). We define $\mathcal{V} = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ and denote the natural coordinates on \mathcal{V} by $(x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4)$. Here v_1 and v_2 (respectively v_3 and v_4) represent the velocity of the first particle (respectively of the second particle). The bundle homomorphism σ is characterized by

$$\dot{x}_i = v_i \quad (i = 1, 2, 3, 4). \quad (32)$$

The unconstrained motion of the particles is determined by their joint Lagrangian $L(x, v)$. The constraint, which fixes the distance between both particles, may be given the following image representation:

$$\begin{aligned} \dot{x}_1 &= u_1, & \dot{x}_3 &= u_1 + (x_2 - x_4)u_3, \\ \dot{x}_2 &= u_2, & \dot{x}_4 &= u_2 - (x_1 - x_3)u_3. \end{aligned} \quad (33)$$

The dynamical behavior of the interconnected particles is governed by the following optimal control problem: among all piecewise continuous control inputs $u_1, u_2, u_3: [t_i, t_f] \rightarrow \mathbb{R}$ that steer x from x_i at t_i to x_f at t_f according to the dynamics (33), find that control that minimizes $\int_{t_i}^{t_f} L(x, v) dt$ where v is given by (see equations (32) and (33))

$$\begin{aligned} v_1 &= u_1, & v_3 &= u_1 + (x_2 - x_4)u_3, \\ v_2 &= u_2, & v_4 &= u_2 - (x_1 - x_3)u_3. \end{aligned} \quad (34)$$

This yields the following implicit Lagrangian description for the interconnected particles:

$$\begin{aligned} \dot{x}_3 - \dot{x}_1 &= (x_2 - x_4)u_3, \\ \dot{x}_4 - \dot{x}_2 &= -(x_1 - x_3)u_3, \end{aligned} \quad (35)$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_1} \right) - \frac{\partial L}{\partial x_1} + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_3} \right) - \frac{\partial L}{\partial x_3} &= 0, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_2} \right) - \frac{\partial L}{\partial x_2} + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_4} \right) - \frac{\partial L}{\partial x_4} &= 0, \\ (x_2 - x_4) \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_3} \right) - \frac{\partial L}{\partial x_3} \right) & \\ - (x_1 - x_3) \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_4} \right) - \frac{\partial L}{\partial x_4} \right) &= 0. \end{aligned} \quad (36)$$

If $\frac{\partial L}{\partial v_i} = p_i$ is solvable for $v = v(x, p)$, we may rewrite this as an implicit Hamiltonian system

$$\dot{x}_1 = \frac{\partial H}{\partial p_1}, \quad \dot{x}_3 = \frac{\partial H}{\partial p_3}, \quad (37)$$

$$\dot{x}_2 = \frac{\partial H}{\partial p_2}, \quad \dot{x}_4 = \frac{\partial H}{\partial p_4},$$

$$\begin{aligned} \dot{x}_3 - \dot{x}_1 &= (x_2 - x_4)u_3, \\ \dot{x}_4 - \dot{x}_2 &= -(x_1 - x_3)u_3, \end{aligned} \quad (38)$$

$$\dot{p}_1 + \dot{p}_3 = -\frac{\partial H}{\partial x_1} - \frac{\partial H}{\partial x_3},$$

$$\dot{p}_2 + \dot{p}_4 = -\frac{\partial H}{\partial x_2} - \frac{\partial H}{\partial x_4},$$

$$(x_2 - x_4) \left(\dot{p}_3 + \frac{\partial H}{\partial x_3} \right) - (x_1 - x_3) \left(\dot{p}_4 + \frac{\partial H}{\partial x_4} \right) = 0. \quad (39)$$

where $H(x, p) = p_i v_i(x, p) - L(x, v(x, p))$.

The constrained models (35)–(36) and (37)–(39) differ from their unconstrained counterparts $(d/dt)(\partial L/\partial \dot{x}_i) = \partial L/\partial x_i$ and $\dot{x}_i = \partial H/\partial p_i$, $\dot{p}_i = -\partial H/\partial x_i$ only by the underlying generalized Dirac structure, but not by the variables in terms of which the Lagrangian or Hamiltonian is expressed. In particular, the state space for the constrained and unconstrained configuration is the same.

A different treatment of the present example is given in [2]: first, a Hamiltonian input-output formulation is given for the forced motion of a particle (with force as input and velocity as output) and secondly, forces and velocities are eliminated by imposing the constraint which fixes the distance between both particles. This results in an implicit Hamiltonian description of the constrained particles.

The implicit Hamiltonian system of [2] differs from equations (37)–(39) of the present paper in that we use an image representation of the constraint, whereas [2] uses the kernel representation: $(x_1 - x_3)(\dot{x}_1 - \dot{x}_3) + (x_2 - x_4)(\dot{x}_2 - \dot{x}_4) = 0$. Notice that, when we impose a zero distance between both particles ($x_1 = x_3$ and $x_2 = x_4$), then the image representation yields $\dot{x}_1 = \dot{x}_3$ and $\dot{x}_2 = \dot{x}_4$ as expected, whereas the kernel representation yields the redundant equation $0 = 0$.

V. APPLICATION 2: MECHANICAL SYSTEMS WITH SINGULARITIES

Consider a horizontal bar with mass m , constrained to move in the vertical direction, interconnected with a *massless* pendulum of length l and moving in a uniform gravitational field characterized by the gravitational constant

g. The configuration of this system is characterized by the angle θ that the pendulum makes with the vertical axis (Figure 1).

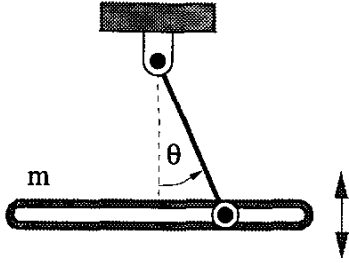


Fig. 1. Singular mechanical system

The Lagrangian expressed as a function of θ and $\dot{\theta}$ is given by

$$L(\theta, \dot{\theta}) = m \frac{(l \sin(\theta) \dot{\theta})^2}{2} + mgl \cos(\theta) \quad (40)$$

and the corresponding classical Euler-Lagrange equation is

$$ml^2 \sin^2(\theta) \ddot{\theta} + ml^2 \sin(\theta) \cos(\theta) \dot{\theta}^2 + mgl \sin(\theta) = 0. \quad (41)$$

The Legendre transformation

$$(\theta, \dot{\theta}) \mapsto (\theta, \xi) = (\theta, ml^2 \sin^2(\theta) \dot{\theta}) \quad (42)$$

is not invertible, which complicates a Hamiltonian description in terms of (θ, ξ) .

Using the approach outlined in the present paper, we rewrite the Lagrangian as

$$L(\theta, v) = m \frac{v^2}{2} + mgl \cos(\theta) \quad (43)$$

with

$$v = l \sin(\theta) \dot{\theta} \quad (44)$$

and we associate the following optimal control problem to the system. Among all piecewise continuous control inputs u that steer θ from θ_i at t_i to θ_f at t_f according to the dynamics

$$\dot{\theta} = u, \quad (45)$$

find that control that minimizes

$$\int_{t_i}^{t_f} L(\theta, v) dt \quad (46)$$

where

$$v = l \sin(\theta) u. \quad (47)$$

This leads to an implicit Lagrangian description

$$l \sin(\theta) \frac{d}{dt} \left(\frac{\partial L}{\partial v} \right) = \frac{\partial L}{\partial \theta} \quad (48)$$

$$v = l \sin(\theta) \dot{\theta}$$

and an implicit Hamiltonian description

$$\begin{pmatrix} \frac{\partial H}{\partial \theta} \\ \frac{\partial H}{\partial p} \end{pmatrix} = \begin{pmatrix} 0 & -l \sin(\theta) \\ l \sin(\theta) & 0 \end{pmatrix} \cdot \begin{pmatrix} \dot{\theta} \\ p \end{pmatrix} \quad (49)$$

where $p = \partial L / \partial v = mv$ and $H(\theta, p) = \frac{p^2}{2m} - mgl \cos(\theta)$.

Whereas the classical approach yields the Euler-Lagrange equation (41) which has no obvious Hamiltonian counterpart, the approach from the present paper yields a Lagrangian model (48) and a Hamiltonian model (49), both of which have a remarkably simple structure.

The mechanical example of this section gives rise to singular differential equations. All the obtained models (41), (48) and (49) are singular when $\sin(\theta) = 0$. This is due to the geometry of the system, but is not related to the precise form of the force acting on the horizontal bar. The purpose of our manipulations was *not* to remove this singularity. Rather, we were interested in obtaining models whose structure truly reflects the geometric nature of the singularity. This is accomplished in equations (48) and (49), where the singularity is captured by the underlying Dirac structure (which is actually presymplectic in this case).

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APPENDIX

GENERALIZED DIRAC STRUCTURES AND IMPLICIT HAMILTONIAN SYSTEMS

Definition 1 (Generalized Dirac structure [7], [9]). A generalized Dirac structure on a smooth m -dimensional manifold M is a collection $\{\mathcal{D}_m\}_{m \in M}$ of m -dimensional linear subspaces $\mathcal{D}_m \subset T_m M \times T_m^* M$ satisfying

$$\langle W, \beta \rangle = 0 \quad \forall (W, \beta) \in \mathcal{D}_m. \quad (50)$$

Definition 2 (Implicit Hamiltonian system [20]). An implicit Hamiltonian system is a triple (M, \mathcal{D}, H) with (i) M a smooth finite-dimensional manifold; (ii) \mathcal{D} a generalized Dirac structure on M ; (iii) H a smooth real-valued function on M (the Hamiltonian). The dynamics of this implicit Hamiltonian system are governed by

$$(\dot{m}, dH(m)) \in \mathcal{D}_m, \quad m \in M. \quad (51)$$

Equation (50) implies that H is conserved along solutions of (51).

Definition 3 (Presymplectic structure). A smooth presymplectic structure \mathcal{D} on a smooth finite-dimensional manifold M is a generalized Dirac structure \mathcal{D} on M that satisfies: (i) \mathcal{D}_m is the graph of a linear map $\mathcal{D}_m^b : T_m M \rightarrow T_m^* M$ for each $m \in M$; (ii) the linear maps $\mathcal{D}_m^b : T_m M \rightarrow T_m^* M$ constitute a smooth bundle homomorphism $\mathcal{D}^b : TM \rightarrow T^* M$; (iii) \mathcal{D}^b (viewed as a 2-form on M) is closed; that is $d\mathcal{D}^b = 0$.

In this case, equation (51) that governs the dynamics may be rewritten as

$$dH(m) = \mathcal{D}_m^b(\dot{m}), \quad m \in M. \quad (52)$$

Definition 4 (Symplectic structure). A smooth symplectic structure \mathcal{D} on a smooth finite-dimensional manifold M is a presymplectic structure \mathcal{D} on M for which \mathcal{D}^b is non-degenerate; that is, for which \mathcal{D}^b is a bundle isomorphism.

In this case, equation (51) that governs the dynamics may be rewritten as

$$\dot{m} = \mathcal{D}_m^\#(dH(m)), \quad m \in M \quad (53)$$

where $\mathcal{D}_m^\#$ is the inverse of \mathcal{D}_m^b .

Lemma 1. Let X be a smooth finite-dimensional manifold and M a smooth vector bundle over X with finite-dimensional fibers. Consider a presymplectic structure ω on M and smooth vector fields E_1, \dots, E_{n_u} ($n_u \in \mathbb{N}$) on X . Associate with every $m \in M$ a subspace \mathcal{D}_m of $T_m M \times T_m^* M$ defined by

$$\mathcal{D}_m = \{(W, \beta) \in T_m M \times T_m^* M : W \in \Sigma_m, \beta - \omega^b(W) \in \text{ann}(\Sigma_m)\}. \quad (54)$$

(Here Σ_m is the linear subspace of $T_m M$ obtained as the inverse image of $\text{span}\{E_1(\Pi(m)), \dots, E_{n_u}(\Pi(m))\}$ under the linear map $T_m \Pi : T_m M \rightarrow T_{\Pi(m)} X$ and $\text{ann}(\Sigma_m)$ is its annihilator, where Π denotes the natural projection from M onto X .) This defines a Dirac structure $\mathcal{D} = \{\mathcal{D}_m\}_{m \in M}$ on M .

Proof: It is easy to see that the dimension of \mathcal{D}_m equals the dimension of M . It is also easy to see that $\langle W, \beta \rangle = 0$ for all $(W, \beta) \in \mathcal{D}_m$. \square