

# Port Hamiltonian Formulation of Infinite Dimensional Systems

## II. Boundary Control by Interconnection

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**Abstract**—In this paper, some new results concerning the boundary control of distributed parameter systems in port Hamiltonian form are presented. The classical finite dimensional port Hamiltonian formulation of a dynamical system has been generalized to the distributed parameter and multi-variable case by extending the notion of finite dimensional Dirac structure in order to deal with an infinite dimensional space of power variables. Consequently, it seems natural that also finite dimensional control methodologies developed for finite dimensional port Hamiltonian systems can be extended in order to cope with infinite dimensional systems. In this paper, the control by interconnection and energy shaping methodology is applied to the stabilization problem of a distributed parameter system by means of a finite dimensional controller. The key point is the generalization of the definition of Casimir function to the hybrid case, i.e. when the dynamical system to be considered results from the power conserving interconnection of an infinite and a finite dimensional part. A simple application concerning the stabilization of the one-dimensional heat equation is presented.

### I. INTRODUCTION

The port Hamiltonian representation of a finite dimensional system [1], [2] has been recently generalized to the infinite dimensional case, [3], [4] by extending the notion of Dirac structure in order to cope with an infinite dimensional space of power variables. From the modeling point of view, the port Hamiltonian formulation of an infinite dimensional system with spatial domain  $\mathcal{Z}$  provides a deep insight of the structure of the system, whose dynamics can be interpreted as the result of the interaction among (at least) two energy domains within  $\mathcal{Z}$  and/or between the system and its environment through  $\partial\mathcal{Z}$ .

From the control perspective, one of the main advantages in adopting the port Hamiltonian approach in both the finite either the infinite dimensional case is that the energy (Hamiltonian) function, which is usually a good Lyapunov function, explicitly appears in the dynamics of the system. Given a desired state of equilibrium, if the Hamiltonian of the system assumes its minimum at this configuration, then asymptotic stability can be assured by introducing a dissipative effect with the controller. In this way, energy decreases until the minimum of energy or, equivalently, the

desired equilibrium configuration is reached. This control methodology is called control by *damping injection*, [5], [2].

On the other hand, if the Hamiltonian function of the system does not assume its minimum in the desired equilibrium state, it is necessary to *shape* the open-loop energy function and to introduce a new minimum in the desired configuration. The idea is to interconnect a controller to the plant and to choose the Hamiltonian of the regulator in order to properly shape the total (closed-loop) energy function. It is important to note that, in general, there is no *a priori* relation between the state of the plant and the state of the controller, so it is not immediate how the controller Hamiltonian function can be chosen in order to correctly shape the total energy. This problem can be solved by choosing the structure of the controller, i.e. its interconnection, damping and input/output matrices, in such a way that the state of the closed-loop system is constrained on certain subspace *independently* of the energy function of both the plant either the controller. Equivalently, this can be done by introducing a set of *Casimir functions* in the system, [6], [7]. Under some technical hypothesis, then, it is possible to introduce an *intrinsic* non-linear state feedback law that will be used in order to choose the energy function of the controller so that the closed-loop Hamiltonian can be properly shaped. Note that, under these hypothesis, this energy function depends on the state variables of the plant. This control methodology is called *invariant function method* or, within the framework of port Hamiltonian systems, control by *interconnection and energy shaping* and it is deeply discussed in [6], [7] and also in [8], [9] for the stabilization of non-linear port Hamiltonian systems.

In this paper, the control by interconnection and energy shaping is extended and applied to the regulation problem of an infinite dimensional system by means of a finite dimensional controller that can act on the system by exchanging power through the boundary. Some preliminary results in this direction have already been presented in [10], [11] where the infinite dimensional system is given by a set of transmission lines, while an application to stabilization of the Timoshenko beam has been discussed in [12], [13]. The main result concerns the necessary and sufficient conditions for a real-valued function defined over the closed-loop state space to be a structural invariant (Casimir function) for the controlled system which is an *hybrid* system since it results from the power conserving interconnection of an infinite and of a finite dimensional system. Once these

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conditions are deduced, by choosing a proper family of Casimir functions, the control by interconnection and energy shaping methodology can be applied as in the finite dimensional case. In this way, the open-loop energy function can be shaped by introducing a new minimum at the desired equilibrium configuration.

This paper is organized as follows. In Sect. II, a short introduction about the control by interconnection and energy shaping for finite dimensional port Hamiltonian systems is given and then the boundary control by interconnection for infinite dimensional systems is discussed in Sect. III. Necessary and sufficient conditions for the existence of Casimir functions in the closed loop system are deduced and their applications in the energy shaping procedure is described. Finally, a simple example concerning the boundary stabilization of the heat equation is discussed in Sect. IV, while conclusions are presented in Sect. V.

## II. CONTROL BY INTERCONNECTION IN FINITE DIMENSIONS

Denote by  $\mathcal{X}$  an  $n$ -dimensional space of state (energy) variables and by  $H : \mathcal{X} \rightarrow \mathbb{R}$  a scalar energy function (Hamiltonian) bounded from below. Denote by  $\mathcal{U}$  an  $m$ -dimensional (linear) space of input variables and by its dual  $\mathcal{Y} \equiv \mathcal{U}^*$  the space of output variables. Then,

$$\begin{cases} \dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x} + G(x)u \\ y &= G^T(x) \frac{\partial H}{\partial x} \end{cases} \quad (1)$$

with  $J(x) = J^T(x)$ ,  $R(x) = R^T(x) \geq 0$  and  $G(x)$  matrices of proper dimensions, is a port Hamiltonian system with dissipation. The  $n \times n$  matrices  $J$  and  $R$  are called *interconnection* and *damping* matrix respectively.

Suppose (1) has to be asymptotically stabilized around the configuration  $x^* \in \mathcal{X}$  by means of the following dynamical controller in port Hamiltonian form:

$$\begin{cases} \dot{x}_c &= [J_c(x_c) - R_c(x_c)] \frac{\partial H_c}{\partial x_c} + G_c(x_c)u_c \\ y_c &= G_c^T(x_c) \frac{\partial H_c}{\partial x_c} \end{cases} \quad (2)$$

Denote by  $\mathcal{X}_c$  the controller state space, with  $\dim \mathcal{X}_c = n_c$ , and by  $H_c : \mathcal{X}_c \rightarrow \mathbb{R}$  the Hamiltonian function, bounded from below. Moreover, suppose that  $J_c(x_c) = -J_c^T(x_c)$  and  $R_c(x_c) = R_c^T(x_c)$  and that  $\dim \mathcal{U}_c = \dim \mathcal{Y}_c = m$ .

If systems (1) and (2) are interconnected in power conserving way, that is if

$$\begin{cases} u &= -y_c \\ y &= u_c \end{cases} \quad (3)$$

the resulting dynamics is given by the following autonomous port Hamiltonian systems, with state space  $\mathcal{X} \times \mathcal{X}_c$  and Hamiltonian  $H + H_c$ :

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} J(x) - R(x) & -G(x)G_c^T(x_c) \\ G_c(x_c)G^T(x) & J_c(x_c) - R_c(x_c) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H_c}{\partial x_c} \end{bmatrix} \quad (4)$$

Given a generic port Hamiltonian system, it is possible to give the following fundamental definition of *structural invariant* or, equivalently, of Casimir function, [6], [7], [2].

*Definition 2.1 (Casimir function):* Consider the port Hamiltonian system (1) with state space  $\mathcal{X}$  and Hamiltonian function  $H : \mathcal{X} \rightarrow \mathbb{R}$ . A function  $\mathcal{C} : \mathcal{X} \rightarrow \mathbb{R}$  is a Casimir function for (1) if and only if  $\dot{\mathcal{C}} = 0$  for every possible choice of Hamiltonian  $H$ .

From Def. 2.1, a scalar function  $\mathcal{C} : \mathcal{X} \times \mathcal{X}_c \rightarrow \mathbb{R}$  is a Casimir function for (4) if and only if the following relations are satisfied:

$$\frac{\partial^T \mathcal{C}}{\partial x} (J - R) + \frac{\partial^T \mathcal{C}}{\partial x_c} G_c G^T = 0 \quad (5a)$$

$$\frac{\partial^T \mathcal{C}}{\partial x_c} (J_c - R_c) - \frac{\partial^T \mathcal{C}}{\partial x} G G_c^T = 0 \quad (5b)$$

These conditions are direct consequence of the interconnection law (3).

The existence of Casimir functions for the closed-loop system (4) plays an important role in the control by interconnection and energy shaping methodology. If  $x^* \in \mathcal{X}$  is the desired equilibrium configuration for (1), asymptotic stability in  $x^*$  can be achieved by properly choosing the Hamiltonian function of (2) in order to shape the closed-loop energy  $H + H_c$  so that a (possibly) global minimum in the desired equilibrium configuration can be introduced. It is important to note that there is no relation between the state of the controller and the state of the system to be controlled. Then, it is not clear how the controller energy, which is freely assignable, has to be chosen in order to solve the regulation problem.

A possible solution can be to constrain the state of the closed-loop system (4) on a certain subspace of  $\mathcal{X} \times \mathcal{X}_c$ , for example given by:

$$\Omega_c := \{(x, x_c) \in \mathcal{X} \times \mathcal{X}_c \mid x_c = S(x) + c\}$$

where  $c \in \mathbb{R}^{n_c}$  and  $S : \mathcal{X} \rightarrow \mathcal{X}_c$  is a function to be computed. In other words, we are looking for a set of Casimir functions  $\mathcal{C}_i : \mathcal{X} \times \mathcal{X}_c \rightarrow \mathbb{R}$ ,  $i = 1, \dots, n_c$  for the closed-loop system (4) such that

$$\mathcal{C}_i(x, x_c) := S_i(x) - x_{c,i} \quad (6)$$

where  $[S_1(x), \dots, S_{n_c}(x)]^T = S(x)$ . Due to the nature of a Casimir function, it is possible to introduce an *intrinsic* non-linear state feedback law that will be used in order to choose the energy function of the controller so that the closed-loop Hamiltonian can be properly shaped. Note that, under these hypothesis, this energy function depends on the state variables of system (1). This control methodology is called *invariant function method*, [6], [7].

From (5), the set of functions (6) are Casimir functions for (4) if and only if

$$-\frac{\partial^T S}{\partial x} G G_c^T = J_c - R_c \quad \frac{\partial^T S}{\partial x} (J - R) = G_c G^T$$

Then, the following proposition can be proved, [9], [2].

*Proposition 2.1:* The functions  $\mathcal{C}_i$ ,  $i = 1, \dots, n_c$ , defined in (6) are Casimir functions for the system (4) if and only if the following conditions are satisfied:

$$\frac{\partial^T S}{\partial x} J(x) \frac{\partial S}{\partial x} = J_c(x_c) \quad (7a)$$

$$R(x) \frac{\partial S}{\partial x} = 0 \quad (7b)$$

$$R_c(x_c) = 0 \quad (7c)$$

$$\frac{\partial^T S}{\partial x} J(x) = G_c(x_c) G^T(x) \quad (7d)$$

Suppose that (7) are satisfied. Then, from (6), the state variables of the controller are *robustly* related to the state variable of the system to be stabilized since

$$x_{c,i} = S_i(x) + c_i, \quad i = 1, \dots, n_c \quad (8)$$

with  $c_i \in \mathbb{R}$  depending on the initial conditions. Moreover, the closed-loop dynamics (4) evolves on the foliation induced by the level sets

$$\mathcal{L}_{\mathcal{C}_i}^{c_i} = \{(x, x_c) \in \mathcal{X} \times \mathcal{X}_c \mid x_{c,i} = S_i(x) + c_i\} \quad (9)$$

with  $i = 1, \dots, n_c$ , which can be expressed as a function of the  $x$  coordinate. If conditions (7b) and (7d) are taken into account, the reduced dynamics of (4) on these level sets is given by

$$\begin{aligned} \dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x} - G(x) G_c^T(x_c) \frac{\partial H_c}{\partial x_c} \\ &= [J(x) - R(x)] \left( \frac{\partial H}{\partial x} + \frac{\partial S}{\partial x} \frac{\partial H_c}{\partial x_c} \right) \end{aligned} \quad (10)$$

From (8), we have that  $H_c(x_c) \equiv H_c(S(x) + c)$ : the controller energy function is finally dependent from  $x_b$  through the non-linear feedback action  $S(\cdot)$ . If

$$H_d(x) := H(x) + H_c(S(x) + c) \quad (11)$$

then (10) can be written as

$$\dot{x} = (J - R) \left( \frac{\partial H}{\partial x} + \frac{\partial S}{\partial x} \frac{\partial H_c}{\partial x_c} \right) = (J - R) \frac{\partial H_d}{\partial x} \quad (12)$$

In conclusion, the following proposition has been proved, [9], [2].

*Proposition 2.2:* Consider the closed-loop port Hamiltonian system (4) and suppose that the vector function  $S(x) = [S_1(x), \dots, S_{n_c}(x)]^T$  satisfies conditions (7). Then, the reduced dynamics on the level sets (9) is given by (12), where the closed-loop energy function  $H_d$  is given by (11).

By properly choosing the controller energy function  $H_c$ , it is possible to shape the closed-loop energy function  $H_d$  defined in (11) so that a new minimum in  $x^*$  is introduced. Then, the desired configuration can be reached with the dynamics given by (12).

### III. BOUNDARY CONTROL BY INTERCONNECTION OF MDPH SYSTEMS

#### A. Introduction

In this section, the control by interconnection and energy shaping, discussed in Sect. II for the finite dimensional case, is generalized to distributed parameter systems in port Hamiltonian form. In particular, it is shown how it is possible to shape the open loop energy function of a distributed parameter system by interconnecting a finite dimensional controller to its boundary. The structure of the controller has to be chosen so that a proper set of structural invariants (Casimir functions) are introduced in the closed loop system. In this way, the energy *variables* of the distributed parameter system can be robustly related to the state variables of the controller, thus introducing an implicit state feedback law. Then, same procedure presented for the finite dimensional case can be applied.

#### B. Existence of Casimir functions

Consider the following multi-variable distributed port Hamiltonian system with spatial domain  $\mathcal{Z} \subset \mathbb{R}^d$  (closed and compact), [3]:

$$\begin{cases} \frac{\partial x}{\partial t} = (J - R) \delta_x \mathcal{H} \\ w = B_{\mathcal{Z}}(\delta_x \mathcal{H}) \end{cases} \quad (13)$$

where  $x \in \mathcal{X}$  is the configuration variable,  $w \in \mathcal{W}$  are the boundary terms defined by the boundary operator  $B_{\mathcal{Z}}$ ,  $\mathcal{H} : \mathcal{X} \rightarrow \mathbb{R}$  is the Hamiltonian function,  $J$  is a skew adjoint differential operator and  $R$  is a non-negative self-adjoint differential operator taking into account the dissipative effects. Both  $\mathcal{X}$  either  $\mathcal{W}$  are spaces of vector value smooth functions of proper dimension.

It is possible to prove that the following energy balance relation holds, [3]:

$$\frac{d\mathcal{H}}{dt} \leq \frac{1}{2} \int_{\partial \mathcal{Z}} B_{\{J, R\}}(w, w) \cdot dA \quad (14)$$

where the integral over  $\partial \mathcal{Z}$  represent the power exchanged with the environment through the boundary and the  $B_{\{J, R\}}$  is a constant operator depending on the differential operators  $J$  and  $R$ . See [3] for more details.

Suppose that (13) has to be stabilized in the configuration  $x^* \in \mathcal{X}$  by means of the finite dimensional controller (2) that has to be interconnected to the system (13) in power conserving way. Then, relation (3) has to be generalized in order to deal with a situation in which the *power port* of the system to be stabilized is not a finite dimensional vector space. A possible solution can be the following. Denote by  $\Psi_u(z)$  and  $\Psi_y(z)$  a couple of matrices depending eventually on  $z \in \partial \mathcal{Z}$  and suppose that it is possible to write the boundary terms in (13) as follows:

$$w = \Psi_y u_c - \Psi_u y_c \quad (15)$$

The interconnection law expressed in (15) is power conserving if and only if

$$y_c^T u_c + \frac{1}{2} \int_{\partial Z} B_{\{J,R\}}(w, w) \cdot dA = 0$$

where, from (15), we have that

$$\begin{aligned} & \int_{\partial Z} B_{\{J,R\}}(w, w) \cdot dA = \\ & = \sum_{i,j=1}^m \left[ \int_{\partial Z} B_{\{J,R\}}(\Psi_{y,i}, \Psi_{y,j}) \cdot dA \right] u_{c,i} u_{c,j} \\ & + \sum_{i,j=1}^m \left[ \int_{\partial Z} B_{\{J,R\}}(\Psi_{u,i}, \Psi_{u,j}) \cdot dA \right] y_{c,i} y_{c,j} \\ & - 2 \sum_{i,j=1}^m \left[ \int_{\partial Z} B_{\{J,R\}}(\Psi_{u,i}, \Psi_{y,j}) \cdot dA \right] u_{c,i} y_{c,j} \end{aligned}$$

and, then, relation (15) can be satisfied if and only if

$$\int_{\partial Z} B_{\{J,R\}}(\Psi_{u,i}, \Psi_{u,j}) \cdot dA = 0 \quad (16a)$$

$$\int_{\partial Z} B_{\{J,R\}}(\Psi_{y,i}, \Psi_{y,j}) \cdot dA = 0 \quad (16b)$$

$$\int_{\partial Z} B_{\{J,R\}}(\Psi_{u,i}, \Psi_{y,j}) \cdot dA = \delta_{ij} \quad (16c)$$

for every  $i, j = 1, \dots, m$  and where  $\delta$  is the Kronecker symbol. Note that, given  $w \in \mathcal{W}$

$$u_{c,i} = \int_{\partial Z} B_{\{J,R\}}(\Psi_{u,i}, w) \cdot dA \quad (17a)$$

$$y_{c,i} = - \int_{\partial Z} B_{\{J,R\}}(\Psi_{y,i}, w) \cdot dA \quad (17b)$$

or equivalently that

$$u_c = \mathcal{B}_{\{J,R\}}^u(w) \quad y_c = -\mathcal{B}_{\{J,R\}}^y(w) \quad (18)$$

where  $\mathcal{B}_{\{J,R\}}^u : \mathcal{W} \rightarrow \mathcal{U}_c$  and  $\mathcal{B}_{\{J,R\}}^y : \mathcal{W} \rightarrow \mathcal{Y}_c$  are two linear operator whose definition is based on (17).

Consider a function  $\mathcal{C} : \mathcal{X} \times \mathcal{X}_c \rightarrow \mathbb{R}$  defined over the state space of the closed loop system resulting from the power conserving interconnection (15) of (13) and (2). From Def. 2.1, we can say that  $\mathcal{C}$  is a Casimir function if and only if

$$\begin{aligned} \frac{d\mathcal{C}}{dt} &= \frac{\partial^T \mathcal{C}}{\partial x_c} (J_c - R_c) \frac{\partial H_c}{\partial x_c} + \frac{\partial^T \mathcal{C}}{\partial x_c} G_c \mathcal{B}_{\{J,R\}}^u(w) \\ &+ \int_{\mathcal{Z}} (\delta_x \mathcal{C})^T (J - R) \delta_x \mathcal{H} dV = 0 \end{aligned}$$

for every Hamiltonian functions  $\mathcal{H}$  and  $H_c$ , where  $u_c$  is expressed as a function of the boundary terms as in (17). Since  $J$  and  $R$  are a skew adjoint and a self adjoint differential operator respectively, we have that (see [3], [14]):

$$\begin{aligned} (\delta_x \mathcal{C})^T (J - R) \delta_x \mathcal{H} &= - (\delta_x \mathcal{H})^T (J + R) \delta_x \mathcal{C} \\ &+ \text{div } B_{\{J,R\}}(B_Z(\delta_x \mathcal{C}), w) \end{aligned}$$

and then

$$\begin{aligned} \frac{d\mathcal{C}}{dt} &= - \frac{\partial^T H_c}{\partial x_c} (J_c + R_c) \frac{\partial \mathcal{C}}{\partial x_c} \\ &- \int_{\mathcal{Z}} (\delta_x \mathcal{H})^T (J + R) \delta_x \mathcal{C} dV \\ &+ \int_{\partial Z} B_{\{J,R\}} \left( w, B_Z(\delta_x \mathcal{C}) + \Psi_u G_c^T \frac{\partial \mathcal{C}}{\partial x_c} \right) \cdot dA \end{aligned}$$

that has to be 0 for every Hamiltonian function of the closed loop system and for every  $w$  given by (15). This means that (15) has to be satisfied for some  $u_c \in \mathcal{U}_c$  and  $y_c \in \mathcal{Y}_c$ , which are related to the boundary variables  $w$  by (18). Consequently, it is possible to verify that the last integral is equal to

$$\begin{aligned} & \left[ \mathcal{B}_{\{J,R\}}^u(B_Z(\delta_x \mathcal{H})) \right]^T \times \\ & \times \left( \mathcal{B}_{\{J,R\}}^y(B_Z(\delta_x \mathcal{C})) + G_c^T \frac{\partial \mathcal{C}}{\partial x_c} \right) \\ & - \frac{\partial^T H_c}{\partial x_c} G_c \mathcal{B}_{\{J,R\}}^u(B_Z(\delta_x \mathcal{C})) \end{aligned}$$

and then

$$\begin{aligned} \frac{d\mathcal{C}}{dt} &= - \frac{\partial^T H_c}{\partial x_c} \left[ (J_c + R_c) \frac{\partial \mathcal{C}}{\partial x_c} + G_c \mathcal{B}_{\{J,R\}}^u(B_Z(\delta_x \mathcal{C})) \right] \\ &- \int_{\mathcal{Z}} (\delta_x \mathcal{H})^T (J + R) \delta_x \mathcal{C} dV \\ &+ \left[ \mathcal{B}_{\{J,R\}}^u(B_Z(\delta_x \mathcal{H})) \right]^T \times \\ & \times \left( \mathcal{B}_{\{J,R\}}^y(B_Z(\delta_x \mathcal{C})) + G_c^T \frac{\partial \mathcal{C}}{\partial x_c} \right) \end{aligned}$$

that has to be 0 for every Hamiltonian function of the closed loop system. This is true if and only if

$$(J_c + R_c) \frac{\partial \mathcal{C}}{\partial x_c} + G_c \mathcal{B}_{\{J,R\}}^u(B_Z(\delta_x \mathcal{C})) = 0 \quad (19a)$$

$$(J + R) \delta_x \mathcal{C} = 0 \quad (19b)$$

$$\mathcal{B}_{\{J,R\}}^y(B_Z(\delta_x \mathcal{C})) + G_c^T \frac{\partial \mathcal{C}}{\partial x_c} = 0 \quad (19c)$$

In conclusion, the following proposition has been proved.

*Proposition 3.1:* Consider the closed loop system resulting from the power conserving interconnection (15) of the infinite dimensional system (13) with the finite dimensional controller (2). Denote by  $\mathcal{X}$  and  $\mathcal{X}_c$  the state space of the distributed parameter system and of the controller respectively. Then, a real value function  $\mathcal{C} : \mathcal{X} \times \mathcal{X}_c \rightarrow \mathbb{R}$  is a Casimir function for the closed loop system with respect to the interconnection law (15) if and only the set of conditions (19) are satisfied.

*Note 3.1:* The set of necessary and sufficient conditions (19) concerning the existence of structural invariants in the closed loop system are the generalization of the analogous conditions (5) in the finite dimensional case. In the hybrid case, the structural invariants have to satisfy the PDEs (19a) and (19c) in the controller/plant variables and the PDE (19b) in the spatial variable of the distributed parameter system.

This PDE provides the variational derivative of the candidate Casimir function with respect to the configuration variable. Note that the boundary conditions for (19b) have to be chosen in such a way that (19a) and (19c) are satisfied. The Casimir functions are a consequence of the interconnection law (15).

### C. Energy shaping via structural invariants

As discussed in finite dimensions, the existence of a particular class of Casimir functions in the controlled system can be of great interest in the energy shaping procedure. Also in the distributed parameter case, a possible solution can be to choose the structure of the controller (2) in order to introduce a set of  $\bar{n} \leq n_c$  structural invariants in the closed loop system in the form

$$\mathcal{C}_i(x, x_c) = \mathcal{S}_i(x) - x_{c,i} \quad (20)$$

where now  $\mathcal{S}_i(x) = \int_{\mathcal{Z}} \mathcal{S}_i(z, x) dV$ , with  $i = 1, \dots, \bar{n}$ . These functions are Casimir function for the closed loop system if and only if the set of conditions (19) are satisfied. In particular, denote by  $\bar{J}_c$ ,  $\bar{R}_c$  and  $\bar{G}_c$  the sub-matrices of the interconnection, damping and input matrices of (2) corresponding to the first  $\bar{n}$  state variables and define  $\mathcal{S} : \mathcal{X} \times \mathcal{X}_c \rightarrow \mathbb{R}^{\bar{n}}$  as  $\mathcal{S} = [\mathcal{S}_1 \cdots \mathcal{S}_{\bar{n}}]^T$ . Then, from (19), we obtain the following set of conditions:

$$G_c [\mathcal{B}^u(\delta_x \mathcal{S}_1) \cdots \mathcal{B}^u(\delta_x \mathcal{S}_{\bar{n}})] = \bar{J}_c + \bar{R}_c \quad (21a)$$

$$(J + R) \delta_x \mathcal{S}_i = 0 \quad (21b)$$

$$[\mathcal{B}^y(\delta_x \mathcal{S}_1) \cdots \mathcal{B}^y(\delta_x \mathcal{S}_{\bar{n}})] = \bar{G}_c^T \quad (21c)$$

with  $i = 1, \dots, \bar{n}$  and where, in order to keep a lighter notation,  $\mathcal{B}^u(\cdot)$  and  $\mathcal{B}^y(\cdot)$  stand respectively for  $\mathcal{B}_{\{J, R\}}^u(B_{\mathcal{Z}}(\cdot))$  and  $\mathcal{B}_{\{J, R\}}^y(B_{\mathcal{Z}}(\cdot))$ . Note that, from (21a) and (21c) only  $(\bar{J}_c + \bar{R}_c)$  is determined by the set of functionals  $\mathcal{S}_i$ , while (21c) gives the expression of the input sub-matrix  $\bar{G}_c$ . Clearly,  $\bar{J}_c$ ,  $\bar{R}_c$  and  $\bar{G}_c$  depend on  $\mathcal{S}_i$ , which have to be solution of the PDE (21b) whose boundary conditions have to be chosen in such a way that (21a) and (21c) are satisfied. If the set of conditions (21) can be satisfied, then the closed loop Hamiltonian function becomes  $\mathcal{H}_{cl}(x, x_c) = \mathcal{H}(x) + H_c(x_{c,1}, \dots, x_{c,n_c}) = \mathcal{H}(x) + H_c(\mathcal{S}_1(x), \dots, \mathcal{S}_{\bar{n}}(x), \dots, x_{c,n_c})$ , thus depending explicitly on the configuration variable of the distributed parameter system.

If  $\bar{n} = n_c$ , then (20) are Casimir functions of the closed loop system if and only if conditions (21) are satisfied and the closed loop Hamiltonian becomes  $\mathcal{H}_{cl}(x, x_c) = \mathcal{H}(x) + H_c(\mathcal{S}_1(x), \dots, \mathcal{S}_{n_c}(x))$ , i.e. only a function of the configuration variable of the distributed parameter system. By properly choosing the controller energy function, it is possible to introduce a minimum at the desired equilibrium configuration that can be reached is some dissipative effect is present in the system. In particular, if in (13)  $R = 0$ , that is no dissipative/diffusion phenomena are present in the infinite dimensional plant, it is convenient to chose the controller structure in order to have  $\bar{n} < n_c$  Casimir

function in the form (20) and then to introduce energy dissipation by acting on the remaining energy variables.

### IV. EXAMPLE: STABILIZATION OF THE HEAT EQUATION

Consider the heat equation that can be written in mdpH form as follows, [3]:

$$\begin{cases} \frac{\partial x}{\partial t} = \frac{\partial^2}{\partial z^2} \delta_x \mathcal{H} \\ w = \left[ \begin{array}{c} \delta_x \mathcal{H} \\ \frac{\partial}{\partial z} \delta_x \mathcal{H} \end{array} \right] \Big|_{\{0, 1\}} \end{cases} \quad (22)$$

where  $\mathcal{Z} = [0, 1]$  is the spatial domain,  $\mathcal{X} = L^2(\mathcal{Z})$  is the space of energy variables,  $\mathcal{H}(x) = \frac{1}{2} \int_0^1 x^2 dz$  is the Hamiltonian function and

$$B_{\{J, R\}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is the constant matrix representing the operator which gives the power through the boundary as in (14). Note that

$$\frac{d\mathcal{H}}{dt} = \int_{\mathcal{Z}} \left[ \frac{\partial}{\partial z} \left( x \frac{\partial x}{\partial z} \right) - \left( \frac{\partial x}{\partial z} \right)^2 \right] dz \leq x \frac{\partial x}{\partial z} \Big|_0^L \quad (23)$$

Denote by  $x^* \in \mathcal{X}$  a desired equilibrium configuration of (22) that the following one dimensional controller (with  $J_c = R_c = 0$ ) should render asymptotically stable.

$$\begin{cases} \dot{x}_c = G_c u_c \\ y_c = G_c^T \frac{\partial H_c}{\partial x_c} \end{cases} \quad (24)$$

Suppose that  $x_c \in \mathbb{R}$  and that  $u_c, y_c \in \mathbb{R}^2$ . From (23), it is easy to verify that (22) and (24) are interconnected in a power conserving way if

$$u_c = \begin{bmatrix} \frac{\partial}{\partial z} \delta_x \mathcal{H}(0) \\ \frac{\partial}{\partial z} \delta_x \mathcal{H}(1) \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial z}(0) \\ \frac{\partial x}{\partial z}(1) \end{bmatrix} \quad y_c = \begin{bmatrix} x(0) \\ -x(1) \end{bmatrix} \quad (25)$$

The first step in the control by interconnection and energy shaping is to choose the controller structure in order to have a set of structural invariants in the form (20). In this case, since the controller is a dynamical system of order 1, it is necessary to determine  $G_c$  such that the function

$$\mathcal{C}(x, x_c) = x_c - \mathcal{S}(x) = x_c - \int_0^1 \mathcal{S}(z, x) dz \quad (26)$$

is a Casimir function for the closed loop system, that is  $\dot{\mathcal{C}} = 0$  for every  $\mathcal{H}$  and  $H_c$ . We have that

$$\begin{aligned} \frac{d\mathcal{C}}{dt} &= (G_c + [\delta_x \mathcal{S}(0) \quad -\delta_x \mathcal{S}(1)]) \begin{bmatrix} \frac{\partial z x(0)}{\partial z x(1)} \end{bmatrix} \\ &\quad - \int_0^1 x \frac{\partial^2}{\partial z^2} (\delta_x \mathcal{S}) dz \\ &\quad - [\partial_z \delta_x \mathcal{S}(0) \quad \partial_z \delta_x \mathcal{S}(1)] G_c^T \frac{\partial H_c}{\partial x_c} \end{aligned}$$

Consequently, (26) is a Casimir function for the controlled system if and only if

$$\frac{\partial^2}{\partial z^2} \delta_x \mathcal{S} = 0 \quad (27a)$$

and

$$\begin{cases} G_c + [\delta_x \mathcal{S}(0) & -\delta_x \mathcal{S}(1)] = 0 \\ [\partial_z \delta_x \mathcal{S}(0) & \partial_z \delta_x \mathcal{S}(1)] G_c^T = 0 \end{cases} \quad (27b)$$

Condition (27a) provides the admissible functionals  $\mathcal{S}$ , while (27b) the input matrix  $G_c$  of the controller. From (27a),  $\delta_x \mathcal{S} = az + b$ , with  $a, b \in \mathbb{R}$ , while, in order to satisfy (27b), it is necessary that  $a = 0$  and it is possible to choose  $b = 1$ . Consequently,

$$G_c = \begin{bmatrix} -1 & 1 \end{bmatrix} \quad (28)$$

and

$$\mathcal{C}(x, x_c) = x_c - \int_0^1 x(z) dz \quad (29)$$

is a Casimir function for the closed loop system. From (28), the controller (24) becomes

$$\begin{cases} \dot{x}_c = \partial_z x(1) - \partial_z x(0) \\ y_c = \begin{bmatrix} -\partial_{x_c} H_c \\ \partial_{x_c} H_c \end{bmatrix} \end{cases}$$

and then, from (25),

$$x(0) = x(1) = -\frac{\partial H_c}{\partial x_c} \quad (30)$$

that is the controller acts on the system by imposing the same *temperature* on both the extremities of the infinite dimensional system. Moreover, the controller internal energy changes, that is  $\dot{x}_c \neq 0$ , only if there is a difference in the gradient of temperature at the extremities of the domain. As a consequence, the controller can stabilize the distributed parameter system only in the configurations for which the temperature is constant along the domain, that is  $x^*(z) = x^*$  for every  $z \in \mathcal{Z}$ . Under the hypothesis that the initial configuration of the system is known, from (29) and from the properties of the Casimir functions, we have that  $x_c = x_c(x) = \int_0^1 x(z) dz$  for the closed loop system. Define  $x_c^* = x_c(x^*) = x^*$ . The configuration  $x^*$  is asymptotically stable if the controller Hamiltonian is  $H_c(x_c) = -x_c x_c^* - x_c x^*$ . In fact, if  $\mathcal{H}_{cl}(x, x_c) = \mathcal{H}(x) + H_c(x_c)$  is the energy function of the closed loop system, taking into account (30), we have that

$$\frac{d\mathcal{H}_{cl}}{dt} = - \int_0^1 \left( \frac{\partial x}{\partial z} \right)^2 dz \leq 0$$

Then,  $\dot{\mathcal{H}}_{cl} = 0$  if  $\partial_z x = 0$  on  $\mathcal{Z}$ , that is if  $x(z)$  is constant on  $\mathcal{Z}$ . Since  $x(0) = x(1) = x^*$ , the only admissible configuration is  $x^*$  that results to be asymptotically stable. The asymptotic stability of  $x^*$  can be alternatively proved by looking at the expression of the closed loop Hamiltonian. We have that

$$\mathcal{H}_{cl}(x) = \frac{1}{2} \int_0^1 (x^2 dz - x_c x^*) dz = \frac{1}{2} \int_0^1 (x - x^*)^2 dz + \kappa$$

which has a global minimum in  $x^*$  and where  $\kappa$  is a constant.

## V. CONCLUSIONS

In this paper, a novel technique for the boundary control of distributed parameter systems in port Hamiltonian form has been developed by extending the well known control by interconnection and energy shaping methodology. The basic result is the generalization of the conditions for obtaining a particular set of *Casimir function* to the *hybrid* case, that is the dynamical system to be considered results from the power conserving interconnection of an infinite dimensional system (the plant) and of a finite dimensional one (the controller). A simple application concerning the stabilization of the one-dimensional heat equation has been presented.

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