Standard diffusive systems are well-posed linear systems.

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1 Introduction

The class of well-posed linear systems as introduced by Salamon [5] has become a well-understood class of systems, see e.g. the work of Weiss [8] and the book of Staffans [7]. Many partial partial differential equations with boundary control and point observation can be formulated as a well-posed linear system. In parallel to the development of well-posed linear systems, the class of diffusive systems has been developed, independently, by [6, § 5] and [4]. This class is used to model systems for which the impulse response has a long tail, i.e., decays slowly, or systems with a diffusive nature, like the Lokshin model in acoustics, see [2]. Another class of models are the fractional differential equations, i.e., a system which has fractional powers of s in its transfer function, see e.g. [3].

2 Summary of the results

A well-posed linear system is a system with space space X, input space U, and output space Y, all Hilbert spaces, of the following form

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) & = & Cx(t) + Du(t). \end{array} \tag{1}$$

for which we have that for all $x_0 \in X$ and $u \in L^2_{\text{loc}}((0,\infty);U)$ there exists a state trajectory $x(t)$ which is continuous with values in X, and an output trajectory $y \in L^2_{\text{loc}}([0,\infty);Y)$. A well-posed system is said to be regular, when the transfer function has a (strong) limit for $s \to \infty$. If the system is regular, then this limit equals D.

A diffusive linear systems on $(0,\infty)$ is of the form

$$\begin{array}{rcl} \frac{\partial \phi}{\partial t}(\xi,t) & = & -\xi\, \phi(\xi,t) + u(t), \quad \phi(0) = \phi_0 \\ y(t) & = & \int_0^\infty \phi(\xi,t)\, d\mu(\xi), \end{array} \tag{2}$$

where μ is a positive measure without singular part and satisfying $\int_0^\infty (1+\xi)^{-1}\, d\mu < +\infty$. Using Engel and Nagel [1], we show that the homogeneous equation generates a self-adjoint, contractive and analytic semigroup on the state space $X = L^2([0,\infty), d\mu)$. Furthermore, we show that any diffusive linear system is a regular well-posed system with D-term equal to zero.

The well-posed system associated to the diffusive linear system is always approximately observable and controllable, and the measure μ determines the stability of the semigroup. That is, it is exponentially stable if and only if $\mu([0,\varepsilon)) = 0$ for some $\varepsilon > 0$ and it is strongly stable if and only if $\lim_{\varepsilon\downarrow 0}\mu([0,\varepsilon)) = 0$.

We illustrate the obtained results on some standard examples of diffusive systems.

The proof of above results is not complicated, and hence diffusive systems form an easy class within the general class of well-posed linear systems. This has two important consequences:

- The theory as developed for well-posed linear systems, like feedback, interconnection, etc, is directly applicable to diffusive systems;
- New notions and results for the class of well-posed systems can be illustrated by means of diffusive systems, without having to use deep results on functional analysis or partial differential equations.

References

[1] K-J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*. Springer Verlag, New York, 2000.

[2] H. Haddar, Th. Hélie, and D. Matignon. A Webster-Lokshin model for waves with viscothermal losses and impedance boundary conditions: strong solutions. In *Sixth international conference on mathematical and numerical aspects of wave propagation phenomena*, pages 66–71, Jyväskylä, Finland, July 2003. INRIA.

[3] D. Matignon. Stability properties for generalized fractional differential systems. *ESAIM: Proceedings*, 5:145–158, December 1998. URL: `http://www.edpsciences.org/articlesproc/Vol.5/`.

[4] G. Montseny. Diffusive representation of pseudo-differential time-operators. *ESAIM: Proceedings*, 5:159–175, December 1998. URL: `http://www.edpsciences.org/articles/proc/Vol.5/`.

[5] D. Salamon. Infinite dimensional linear systems with unbounded control and observation: a functional analytic approach. *Trans. Amer. Math. Soc.* 300:383–431, 1987.

[6] O. J. Staffans. Well-posedness and stabilizability of a viscoelastic equation in energy space. *Trans. Amer. Math. Soc.* 345(2):527–575, October 1994.

[7] O.J. Staffans. *Well-Posed Linear Systems I: General Theory.* Cambridge University Press, Cambridge, 2004. Manuscript available at `http://www.abo.fi/ staffans/publ.htm`

[8] G. Weiss. Regular Linear Systems with Feedback. *Mathematics of Control, Signals and Systems*, 7:23-57, 1994.