A functional analytic approach towards nonlinear dissipative well-posed systems

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Abstract

The aim of this paper is to develop a functional analytic approach towards nonlinear systems. For linear systems this is well known and the resulting class of well-posed and regular linear systems is well studied. Our approach is based on the theory of nonlinear semigroup and we explain it by means of an example, namely equations of quasi-hyperbolic type.

1 Equations of quasi-hyperbolic type

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a strictly monotone decreasing surjective function with $\psi(0) = 0$. Associated to this function, we consider the following partial differential equation on the spatial interval $[0, 1]$,

$$\begin{aligned} \frac{\partial f}{\partial t}(t,\eta) &= \frac{\partial \psi(f)}{\partial \eta}(t,\eta), \quad 0 < \eta < 1, t > 0, \\ f(0,\eta) &= f_0(\eta), \quad 0 < \eta < 1, \\ f(t,0) &= 0, \quad t > 0. \end{aligned} \tag{1}$$

As output we define

$$y(t) = \psi(f(t,1)). \tag{2}$$

We want to show that (1)–(2) defines a dissipative system. In particular, this will imply that the output y is a function in $L^1(0,\infty)$.

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As state space we choose $X = L^1(0,1)$ and we choose $L^1_{\mathrm{loc}}(0,\infty)$ as our space of output signals. In [4, p.110] it is shown that $A : D(A) \subset L^1(0,1) \to L^1(0,1)$, given by

$$(Af)(\eta) := \frac{d\psi(f(\eta))}{d\eta}, \qquad f \in D(A),$$

$$D(A) := \{f \in C[0,1] \mid f(0) = 0, \psi(f) \text{ is absolutely continuous on } [0,1]\}$$

is an m-dissipative operator on X. Associated to this m-dissipative operator we can define a semigroup $(T(t))_{t\geq 0}$ of nonlinear contractions on X. This semigroup is defined as

$$T(t)f_0 = \lim_{\lambda\downarrow 0}(I - \lambda A)^{-[t/\lambda]} f_0. \tag{3}$$

Unfortunately, in general A is not the infinitesimal generator of this semigroup, see [4, p. 104]. However, if f is a classical solution of the Cauchy problem $\dot{f} = Af$, $f(0) = f_0$, then $f(t) = T(t)f_0$. Furthermore, if $f_0 \in D(A)$ and $(T(t)f_0)_{t\geq 0}$ is strongly differentiable a.e., then this function is the unique solution of the Cauchy problem $\dot{f} = Af$, $f(0) = f_0$. In general, $(T(t)f_0)_{t\geq 0}$ only is an integral solution.
Using the method of characteristics, it can be shown that for smooth functions f_0 the partial differential equation (1) has on some interval $[0, t_0)$ a classical solution. This solution equals $T(t)f_0$ for $t \in [0, t_0)$, [2, p. 189]. Thus the output — as given in (2) — is well-defined on some interval $[0, t_0)$ provided f_0 is smooth. Dissipativity of a system now guarantees L^1-outputs and that the system generates no energy.

Definition 1.1 *We say that the system (1)–(2) is* dissipative*, if there exists a mapping $\mathcal{C} : L^1(0,1) \to L^1(0,\infty)$ satisfying*

1. *If for $f_0 \in L^1(0,1)$ and $t_0 > 0$ the partial differential equation (1) possesses a classical solution on $[0, t_0]$, then $(\mathcal{C}f_0)(t) = \psi((T(t)f_0)(1))$ for $t \in [0, t_0]$.*

2. *For any $t, s \geq 0$ and every $f_0 \in L^1(0,1)$ we have*

$$P_{t+s}\mathcal{C}f_0 = P_s\mathcal{C}f_0 + S_r(s)P_t\mathcal{C}T(s)f_0.$$

3. *For $f_0, f_1 \in L^1(0,1)$ and for every $t_0 > 0$ we have*

$$\|\mathcal{C}f_0 - \mathcal{C}f_1\|_{L^1(0,t_0)} \leq \|f_0 - f_1\|_{L^1(0,1)} - \|T(t_0)f_0 - T(t_0)f_1\|_{L^1(0,1)}. \tag{4}$$

Here $P_t \in \mathcal{L}(L^1(0,\infty))$ is the projection onto the interval $[0,t]$ and $S_r(t) \in \mathcal{L}(L^1(0,\infty))$ is the right-shift by t. As mentioned above, if f_0 is smooth then there exists an interval on which (1) has a classical solution. On this interval this classical solution equals $T(t)f_0$. Hence from item 1. and equation (2) we see that y equals $\mathcal{C}f_0$ on this time-interval. For general f_0 we call the function $\mathcal{C}f_0$ the *"generalized" output* of system (1)–(2). The main result of this article is as follows.

Theorem 1.2 *System (1)–(2) is dissipative.*

The proof of this theorem will be given at the end of the section. The following lemma will be useful.

Lemma 1.3 *Let ϕ be a strictly monotone decreasing surjective function from $\mathbb{R}$ to $\mathbb{R}$ with $\phi(0) = 0$, let $-\infty \leq a < b \leq \infty$ and let $A_\phi : D(A_\phi) \subset L^1(a, b) \to L^1(a, b)$ be defined by*

$$(A_\phi f)(\eta) := \frac{d\phi(f(\eta))}{d\eta}, \qquad f \in D(A_\phi),$$

$D(A_\phi) := \{f \in L_1(a, b) \mid \phi(f)$ *is absolutely continuous on* $[a, b]$ *and* $\phi(f)' \in L_1(a, b)\}$.

Then we have

$$\begin{aligned} &\|f_1 - f_2 - \lambda(A_\phi f_1 - A_\phi f_2)\|_{L^1(a,b)} \\ &\quad \geq \ \|f_1 - f_2\|_{L^1(a,b)} + \lambda|\phi(f_1(b)) - \phi(f_2(b))| - \lambda|\phi(f_1(a)) - \phi(f_2(a))|. \end{aligned}$$

for all $\lambda > 0$ and all $f_1, f_2 \in D(A)$.
If $a = -\infty$, then $f_1(a) = f_2(a) = 0$ in the above inequality. A similar remark holds if $b = \infty$.

The proof of this lemma follows largely [4, page 110–111].
Proof: We define the functions $p_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, by

$$p_n(\eta) := \begin{cases} -n\eta & |\eta| \leq 1/n \\ -\operatorname{sign} \eta & |\eta| > 1/n \end{cases}.$$

It is easy to see that the functions p_n are monotone decreasing, Lipschitz continuous, $p_n(0) = 0$ and $|p_n(\eta)| \leq 1$ for every $\eta \in \mathbb{R}$ and every $n \in \mathbb{N}$. Let $f_1, f_2 \in D(A)$ and $\lambda > 0$ be arbitrarily. Then we have for every $n \in \mathbb{N}$

$$\begin{aligned} &\int_a^b (Af_1 - Af_2)\, p_n(\phi(f_1(\eta)) - \phi(f_2(\eta)))\, d\eta \\ &= \int_a^b (\phi(f_1(\eta)) - \phi(f_2(\eta)))'\, p_n(\phi(f_1(\eta)) - \phi(f_2(\eta))\, d\eta \\ &= \int_a^b \frac{d}{d\eta}(q_n(\phi(f_1(\eta)) - \phi(f_2(\eta))))\, d\eta \\ &= q_n(\phi(f_1(b)) - \phi(f_2(b))) - q_n(\phi(f_1(a)) - \phi(f_2(a))), \end{aligned}$$

where $q_n(s) := \int_0^s p_n(\eta)\, d\eta$, and thus

$$\begin{aligned}
&\int_a^b |f_1 - f_2 - \lambda(Af_1 - Af_2)|\, d\eta \\
&\quad \geq \int_a^b |f_1 - f_2 - \lambda(Af_1 - Af_2)|\, |p_n(\phi(f_1) - \phi(f_2))|\, d\eta \\
&\quad \geq \int_a^b (f_1 - f_2 - \lambda(Af_1 - Af_2))\, p_n(\phi(f_1) - \phi(f_2))\, d\eta
\end{aligned}$$

$$\begin{aligned}
&\geq \int_a^b (f_1 - f_2)\, p_n(\phi(f_1) - \phi(f_2))\, d\eta - \lambda \int_a^b (Af_1 - Af_2)\, p_n(\phi(f_1) - \phi(f_2))\, d\eta \\
&\geq \int_a^b (f_1(\eta) - f_2(\eta))\, p_n(\phi(f_1(\eta)) - \phi(f_2(\eta)))\, d\eta \\
&\quad +\lambda q_n(\phi(f_1(a)) - \phi(f_2(a))) - \lambda q_n(\phi(f_1(b)) - \phi(f_2(b))).
\end{aligned}$$

Now p_n converges pointwise to $p(\eta) = -\operatorname{sign}\eta$ and thus by Lebesgue Dominated Convergence Theorem we finally have

$$\begin{aligned}
&\int_a^b |f_1 - f_2 - \lambda(Af_1 - Af_2)|\, d\eta \\
&\quad \geq \int_a^b |f_1(\eta) - f_2(\eta)|\, d\eta + \lambda|\phi(f_1(b)) - \phi(f_2(b))| - \lambda|\phi(f_1(a)) - \phi(f_2(a))|.
\end{aligned}$$

In the last step we used that ϕ is strictly monotone decreasing. ■

Now we define the mapping $A_e : D(A_e) \subset X \times L^1(0,\infty) \to X \times L^1(0,\infty)$ by

$$A_e \begin{pmatrix} f \\ y \end{pmatrix} = \begin{pmatrix} Af \\ -y' \end{pmatrix} \tag{5}$$

$$D(A_e) = \left\{ \left(\begin{smallmatrix} f \\ y \end{smallmatrix} \right) \in D(A) \times W^{1,1}(0,\infty) \mid y(0) = \psi(f(1)) \right\}. \tag{6}$$

This operator is m-dissipative on $L^1(0,1) \times L^1(0,\infty)$, where the norm on this product space is the sum of the separate norms.

Theorem 1.4 *The operator A_e as defined in (5) and (6) is m-dissipative on $L^1(0,1) \times L^1(0,\infty)$.*

Proof First we prove that A_e is dissipative, that is, for all $\lambda > 0$ and all $z_1, z_2 \in D(A_e)$ we have

$$\|z_1 - z_2 - \lambda(A_e z_1 - A_e z_2)\| \geq \|z_1 - z_2\|.$$

Thus let $\lambda > 0$ and $\binom{f_1}{y_1}$, $\binom{f_2}{y_2} \in D(A_e)$ be arbitrarily. Applying Lemma 1.3 to the operator A and noting that f_1 $f_2 \in D(A)$ we get

$$\begin{aligned} &\|f_1 - f_2 - \lambda(Af_1 - Af_2)\|_{L^1(0,1)} \\ &\quad \geq \ \|f_1 - f_2\|_{L^1(0,1)} + \lambda|\psi(f_1(1)) - \psi(f_2(1))| - \lambda|\psi(f_1(0)) - \psi(f_2(0))| \\ &\quad = \ \|f_1 - f_2\|_{L^1(0,1)} + \lambda|y_1(0) - y_2(0)|, \end{aligned} \tag{7}$$

since $f_1, f_2 \in D(A)$ and $\binom{f_1}{y_1}$, $\binom{f_2}{y_2} \in D(A_e)$. Choosing $a = 0$, $b = \infty$, $\phi(s) := -s$, $s \in \mathbb{R}$ in Lemma 1.3 and noting that y_1, $y_2 \in W^{1,1}(0, \infty)$ we get

$$\|y_1 - y_2 - \lambda(-\dot{y_1} + \dot{y_2})\|_{L^1(0,\infty)} \geq \|y_1 - y_2\|_{L^1(0,\infty)} - \lambda|y_1(0) - y_2(0)|. \tag{8}$$

Finally, combining (7) and (8) gives for $\binom{f_1}{y_1}$, $\binom{f_2}{y_2} \in D(A_e)$

$$\begin{aligned} &\left\|\binom{f_1}{y_1} - \binom{f_2}{y_2} - \lambda\left(A_e\binom{f_1}{y_1} - A_e\binom{f_2}{y_2}\right)\right\| \\ &\quad = \ \|f_1 - f_2 - \lambda(Af_1 - Af_2)\|_{L^1(0,1)} + \|y_1 - y_2 - \lambda(-\dot{y_1} + \dot{y_2})\|_{L^1(0,\infty)} \\ &\quad \geq \ \|f_1 - f_2\|_{L^1(0,1)} + \|y_1 - y_2\|_{L^1(0,\infty)} \\ &\quad = \ \left\|\binom{f_1}{y_1} - \binom{f_2}{y_2}\right\|. \end{aligned}$$

Thus A_e is dissipative on $L^1(0, 1) \times L^1(0, \infty)$.
In order to show that A_e is m-dissipative on $L^1(0, 1) \times L^1(0, \infty)$ it remains to prove that the range of $I - A_e$ equals $L^1(0, 1) \times L^1(0, \infty)$. Let $\binom{\tilde{f}}{\tilde{y}} \in L^1(0, 1) \times L^1(0, \infty)$ be arbitrarily. Since A is m-dissipative on $L^1(0, 1)$, see [4], we have that the range of $(I - A)$ equals $L^1(0, 1)$. Thus there exists $f \in D(A)$ such that

$$(I - A)f = \tilde{f}.$$

We now define $y \in L^1(0, \infty)$ by

$$y(t) := e^{-t}\psi(f(1)) + \int_0^t e^{-(t-s)}\tilde{y}(s)\, ds.$$

It is easy to see that

$$y(t) + \dot{y}(t) = \tilde{y}(t) \quad \text{and} \quad y(0) = \psi(f(1)).$$

Thus $\binom{f}{y} \in D(A_e)$ and $(I - A_e)\binom{f}{y} = \binom{\tilde{f}}{\tilde{y}}$. This proves that A_e is m-dissipative on $L^1(0, 1) \times L^1(0, \infty)$. This completes the proof. ■

The following lemma is easy to verify.

Lemma 1.5 *Let* $\binom{f_0}{y_0} \in D(A_e)$ *and* $t_0 > 0$. *Then the partial differential equation (1) possesses a classical solution on the interval* $t \in [0, t_0]$ *if and only if the Cauchy problem*

$$\frac{d}{dt}\begin{pmatrix} f(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} Af(t) \\ -\frac{d}{ds}(y(t)) \end{pmatrix}.$$

has a classical solution on the interval $[0, t_0]$.

Let $T_e(t)$ be the semigroup associated to A_e, i.e.,

$$T_e(t)\begin{pmatrix} f_0 \\ y \end{pmatrix} = \lim_{\lambda\downarrow 0}(I-\lambda A_e)^{-[t/\lambda]}\begin{pmatrix} f_0 \\ y \end{pmatrix}. \tag{9}$$

and let $T(t)$ be the semigroup associated to A, see (3). An easy calculation shows that for $\lambda > 0$ and $\binom{f_0}{y_0} \in L^1(0,1)\times L^1(0,\infty)$ we have

$$(I-\lambda A_e)^{-1}\begin{pmatrix} f_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} (I-\lambda A)^{-1}f_0 \\ e^{-\cdot/\lambda}\psi(((I-\lambda A)^{-1}f_0)(1)) + (I-\lambda D_R)^{-1}y_0 \end{pmatrix},$$

where $D_R f := -f'$. Thus similar as the non-linear operator A_e, the resolvent operator has the nice property that

$$(I-\lambda A_e)^{-1}\begin{pmatrix} f_0 \\ y_0 \end{pmatrix} = (I-\lambda A_e)^{-1}\begin{pmatrix} f_0 \\ 0 \end{pmatrix} + (I-\lambda A_e)^{-1}\begin{pmatrix} 0 \\ y_0 \end{pmatrix}.$$

Furthermore, since D_R is a linear operator, the resolvent is linear in the second component. Using this, equation (9), (3), and the fact that D_R is the infinitesimal generator of the right-shift semigroup $(S_r(t))_{t\geq 0}$ we obtain that $T_e(t)$ is given by

$$T_e(t)\begin{pmatrix} f_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} T(t)f_0 \\ \mathcal{Q}_t f_0 + S_r(t)y_0 \end{pmatrix} \tag{10}$$

for some mapping $\mathcal{Q}_t$ from $L^1(0,1)$ to $L^1(0,\infty)$. Since $(T_e(t))_{t\geq 0}$ is a contraction semigroup, we have

$$\|\mathcal{Q}_t f_0 - \mathcal{Q}_t f_1\|_{L^1(0,\infty)} \leq \|f_0 - f_1\|_{L^1(0,1)} - \|T(t)f_0 - T(t)f_1\|_{L^1(0,1)}. \tag{11}$$

Moreover, the semigroup property implies

$$\mathcal{Q}_{t+s}f_0 = \mathcal{Q}_t T(s) f_0 + S_r(t)\mathcal{Q}_s f_0 \tag{12}$$

Theorem 1.6 *For $\binom{f_0}{y_0} \in D(A_e)$ such that the partial differential equation (1) possesses a classical solution on some interval $[0,t_0]$ we have that for $t \in [0,t_0]$*

$$(\mathcal{Q}_t f_0)(\rho) = \begin{cases} \psi((T(t-\rho)f_0)(1)), & \text{for } \rho \in [0,t] \\ 0 & \text{for } \rho > t. \end{cases}$$

Proof Let $\binom{f_0}{y_0} \in D(A_e)$ arbitrarily such that the partial differential equation (1) possesses a classical solution on some interval $[0,t_0]$. Then $\binom{f(t)}{y(t)} := T_e(t)\binom{f_0}{y_0}$ is a classical solution of $\dot{z}(t) = A_e z(t)$ on $[0,t_0]$ and thus $f(t) = T(t)f_0$. The second equation equals the partial differential equation

$$\frac{\partial y(t)(s)}{\partial t} = -\frac{\partial y(t)(s)}{\partial s}, \qquad 0 \leq t \leq t_0, \quad s \geq 0.$$

with initial condition $(y(0))(s) = y_0(s)$. The solution equals $(y(t))(s) = g(t-s)$, $s > 0$ and $t \in [0, t_0]$, for some function $g \in W^{1,1}(-\infty, t_0)$ with $g(-s) = y_0(s)$ for $s > 0$. Since $\binom{f(t)}{y(t)} \in D(A_e)$, we see that $(y(t))(0) = \psi((f(t)(1)))$ for $t \in [0, t_0]$. Thus

$$g(t) = \psi((f(t))(1)) \qquad t \in [0, t_0].$$

By equation (10) we have that $y(t) = \mathcal{Q}_t f_0 + S_r(t) y_0$. Combining this with the results above, implies that for $s \in [0, t]$ and $t \leq t_0$

$$\begin{aligned} \psi((f(t-s))(1)) &= g(t-s) = (y(t))(s) \\ &= (\mathcal{Q}_t f_0 + S_r(t) y_0)(s) = (\mathcal{Q}_t f_0)(s). \end{aligned} \tag{13}$$

For $s > t$ and $t \leq t_0$, we have that

$$\begin{aligned} y_0(s-t) &= g(t-s) = (y(t))(s) \\ &= (\mathcal{Q}_t f_0 + S_r(t) y_0)(s) = (\mathcal{Q}_t f_0)(s) + y_0(s-t). \end{aligned} \tag{14}$$

This implies that $(\mathcal{Q}_t f_0)(s) = 0$ for $s > t$, whenever $t \in [0, t_0]$. Combining this with equation (13) gives the expression for $\mathcal{Q}_t$. ■

Lemma 1.7 *If $f_0 \in L^1(0,1)$ and $t > 0$, then we have $(\mathcal{Q}_t f_0)(\rho) = 0$ for a.e. $\rho > t$.*

Proof: Let $f_0 \in L^1(0,1)$ and $t > 0$ be arbitrarily. Then definitions of A and A_e imply

$$(I - \lambda A)^{-1} 0 = 0 \text{ and } (I - \lambda A_e)^{-1} \binom{0}{0} = \binom{0}{0}$$

for every $\lambda > 0$. This together with the fact that A_e is m-dissipative shows

$$\left\| (I - \lambda A_e)^{-1} \binom{f}{y} \right\| \leq \left\| \binom{f}{y} \right\|, \qquad \lambda > 0, f \in L^1(0,1), y \in L^1(0, \infty). \tag{15}$$

Moreover, for $\binom{f}{y} \in L^1(0,1) \times L^2(0, \infty)$ we have

$$(I - \lambda A_e)^{-1} \begin{pmatrix} f \\ y \end{pmatrix} = \begin{pmatrix} (I - \lambda A)^{-1} f \\ e^{-\cdot/\lambda} \Psi(((I - \lambda A)^{-1} f)(1)) + (I - \lambda D_R)^{-1} y \end{pmatrix}$$

and

$$(I - \lambda A_e)^{-n} \begin{pmatrix} f \\ y \end{pmatrix} = \begin{pmatrix} (I - \lambda A)^{-n} f \\ e^{-\cdot/\lambda} \sum_{k=0}^{n-1} \frac{(\cdot)^k}{k! \lambda^k} \Psi(((I - \lambda A)^{-(n-k)} f)(1)) + (I - \lambda D_R)^{-n} y \end{pmatrix}.$$

We note, that $(I - \lambda D_R)^{-k} e^{-\cdot/\lambda} = \frac{(\cdot)^k}{k! \lambda^k} e^{-\cdot/\lambda}$. Thus equation (15) implies

$$\begin{aligned} \|f_0\|_{L^1(0,1)} &\geq \|(I - \lambda A)^{-1} f_0\|_{L^1(0,1)} + \lambda |\Psi(((I - \lambda A)^{-1} f_0)(1)| \\ &\overset{(15)}{\geq} \|(I - \lambda A)^{-2} f_0\|_{L^1(0,1)} + \lambda |\Psi(((I - \lambda A)^{-2} f_0)(1)| \\ &\quad + \lambda |\Psi(((I - \lambda A)^{-1} f_0)(1)| \\ &\geq \|(I - \lambda A)^{-n} f_0\|_{L^1(0,1)} + \lambda \sum_{j=1}^{n} |\Psi(((I - \lambda A)^{-j} f_0)(1)|. \end{aligned}$$

In particular, for $\lambda = t/n$ we get

$$\frac{t}{n}\sum_{j=1}^{n} |\Psi(((I-\lambda A)^{-j} f_0)(1)| \leq \|f_0\|. \tag{16}$$

We now define the function $g_n \in L^1(0,\infty)$ by

$$g_n(s) = e^{-\frac{sn}{t}} \sum_{k=0}^{n-1} \frac{n^k s^k}{k! t^k} \Psi(((I-\frac{t}{n}A)^{-(n-k)} f_0)(1)).$$

We know already that

$$\lim_{n\to\infty} \|g_n - Q_t f_0\|_{L^1(0,\infty)} = 0.$$

Thus by Lebesgue's theorem it remain to show that

$$g_n(s) \to 0 \text{ for almost every } s > t.$$

For $s > t$ we have

$$\begin{aligned}
|g_n(s)| &\leq e^{-\frac{sn}{t}} \sum_{k=0}^{n-1} \frac{n^k s^k}{k! t^k} |\Psi(((I-\frac{t}{n}A)^{-(n-k)} f_0)(1))| \\
&\leq e^{-\frac{sn}{t}} \sup_{k=0}^{n-1} \left(\frac{n^k s^k}{k! t^k}\right) \sum_{k=0}^{n-1} |\Psi(((I-\frac{t}{n}A)^{-(n-k)}| \\
&\overset{(16)}{\leq} \|f_0\| \frac{n}{t} e^{-\frac{sn}{t}} \frac{n^{n-1} s^n}{n! t^n} \quad \text{since } s > t \\
&\leq \|f_0\| \frac{n}{s} e^{-\frac{sn}{t}} \frac{n^n}{n!} \frac{s^n}{t^n} \\
&\leq \frac{\|f_0\|}{\sqrt{2\pi} s} \sqrt{n} e^n e^{-\frac{sn}{t}} \left(\frac{s}{t}\right)^n \quad \text{(by Stirlings formula } n^n/n! \leq e^n/\sqrt{2\pi n}) \\
&= \frac{\|f_0\|}{\sqrt{2\pi} s} \sqrt{n} \exp\left(n\left\{1 - \frac{s}{t} + \log\frac{s}{t}\right\}\right) \\
&\quad \to 0 \text{ for } n \to \infty \quad \text{since } s > t).
\end{aligned}$$

■

Proof of Theorem 1.2: We define the mapping $\mathcal{C}_t$ as

$$\mathcal{C}_t = R(t)\mathcal{Q}_t, \tag{17}$$

where $R(t) \in \mathcal{L}(L^1(0,\infty))$ is defined by $(R(t)f)(s) = f(t-s)$ if $0 \leq s \leq t$ and $(R(t)f)(s) = 0$, otherwise. Since $\mathcal{Q}_t$ has its support in $[0,t]$, we have that $\mathcal{Q}_t = R(t)\mathcal{C}_t$. Using this and (12) we find that

$$\mathcal{C}_{t+s} f_0 = \mathcal{C}_s f_0 + S_r(s)\mathcal{C}_t T(s) f_0.$$

Since by the definition of $\mathcal{C}_t$ we have that $P_t\mathcal{C}_t = \mathcal{C}_t$ we have that the above equality is equivalent to

$$P_{t+s}\mathcal{C}_{t+s}f_0 = P_s\mathcal{C}_s f_0 + S_r(s)P_t\mathcal{C}_t T(s)f_0. \tag{18}$$

In particular, this shows $(\mathcal{C}_{t+s}f_0)(\rho) = (\mathcal{C}_s f_0)(\rho)$, a.e. $\rho \in [0, s]$ for any $t, s > 0$. Thus we can define $\mathcal{C} : L^1(0, 1) \to L^1(0, \infty)$ by

$$(\mathcal{C}f_0)(\rho) = (\mathcal{C}_t f_0)(\rho), \qquad t > \rho. \tag{19}$$

Using this and equations (17), (11) it is easy to see that

$$\|\mathcal{C}f_0 - \mathcal{C}f_1\|_{L^1(0,t)} \leq \|f_0 - f_1\|_{L^1(0,1)} - \|T(t)f_0 - T(t)f_1\|_{L^1(0,1)}. \tag{20}$$

Hence we have shown that item 2. and 3. of Definition 1.1 hold. In Theorem 1.6 we have shown that it also satisfies item 1. Thus our system is dissipative. ■

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