Exact Membership Functions for the Fuzzy Weighted Average

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Abstract. The problem of computing the fuzzy weighted average, where both attributes and weights are fuzzy numbers, is well studied in the literature. Generally, the approach is to apply Zadeh's extension principle to compute α-cuts of the fuzzy weighted average from the α-cuts of the attributes and weights for fixed values of $\alpha \in [0..1]$; this means that all values of the membership functions of the fuzzy weighted average are computed separately. In this paper, we generalise this approach in such a way that α is considered to be a parameter; this enables us to compute exact analytical membership functions for the fuzzy weighted average in case the attributes and weights are triangular or trapeizodal fuzzy numbers. To illustrate the power of our algorithms, they are applied to the examples from the literature, providing exact membership functions in each case.

Keywords: Fuzzy weighted average, membership functions

1   Introduction

In multiple criteria decision making problems, values of decision variables are weighted averages of criteria ratings. Often the rating criteria and their corresponding importance weights are vague, and are therefore represented by fuzzy numbers. Then the values of the decision variables which are determined by them are fuzzy numbers as well; they are fuzzy weighted averages of the criteria ratings.

The standard approach to the calculation of fuzzy weighted averages [1-11] is to apply the extension principle to the following weighted average function:

$$\text{wa}(x_1, ..,x_n,w_1,..,w_n) = \sum_{i=1}^{n}(x_i * w_i) \Big/ \sum_{i=1}^{n} w_i \qquad (1)$$

Here $x_1,..,x_n$ are real numbers, called attributes, and $w_1,..,w_n$ are non–negative real numbers, called weights.

Let $A_1,..,A_n$ and $W_1,..,W_n$ be triangular or trapezoidal fuzzy numbers. Their α-cuts, denoted by $[A_1]^\alpha,..,[A_n]^\alpha$, $[W_1]^\alpha,..,[W_n]^\alpha$, are closed intervals; the elements of $[W_1]^\alpha,..,[W_n]^\alpha$ are non-negative. The α-cut of the fuzzy weighted average of attributes $A_1,..,A_n$ with weights $W_1,..,W_n$ is given by the set

$$\{\text{wa}(x_1,..,x_n,w_1,..,w_n) \mid x_1 \in [A_1]^\alpha,..,x_n \in [A_n]^\alpha, w_1 \in [W_1]^\alpha,..,w_n \in [W_n]^\alpha\} \qquad (2)$$

This set is a closed interval and is computed by computing its extremal values. Algorithms for computing these extremal values, for fixed value of α, are given in the literature. In this paper we will give an algorithm to solve this problem analytically, i.e. we will show how to compute analytical expressions for the extremal values of eq. (2). Also an algorithm for the computation of the inverses of these expressions is given; this enables us to calculate analytically the exact membership functions of the fuzzy weighted average.

In Section 2 we will present an algorithm for the computation of the extremal values of eq. (2) for a fixed value of α, which is based on previous algorithms. This algorithm has an intuitive geometrical interpretation, and therefore it can be easily understood. In Section 3 this algorithm is generalised to give analytical solutions, for triangular or trapezoidal weights and attributes, and it is shown how these solutions can be reversed to give exact membership functions of the fuzzy weighted average. In Section 4, the power of the algorithm is demonstrated by applying it to examples from the literature, giving exact membership functions of the fuzzy weighted average in all cases.

2   Computation of α-Cuts of the Fuzzy Weighted Average

Let us start with the computation of the minimal value of the set of eq. (2). It is shown by Dong and Wong [3] and by Liou and Wang [10] that the minimum of eq. (2) is among the elements where x_i is equal to $\min[A_i]^\alpha$ and w_i equals either $\min[W_i]^\alpha$ or $\max[W_i]^\alpha$. So the problem can be reformulated as the problem of finding the minimal element of the finite set

$$\{\sum_{i=1}^{n} (\min[A_i]^\alpha * w_i) \,\Big/\, \sum_{i=1}^{n} w_i \,\Big|\, w_1 \in \{\min[W_1]^\alpha, \max[W_1]^\alpha\},..,w_n \in \{\min[W_n]^\alpha, \max[W_n]^\alpha\}\} \qquad \textbf{(3)}$$

Let Q be the set of all 2^n mappings from the set $\{1,2,..,n\}$ to the set $\{+,-\}$. These mappings can be seen as states. A state is a partition of the attributes in attributes with maximal weight and attributes with minimal weight. The set of eq. (3) can be denoted as

$$\{\sum_{i=1}^{n} (\min[A_i]^\alpha * W_i^\alpha(q)) \,\Big/\, \sum_{i=1}^{n} W_i^\alpha(q) \,\Big|\, q \in Q\} \qquad \textbf{(4)}$$

where $W_i^\alpha(q)$ stands for $\max[W_i]^\alpha$ if $q(i) = +$ and for $\min[W_i]^\alpha$ if $q(i) = -$.

We define the mapping FWA from states to the real numbers by

$$FWA(q) = \sum_{i=1}^{n} (\min[A_i]^\alpha * W_i^\alpha(q)) \,\Big/\, \sum_{i=1}^{n} W_i^\alpha(q) \quad (5) \qquad \textbf{(5)}$$

Now the problem has become the problem of finding the state q for which FWA(q) is minimal. The first step of the algorithm consists of sorting the set $\{\min[A_i]^\alpha \,|\, 1 <= i <= n\}$. From now on we assume this set is sorted. As a consequence, $\min[A_1]^\alpha <= FWA(q) <= \min[A_n]^\alpha$ for all $q \in Q$. Now let us consider what happens when q changes such that, for some i with $1 <= i <= n$, $q(i)$ changes from $-$ into $+$. This change means that the weight of $\min[A_i]^\alpha$ increases, whereas the other weights remain the same. As a consequence, FWA(q) will change in such a way that it moves towards $\min[A_i]^\alpha$: the absolute value of $FWA(q) - \min[A_i]^\alpha$ decreases, but the sign of $FWA(q) - \min[A_i]^\alpha$ remains the same.

Let q_{min} be the state for which FWA takes its minimal value. Then $q_{min}(i) = +$ if $\min[A_i]^\alpha < FWA(q_{min})$ and $q_{min}(i) = -$ if $\min[A_i]^\alpha > FWA(q_{min})$. We know that $q_{min}(1) = +$ and $q_{min}(n) = -$. Our algorithm to obtain q_{min} is as follows. Let q_i be the element of Q defined by $q_i(j) = +$ for $1 <= j < i$ and $q_i(j) = -$ for $i <= j <= n$. Then $q_{min} = q_i$ for some i with $2 <= i <= n$. To determine q_{min}, start with state q_2. Let us consider a situation with n=6, which is depicted as follows:

```
•------- •-------- •------- ▽-------- •------ •------- •
1+       2-        3-       Fwa        4-      5-        6-
```
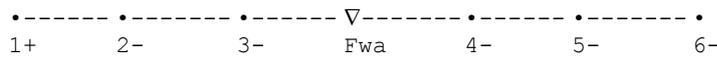
Here the first line shows the real axis with $\min[A_i]^\alpha$ for i=1..6, and the fuzzy weighted average (Fwa) in the present state q_2. The second line shows the indices i, and whether the corresponding weights are maximal (+) or minimal (-) in the present state. We have to change the state in such a way that Fwa becomes as small as possible. Changing the state means changing the weights from maximal to minimal or vice versa. Changing weights 1,4,5, or 6 would increase Fwa. Changing weights 2 or 3 decreases Fwa. Suppose we change weight 3. The value of Fwa decreases, but remains greater than $\min[A_3]^\alpha$. Next we change weight 2. The value of Fwa is further decreased, and can become less than $\min[A_3]^\alpha$. In this case Fwa would decrease even further by restoring the original weight 3. This situation would have been avoided if we had first changed weight 2. If this causes Fwa to become less than $\min[A_3]^\alpha$, the minimum is reached in state q_3. Otherwise, weight 3 is changed as well, and the minimum is reached in state q_4. So it is important that the weight changes are performed from left to right. In Liou & Wang [10] that weight is changed first that causes Fwa to decrease most, which is incorrect.

Let us now return to the general case. The initial state is q_2. Compare $FWA(q_2)$ with $\min[A_2]^\alpha$. If $FWA(q_2) <= \min[A_2]^\alpha$ then $q_{min} = q_2$, else continue with q_3. If $FWA(q_3) <= \min[A_3]^\alpha$ then $q_{min} = q_3$, else continue with q_4, et cetera. This iteration will terminate, since $FWA(q_n) <= \min[A_n]^\alpha$. So, the algorithm to compute the minimum of FWA(q) for $q \in Q$ is

```
sort { min[A_i]^α | 1<=i<=n};
i = 2;
while (FWA(q_i) > min[A_i]^α) i = i+1;
return FWA(q_i);
```

The computation of the maximal value of the set of eq. (2) is quite similar. We now assume that the set $\{max[A_i]^\alpha | 1<=i<=n\}$ is sorted. Note that this ordering may be different from the one above. In the definition of FWA (eq. (5)) we replace $min[A_i]^\alpha$ by $max[A_i]^\alpha$. Let q_{max} be the value of q for which FWA(q) is maximal. Then $q_{max}(i) = -$ if $max[A_i]^\alpha < FWA(q_{max})$ and $q_{max}(i) = +$ if $max[A_i]^\alpha > FWA(q_{max})$. In particular, $q_{max}(1) = -$ and $q_{max}(n) = +$. Let q^i be the element of Q defined by $q^i(j) = -$ for $1<=j<=i$ and $q^i(j) = +$ for $i<j<=n$. Then $q_{max} = q^i$ for some i with $1<=i<=n-1$. To determine q_{max}, start with $q^{(n-1)}$. Compare $FWA(q^{(n-1)})$ with $max[A_{(n-1)}]^\alpha$. If $FWA(q^{(n-1)}) >= max[A_{(n-1)}]^\alpha$ then $q_{max} = q^{(n-1)}$, else continue with $q^{(n-2)}$, et cetera. So, the algorithm to compute the maximum of FWA(q) for $q \in Q$ is

```
sort { max[A_i]^α |1<=i<=n};
i = n-1;
while (FWA(q^i) < max[A_i]^α) i = i-1;
return FWA(q^i);
```

Let us illustrate the algorithm with a small example, with n=6.

i	$[A_i]^\alpha$	$[W_i]^\alpha$	$FWA(q_i)$		i	$[A_i]^\alpha$	$[W_i]^\alpha$	$FWA(q^i)$
1	[1,4]	[1,3]			1	[1,4]	[1,3]	109/16
2	[2,6]	[1,3]	23/8		2	[4,5]	[1,3]	99/14
3	[3,7]	[1,3]	27/10		3	[2,6]	[1,3]	87/12
4	[4,5]	[1,3]	33/12		4	[3,7]	[1,3]	73/10
5	[5,8]	[1,3]	41/14		5	[5,8]	[1,3]	57/8
6	[6,9]	[1,3]	51/16		6	[6,9]	[1,3]	

In the left-hand table the α-cuts $[A_i]^\alpha$ and $[W_i]^\alpha$ for some fixed α are shown in the second and the third column, sorted according to $min[A_i]^\alpha$. The fourth column shows the values of $FWA(q_i)$. The minimum is seen to be 27/10, being the first value from above in the fourth column which is not greater than the corresponding value for $min[A_i]^\alpha$ (in the second column). In the right-hand table the α-cuts $[A_i]^\alpha$ are sorted according to $max[A_i]^\alpha$. The fourth column shows the values of $FWA(q^i)$. The maximum is seen to be 73/10, being the first value from below in the fourth column which is not less than the corresponding value for $max[A_i]^\alpha$.

The extremal values obtained above could also have been found by taking the smallest resp. highest values in the fourth columns of the tables; this is indeed the approach of Chiao [2]. So apparently we did not gain anything, except some geometrical insight, as explained above. However, as it will turn out in the next section, for obtaining an analytical solution it is crucial not to compare values of the fourth column ($FWA(q_i)$ or $FWA(q^i)$) among each other, but instead compare elements of the fourth column with elements in the second column ($[A_i]^\alpha$).

The computational complexity of our algorithm is $O(n*ln(n))$, due to the first step, the sorting of the elements. The second phase, whose computational complexity is $O(n)$, could be optimized by replacing the linear search by binary search, resulting in computational complexity $O(ln(n))$, as in Lee & Park [8]; the overall computational complexity would remain $O(n*ln(n))$, however. Guu [6] has given an algorithm with computational complexity $O(n)$, in which the sorting of the elements is avoided. We have tried to keep our algorithm as simple as possible, in order to be able to generalize it to obtain an analytical solution.

3   Analytical Solution for the Fuzzy Weighted Average

In this section we will show that the algorithm given in the previous section can be generalized to obtain an analytical solution for the membership function of the fuzzy weighted average. There have been two previous

attempts to obtain an analytical solution. Dong and Wong [3] obtained an analytical solution for two small examples. A general method was not given, however. Their approach was to consider the partial derivatives with respect to w_i of eq. (1) in order to obtain the extremal values of this equation. Kao and Liu [7] followed the same approach, and applied it to the same two examples, but also failed to provide a general solution. Our approach is different. We will generalize the algorithm of the previous section, by considering α to be a parameter which ranges over the interval [0,1], instead of being some fixed value. Then, taking the values of the fuzzy weights and fuzzy attributes to be triangular and trapezoidal fuzzy numbers, the extremal values of their α-cuts are linear functions of α.

A trapezoidal fuzzy number will be denoted as a 4–tuple (a,b,c,d) where a,b,c and d are real numbers with $a<=b<=c<=d$. The trapezoidal number (a,b,c,d) has membership function μ, given by

$$
\begin{aligned}
&\mu(x) = 0, &&\text{if } x<=a \\
&\mu(x) = (x-a)/(b-a), &&\text{if } a<x<b \\
&\mu(x) = 1, &&\text{if } b<=x<=c \quad\quad\quad\quad \textbf{(6)} \\
&\mu(x) = (d-x)/(d-c), &&\text{if } c<x<d \\
&\mu(x) = 0, &&\text{if } x>=d
\end{aligned}
$$

The restriction of μ to [a,b] and [c,d] will be referred to as the left-hand side resp. the right-hand side of the trapezoidal number. A triangular fuzzy number is a trapezoidal number of the form (a,b,b,c), and will be denoted by the 3–tuple (a,b,c).

Let the fuzzy weights be given by $W_i = (w_i,x_i,y_i,z_i)$ and the fuzzy attributes by $A_i = (a_i,b_i,c_i,d_i)$. Then we have, for $0<=\alpha<=1$:

$$
\begin{aligned}
&\min[A_i]^\alpha = a_i + \alpha\,(b_i - a_i) \\
&\max[A_i]^\alpha = d_i - \alpha\,(d_i - c_i) \quad\quad\quad\quad \textbf{(7)} \\
&\min[W_i]^\alpha = w_i + \alpha\,(x_i - w_i) \\
&\max[W_i]^\alpha = z_i - \alpha\,(z_i - y_i)
\end{aligned}
$$

Our first aim is to find a function of α which is the minimum of the set of eq. (4) for all α in the interval [0,1]. The first step of the algorithm consists of sorting the set of left-hand sides of the attributes $\{\min[A_i]^\alpha \mid 1<=i<=n\}$. However, this sorting is the same for each α only if these left-hand sides do not intersect. So, we compute all the values of α for which two left-hand sides intersect. Note that coinciding left-hand sides present no problem; therefore they are considered not to intersect each other. Since each left-hand side is a linear function of α (eq. 6a), each pair of left-hand sides can intersect for at most one value of α with $0<=\alpha<=1$. So, there can be at most $n(n-1)/2$ such intersections. In practice, however, it turns out that there are only few intersections, if any at all. The values of α where the intersections occur partition the interval [0,1] in at most $n(n-1)/2+1$ subintervals.

On each of these subintervals the left-hand sides of the A_i do not intersect and the set $\{\min[A_i]^\alpha \mid 1<=i<=n\}$ can be sorted independent of α. We will consider each of these subintervals separately. So, in this step the problem has been reduced to the problem of finding the minimum of the set of eq. (4) for all α in some subinterval [min,max] of [0,1] where the ordering of the set $\{\min[A_i]^\alpha \mid 1<=i<=n\}$ is independent of α.

The next step of the algorithm is to compare $FWA(q_2)$ with $\min[A_2]^\alpha$. For values of α for which $FWA(q_2) <= \min[A_2]^\alpha$ the minimum is $FWA(q_2)$, for the other values of α the computation will continue with q_3. This is done by determining the values of α with $0<=\alpha<=1$ for which $FWA(q_2) = \min[A_2]^\alpha$. From the definition of FWA (eq. (5)) it follows that this equation can be written as:

$$
\sum_{i=1}^{n} (\min[A_i]^\alpha {}_* W_i^\alpha(q_2)) = \min[A_2]^\alpha {}_* \sum_{i=1}^{n} W_i^\alpha(q_2) \quad\quad\quad\quad \textbf{(8)}
$$

From the eq. (7) we find that both sides of this equation are second order polynomials in α. Therefore, solving eq. (8) is trivial, and there are at most two solutions. Those solutions partition the interval [min,max] in at most three subintervals. On each of these subintervals the result of the comparison $FWA(q_2) <= \min[A_2]^\alpha$ is independent of α. On intervals where $FWA(q_2) <= \min[A_2]^\alpha$, the analytical solution is obtained, which is equal to $FWA(q_2)$. On intervals where $FWA(q_2) >= \min[A_2]^\alpha$, the computation continues in state q_3. Iteration of this

process leads to the analytical solution of the minimum of the set of eq. (4) for all α in the interval [min,max]. This solution generally consists of separate solutions for a finite number of subintervals of [min,max]. Repeating this procedure for each of the intervals which were determined in the first step of the algorithm gives the analytical solution of the minimum of the set of eq. (4) for all α in the interval [0,1].

The algorithm can be summarized as follows:

```
Calculate the intersections of the left-hand sides of the attributes;
Partition [0,1] into subintervals according to these intersections;
For each subinterval [min,max]
    Adapt the numbering such that { min[Aᵢ]ᵅ |1<=i<=n} is sorted;
    Exact solution on [min,max] is Proc ([min,max], 2);
```

where `Proc` is defined by

```
Proc (interval, i) ==
    Partition the interval into subintervals according to solutions of FWA(qᵢ) =
    min[Aᵢ]ᵅ ;
    On subintervals where FWA(qᵢ) <= min[Aᵢ]ᵅ the exact solution is FWA(qᵢ);
    On subintervals where FWA(qᵢ) >= min[Aᵢ]ᵅ the exact solution is Proc
    (subinterval, i+1);
```

Note that the key element in this algorithm is the comparison of FWA(q_i) with min$[A_i]^\alpha$, which leads to a second order polynomial equation to be solved. An algorithm which compares values of FWA(q_i) for different values of i, would have led to a third order polynomial equation, which is much more difficult to solve.

The algorithm to calculate a function of α which is the maximum of the set of eq. (4) for all α in the interval [0,1] is quite similar. It can be summarized as follows:

```
Calculate the intersections of the right-hand sides of the attributes;
Partition [0,1] into subintervals according to these intersections;
For each subinterval [min,max]
    Adapt the numbering such that { max[Aᵢ]ᵅ |1<=i<=n} is sorted;
    Exact solution on [min,max] is Proc ([min,max], n-1);
```

where `Proc` is defined by

```
Proc (interval, i) ==
    Partition the interval into subintervals according to solutions of FWA(qⁱ) =
    min[Aᵢ]ᵅ ;
    On subintervals where FWA(qⁱ) <= min[Aᵢ]ᵅ the exact solution is FWA(qⁱ);
    On subintervals where FWA(qⁱ) >= min[Aᵢ]ᵅ the exact solution is Proc
    (subinterval, i-1);
```

The second aim in this section is to show that the exact solutions for the minimum and maximum values of the α-cuts of the fuzzy weighted average can be inverted to give the exact membership function of the fuzzy weighted average. First we will show how to invert the exact solution for the minimum values. In the preceding step of the algorithm, the interval [0,1] has been partioned in a finite number of subintervals, and on each subinterval [min,max] the exact solution is given by FWA(q) for some q∈Q. When the eqs. (7a-7b) are substituted in eq. (5), we find that FWA(q) is a function f of α which takes the form

$$f(\alpha) = (a\alpha^2 + b\alpha + c)/(d\alpha + e) \tag{9}$$

where a,b,c,d,and e are real numbers. The inverse of f is the exact left-hand side of the membership function of the fuzzy weighted average on the interval [f(min), f(max)]; it can be computed by solving α from the equation x = f(α), which can be written as

$$a\alpha^2 + (b - dx)\alpha + c - ex = 0 \tag{10}$$

Let us first consider the case where $a \neq 0$. Here the solution of eq. (10) is given by

$$\mu(x) = (dx - b \pm \sqrt{(dx-b)^2 - 4a(c-ex)}\ )/(2a) \tag{11}$$

where the ambiguity in the sign can be solved with the conditions

$$\mu(f(min)) = min$$
$$\mu(f(max)) = max \tag{12}$$

Next consider the case where $a = 0$ and $dc \neq eb$. Here the eq. (10) is solved by

$$\mu(x) = (ex - c)/(b - dx) \tag{13}$$

on the interval

$$[(b*min + c)/(d*min + e), \quad (b*max + c)/(d*max + e)] \tag{14}$$

Finally consider the case where $a = 0$ and $dc = eb$. In this case $(b\alpha + c)/(d\alpha + e)$ is independent of α, so the inverse does not exist. Then the membership function is non–continuous in $x = c/e$. This occurs for instance when all attributes and weights are crisp numbers (i.e. of the form (a,a,a)), leading to a crisp weighted average, whose membership function is not continuous.

This shows that in each case the exact solution for the minimum values of the α-cuts of the fuzzy weighted average can be inverted, giving the exact solution the left-hand side of the membership function of the fuzzy weighted average. The computation of the exact solution of the right-hand side of the membership function is similar.

4   Examples

In this section we will apply our algorithms to derive exact membership functions to the examples of fuzzy weighted averages which have appeared in the literature. In each case, the fuzzy attributes and fuzzy weights are listed, as well as their α-cuts. This listing, and the assignment of indices from 1 to n, is such that the left-hand sides of the fuzzy attributes are properly ordered on the leftmost subinterval of [0,1]. In most of the cases this is the proper ordering on [0,1], both for the left-hand sides and for the right-hand sides of the fuzzy attributes. On subintervals where the ordering is different, this is clearly indicated. Renumbering fuzzy attributes and fuzzy weights has not been given explicitly, however, so the reader should be aware of the fact that the indices in eq. (5) are with respect to the proper ordering, and not necessarily with respect to the listed ordering.

We have avoided the rounding of real numbers as much as seems reasonable. So, quotients and square roots have not been evaluated to decimal form, except where expressions would otherwise become too unwieldy. To discriminate between decimal notations which are exact and those which are approximations, the latter are followed by an asterisk ([*]).

Example 1

For this example exact membership functions have been derived by Dong and Wong [3] and by Kao and Liu [7]. There are 2 attributes and 2 weights; all of these are triangular fuzzy numbers.

$A_1 = (0,1,2)$ $[A_1]^\alpha = [\alpha, 2-\alpha]$ $W_1 = (0,0.3,0.9)$ $[W_1]^\alpha = [0.3\alpha, 0.9-0.6\alpha]$
$A_2 = (2,3,4)$ $[A_2]^\alpha = [2+\alpha, 4-\alpha]$ $W_2 = (0.4,0.7,1)$ $[W_2]^\alpha = [0.4+0.3\alpha, 1-0.3\alpha]$

Since there are only two attributes, and there are no intersections, the calculation is trivial: the minimum of the α-cuts of the fuzzy weighted average is $FWA(q_2) = (-0.3\alpha^2 + 1.9\alpha + 0.8)/(-0.3\alpha + 1.3)$ and the maximum is $FWA(q^1) = -1.6\alpha + 4$.

This leads to the following membership function for the fuzzy weighted average:

$$\mu(x) = 0 \qquad\qquad\qquad\qquad\qquad \text{if } x \le 8/13 \text{ or } x \ge 4$$

$$\mu(x) = x/2 + 19/6 - (5/3)\sqrt{0.09x^2 - 0.42x + 4.57} \qquad \text{if } 8/13 \le x \le 12/5$$

$$\mu(x) = -5x/8 + 5/2 \qquad\qquad\qquad\qquad \text{if } 12/5 \le x \le 4$$

Example 2

For this example exact membership functions have been derived by Dong and Wong [3] and by Kao and Liu [7]. The example is also treated by Guh, Hon and Lee [4], by Guh, Hon, Wang and Lee [5] and by Liou and Wang [10]. There are 3 attributes and 3 weights; all of these are triangular fuzzy numbers.

$A_1 = (0,1,2)$ $[A_1]^\alpha = [\alpha, 2-\alpha]$ $W_1 = (0,0.3,0.9)$ $[W_1]^\alpha = [0.3\alpha, 0.9-0.6\alpha]$

$A_2 = (2,3,4)$ $[A_2]^\alpha = [2+\alpha, 4-\alpha]$ $W_2 = (0.4,0.7,1)$ $[W_2]^\alpha = [0.4+0.3\alpha, 1-0.3\alpha]$

$A_3 = (4,5,6)$ $[A_3]^\alpha = [4+\alpha, 6-\alpha]$ $W_3 = (0.6,0.8,1)$ $[W_2]^\alpha = [0.6+0.2\alpha, 1-0.2\alpha]$

First, we calculate the minimum of the α-cuts of the fuzzy weighted average. $FWA(q_2) = (-0.1\alpha^2 + 3.3\alpha + 3.2)/(-0.1\alpha + 1.9)$. $FWA(q_2) \le 2+\alpha$ for $0 \le \alpha \le 3/8$, so $FWA(q_2)$ is the minimum for $0 \le \alpha \le 3/8$. $FWA(q_2) \ge 2+\alpha$ for $3/8 \le \alpha \le 1$. For $3/8 \le \alpha \le 1$ the minimum is $FWA(q_3) = (-0.7\alpha^2 + 2.7\alpha + 4.4)/(-0.7\alpha + 2.5)$.

Next we calculate the maximum of the α-cuts of the fuzzy weighted average. $FWA(q^2) = (-0.4\alpha^2 - 0.8\alpha + 7.6)/(0.4\alpha + 1.4)$. $FWA(q^2) \ge 4-\alpha$ for $0 \le \alpha \le 1$, so $FWA(q^2)$ is the maximum for $0 \le \alpha \le 1$.

This leads to the following membership function for the fuzzy weighted average:

$$\mu(x) = 0 \qquad\qquad\qquad\qquad\qquad \text{if } x \le 32/19 \text{ or } x \ge 38/7$$

$$\mu(x) = x/2 + 16.5 - 5\sqrt{0.01x^2 - 0.1x + 12.17} \qquad \text{if } 32/19 \le x \le 19/8$$

$$\mu(x) = x/2 + 27/14 - (5/7)\sqrt{0.49x^2 - 3.22x + 19.61} \qquad \text{if } 19/8 \le x \le 32/9$$

$$\mu(x) = -x/2 - 1 + (5/4)\sqrt{0.16x^2 - 1.6x + 12.8} \qquad \text{if } 32/9 \le x \le 38/7$$

Example 3

This example is treated by Guh, Hon, Wang and Lee [5]. There are 4 attributes and 4 weights; all of these are triangular fuzzy numbers.

$A_1 = (0,1,2)$ $[A_1]^\alpha = [\alpha, 2-\alpha]$ $W_1 = (0,0.3,0.9)$ $[W_1]^\alpha = [0.3\alpha, 0.9-0.6\alpha]$

$A_2 = (2,3,4)$ $[A_2]^\alpha = [2+\alpha, 4-\alpha]$ $W_2 = (0.4,0.7,1)$ $[W_2]^\alpha = [0.4+0.3\alpha, 1-0.3\alpha]$

$A_3 = (4,5,6)$ $[A_3]^\alpha = [4+\alpha, 6-\alpha]$ $W_3 = (0.6,0.8,1)$ $[W_3]^\alpha = [0.6+0.2\alpha, 1-0.2\alpha]$

$A_4 = (5,6,7)$ $[A_4]^\alpha = [5+\alpha, 7-\alpha]$ $W_4 = (0.5,0.8,1)$ $[W_4]^\alpha = [0.5+0.3\alpha, 1-0.2\alpha]$

First, we calculate the minimum of the α-cuts of the fuzzy weighted average. $FWA(q_2) = (0.2\alpha^2 + 5.3\alpha + 5.7)/(0.2\alpha + 2.4)$. $FWA(q_2) \le 2+\alpha$ for $0 \le \alpha \le 1$. $FWA(q_3) = (-0.4\alpha^2 + 4.7\alpha + 6.9)/(-0.4\alpha + 3)$. $FWA(q_3) \ge 4+\alpha$ for $0 \le \alpha \le 1$, so $FWA(q_3)$ is the minimum for $0 \le \alpha \le 1$.

Next, we calculate the maximum of the α-cuts of the fuzzy weighted average. $FWA(q^3) = (-0.6\alpha^2 - 0.4\alpha + 12.2)/(0.6\alpha + 2)$. $FWA(q^3) \ge 6-\alpha$ for $0 \le \alpha \le 0.1$, so $FWA(q^3)$ is the maximum for $0 \le \alpha \le 0.1$. $FWA(q^3) \le 6-\alpha$ for $0.1 \le \alpha \le 1$. $FWA(q^2) = (-0.2\alpha^2 - 3.2\alpha + 14.6)/(0.2\alpha + 2.4)$. $FWA(q^2) \ge 4-\alpha$ for $0.1 \le \alpha \le 1$, so $FWA(q^2)$ is the maximum for $0.1 \le \alpha \le 1$.

This leads to the following membership function for the fuzzy weighted average:

$$\mu(x) = 0 \qquad\qquad\qquad\qquad\qquad \text{if } x \le 23/10 \text{ or } x \ge 61/10$$

$$\mu(x) = x/2 + 47/8 - (5/4)\sqrt{0.16x^2 - 1.04x + 33.13} \qquad \text{if } 23/10 \le x \le 112/26$$

$$\mu(x) = -x/2 - 8 + (5/2)\sqrt{0.04x^2 - 0.64x + 21.92} \qquad \text{if } 112/26 \le x \le 59/10$$

$$\mu(x) = -x/2 - 1/3 + (5/6)\sqrt{0.36x^2 - 4.32x + 29.44} \qquad \text{if } 59/10 \le x \le 61/10$$

Example 4

This example is treated by Lee and Park [8]. There are 5 attributes and 5 weights; all of these are triangular fuzzy numbers.

$A_1 = (1,2,3)$ $[A_1]^\alpha = [1+\alpha, 3-\alpha]$ $W_1 = (1,2,5)$ $[W_1]^\alpha = [1+\alpha, 5-3\alpha]$
$A_2 = (2,5,7)$ $[A_2]^\alpha = [2+3\alpha, 7-2\alpha]$ $W_2 = (2,2.5,3)$ $[W_2]^\alpha = [2+\alpha/2, 3-\alpha/2]$
$A_3 = (6,8,9)$ $[A_3]^\alpha = [6+2\alpha, 9-\alpha]$ $W_3 = (4,7,9)$ $[W_3]^\alpha = [4+3\alpha, 9-2\alpha]$
$A_4 = (7,9,10)$ $[A_4]^\alpha = [7+2\alpha, 10-\alpha]$ $W_4 = (3,4,7)$ $[W_4]^\alpha = [3+\alpha, 7-3\alpha]$
$A_5 = (10,11,12)$ $[A_5]^\alpha = [10+\alpha, 12-\alpha]$ $W_5 = (2,3,4)$ $[W_5]^\alpha = [2+\alpha, 4-\alpha]$

First, we calculate the minimum of the α-cuts of the fuzzy weighted average. $FWA(q_2) = (7.5\alpha^2 + 60\alpha + 74)/(2.5\alpha + 16)$. $FWA(q_2) \geq 2+3\alpha$ for $0<=\alpha<=1$. $FWA(q_3) = (4.5\alpha^2 + 61\alpha + 76)/(1.5\alpha + 17)$. $FWA(q_3) <= 6+2\alpha$ for $0<=\alpha<=1$, so $FWA(q_3)$ is the minimum for $0<=\alpha<=1$.

Next, we calculate the maximum of the α-cuts of the fuzzy weighted average. $FWA(q^4) = (-5\alpha^2 + 15.5\alpha + 131)/(4.5\alpha + 14)$. $FWA(q^4) <= 10-\alpha$ for $0<=\alpha<=1$. $FWA(q^3) = (-\alpha^2 - 28.5\alpha + 171)/(0.5\alpha + 18)$. $FWA(q^3) \geq 9-\alpha$ for $0 <= \alpha <= 9\sqrt{3}-15$, so $FWA(q^3)$ is the maximum for $0<=\alpha <= 9\sqrt{3}-15$. $FWA(q^3) <= 9-\alpha$ for $9\sqrt{3}-15<=\alpha<=1$. $FWA(q^2) = (4\alpha^2 - 78.5\alpha + 216)/(-4.5\alpha + 23)$. $FWA(q^2) \geq 7-2\alpha$ for $9\sqrt{3}-15<=\alpha<=1$, so $FWA(q^2)$ is the maximum for $9\sqrt{3}-15 <= \alpha <= 1$.

These results are in accordance with the results by Lee and Park in [7] for α=0 and α=1. However, their claim that the fuzzy weighted average is a fuzzy triangular number is incorrect. Instead, the membership function of the fuzzy weighted average is calculated to be:

$\mu(x) = 0$ if $x <= 76/17$ or $x >= 19/2$
$\mu(x) = x/6 - 61/9 + (1/9)\sqrt{2.25x^2 + 123x + 2353}$ if $76/17 <= x <= 283/37$
$\mu(x) = -9x/16 + 157/16 - (1/8)\sqrt{20.25x^2 - 338.5x + 2706.25}$ if $283/37 <= x <= 24-9\sqrt{3}$
$\mu(x) = -x/4 - 57/4 + (1/2)\sqrt{0.25x^2 - 43.5x + 1496.25}$ if $24-9\sqrt{3} <= x <= 19/2$

Example 5

This example is treated by Kao and Liu [7]. There are 3 attributes and 3 weights; the weights are are triangular fuzzy numbers and the attributes are trapeizoidal fuzzy numbers.

$A_1 = (-2,1,2,3)$ $[A_1]^\alpha = [-2+3\alpha, 3-\alpha]$ $W_1 = (0,0.3,0.9)$ $[W_1]^\alpha = [0.3\alpha, 0.9-0.6\alpha]$
$A_2 = (1,2,3,5)$ $[A_2]^\alpha = [1+\alpha, 5-2\alpha]$ $W_2 = (0.4,0.7,1)$ $[W_2]^\alpha = [0.4+0.3\alpha, 1-0.3\alpha]$
$A_3 = (2,3,6,7)$ $[A_3]^\alpha = [2+\alpha, 7-\alpha]$ $W_3 = (0.6,0.8,1)$ $[W_3]^\alpha = [0.6+0.2\alpha, 1-0.2\alpha]$

First, we calculate the minimum of the α-cuts of the fuzzy weighted average. $FWA(q_2) = (-1.3\alpha^2 + 5.6\alpha - 0.2)/(-0.1\alpha + 1.9)$. $FWA(q_2) <= 1+\alpha$ for $0 <= \alpha <= (19-\sqrt{109})/12$, so $FWA(q_2)$ is the minimum for $0 <= \alpha <= (19-\sqrt{109})/12$. $FWA(q_2) \geq 1+\alpha$ for $(19-\sqrt{109})/12 <= \alpha <= 1$, so for $(19-\sqrt{109})/12 < =\alpha < =1$ the minimum is $FWA(q_3) = (-1.9\alpha^2 + 5.6\alpha + 0.4)/(-0.7\alpha + 2.5)$.

Next we calculate the maximum of the α-cuts of the fuzzy weighted average. $FWA(q^3) = (-0.7\alpha^2 - 0.8\alpha + 9)/(0.4\alpha + 1.4)$. $FWA(q^3) \geq 5-2\alpha$ for $0<= <=1$, so $FWA(q^3)$ is the maximum for $0<=\alpha<=1$.

This leads to the following membership function for the fuzzy weighted average:

$\mu(x) = 0$ if $x <= -2/19$ or $x >= 45/7$
$\mu(x) = x/26 + 28/13 - (5/13)\sqrt{0.01x^2 - 8.76x + 30.32}$ if $-2/19 <= x <= (31-\sqrt{109})/12$
$\mu(x) = 7x/38 + 28/19 - (5/19)\sqrt{0.49x^2 - 11.16x + 34.4}$ if $(31-\sqrt{109})/12 <= x <= 41/18$
$\mu(x) = 1$ if $41/18 <=x <= 75/18$
$\mu(x) = -2x/7 - 4/7 + (5/7)\sqrt{0.16x^2 - 3.28x + 25.84}$ if $75/18 <= x <= 45/7$

Example 6

This example is treated by Chiao[2]. There are 8 attributes and weights; all of these are fuzzy triangular numbers.

$A_1 = (0,0.1,0.53)$ $\qquad [A_1]^\alpha = [0.1\alpha, 0.53–0.43\alpha]$ $\qquad W_1 = (0.55,0.66,0.67)$ $\qquad [W_1]^\alpha = [0.55+0.11\alpha, 0.67–0.01\alpha]$

$A_2 = (3.78,6.12,8.63)$ $\qquad [A_2]^\alpha = [3.78+2.34\alpha, 8.63–2.51\alpha]$ $\qquad W_2 = (0.52,0.80,0.85)$ $\qquad [W_2]^\alpha = [0.52+0.28\alpha, 0.85–0.05\alpha]$

$A_3 = (4.56,5.85,7.17)$ $\qquad [A_3]^\alpha = [4.56+1.29\alpha, 7.17–1.32\alpha]$ $\qquad W_3 = (0.62,0.75,0.8)$ $\qquad [W_3]^\alpha = [0.62+0.13\alpha, 0.8–0.05\ \alpha]$

$A_4 = (5.08,6.11,7.88)$ $\qquad [A_4]^\alpha = [5.08+1.03\alpha, 7.88–1.77\alpha]$ $\qquad W_4 = (0,0.07,0.1)$ $\qquad [W_4]^\alpha = [0.07\alpha, 0.1–0.03\alpha]$

$A_5 = (5.51,7.22,9.59)$ $\qquad [A_5]^\alpha = [5.51+1.71\alpha, 9.59–2.37\alpha]$ $\qquad W_5 = (0.09,0.38,0.46)$ $\qquad [W_5]^\alpha = [0.09+0.29\alpha, 0.46–0.08\alpha]$

$A_6 = (7.14,7.21,7.27)$ $\qquad [A_6]^\alpha = [7.14+0.07\alpha,7.27–0.06\alpha]$ $\qquad W_6 = (0.44,0.63,0.91)$ $\qquad [W_6]^\alpha = [0.44+0.19\alpha,0.91–0.28\alpha]$

$A_7 = (8.14,9.85,10)$ $\qquad [A_7]^\alpha = [8.14+1.71\alpha, 10–0.15\alpha]$ $\qquad W_7 = (0.59,0.61,0.85)$ $\qquad [W_7]^\alpha = [0.59+0.02\alpha, 0.85–0.24\alpha]$

$A_8 = (8.67,9.26,10)$ $\qquad [A_8]^\alpha = [8.67+0.59\alpha, 10–0.74\alpha]$ $\qquad W_8 = (0.03,0.18,0.29)$ $\qquad [W_8]^\alpha = [0.03+0.15\alpha, 0.29–0.11\alpha]$

First, we calculate the minimum of the α-cuts of the fuzzy weighted average. There are 4 intersections of left-hand sides: A_2 and A_3 intersect at 78/105, A_2 and A_4 intersect at 130/131, A_5 and A_6 intersect at 163/164 and A_7 and A_8 intersect at 53/112. This means that we have to deal with subintervals [0,53/112], [53/112,78/105], [78/105,130/131], [130/131,163/164] and [163/164,1] separately.

Subinterval [0, 53/112]: Ordering $[A_1,A_2,A_3,A_4,A_5,A_6,A_7,A_8]$.
$FWA(q_2) = (1.5259\alpha^2 + 9.7195\alpha + 13.493)/(1.12\alpha + 2.96)$.
$FWA(q_2) >= 3.78 + 2.34\alpha$ for $0 <= \alpha <= 53/112$.
$FWA(q_3) = (0.7537\alpha^2 + 9.2443\alpha + 14.7404)/(0.79\alpha + 3.29)$.
$FWA(q_3) <= 4.56+1.29\alpha$ for $0 <= \alpha <= 0.19463^*$.
$FWA(q_3)$ is the minimum for $0 <= \alpha <= 0.19463^*$.
$FWA(q_3) >= 4.56+1.29\alpha$ for $0.19463^* <= \alpha <= 53/112$.
$FWA(q_4) = (0.5215\alpha^2 + 8.6557\alpha + 15.5612)/(0.61\alpha + 3.47)$.
$FWA(q_4) <= 5.08 + 1.03\alpha$ for $0.19463^* <= \alpha <= 53/112$.
$FWA(q_4)$ is the minimum for $0.19463^* <= \alpha <= 53/112$.

Subinterval [53/112,78/105]: Ordering $[A_1,A_2,A_3,A_4,A_5,A_6,A_8,A_7]$.
$FWA(q_2) = (1.5259\alpha^2 + 9.7195\alpha + 13.493)/(1.12\alpha + 2.96)$.
$FWA(q_2) >= 3.78+2.34\alpha$ for $53/112<=\alpha<=78/105$.
$FWA(q_3) = (0.7537\alpha^2 + 9.2443\alpha + 14.7404)/(0.79\alpha + 3.29)$.
$FWA(q_3) >= 4.56+1.29\alpha$ for $53/112 <= \alpha <= 78/105$.
$FWA(q_4) = (0.5215\alpha^2 + 8.6557\alpha + 15.5612)/(0.61\alpha + 3.47)$.
$FWA(q_4) <= 5.08+1.03\alpha$ for $53/112 <= \alpha <= 78/105$.
$FWA(q_4)$ is the minimum for $53/112 <= \alpha <= 78/105$.

Subinterval [78/105, 130/131]: Ordering $[A_1,A_3,A_2,A_4,A_5,A_6,A_8,A_7]$.
$FWA(q_2) = (1.5259\alpha^2 + 9.7195\alpha + 13.493)/(1.12\alpha + 2.96)$.
$FWA(q_2) >= 4.56 + 1.29\alpha$ for $78/105 <= \alpha <= 130/131$.
$FWA(q_3) = (1.2937\alpha^2 + 9.1309\alpha + 14.3138)/(0.94\alpha + 3.14)$.
$FWA(q_3) <= 3.78+2.34\alpha$ for $0.93436^* <= \alpha <= 130/131$.
$FWA(q_3)$ is the minimum for $0.93436^* <= \alpha <= 130/131$.
$FWA(q_3) >= 3.78+2.34\alpha$ for $78/105 <= \ \alpha <= 0.93436^*$.
$FWA(q_4) = (0.5215\alpha^2 + 8.6557\alpha + 15.5612)/(0.61\alpha + 3.47)$.
$FWA(q_4) <=5.08 + 1.03\alpha$ for $78/105 <= \alpha <= 0.93436^*$
$FWA(q_4)$ is the minimum for $78/105 <= \alpha <= 0.93436^*$.

Subinterval [130/131, 163/164]: Ordering $[A_1, A_3, A_4, A_2, A_5, A_6, A_8, A_7]$.

$FWA(q_2) = (1.5259\alpha^2 + 9.7195\alpha + 13.493)/(1.12\alpha + 2.96)$.

$FWA(q_2) >= 4.56 + 1.29\alpha$ for $130/131 <= \alpha <= 163/164$.

$FWA(q_3) = (1.2937\alpha^2 + 9.1309\alpha + 14.3138)/(0.94\alpha + 3.14)$.

$FWA(q_3) <= 3.78 + 2.34\alpha$ for $130/131 <= \alpha <= 163/164$.

$FWA(q_3)$ is the minimum for $130/131 <= \alpha <= 163/164$.

Subinterval [163/164, 1]: Ordering $[A_1, A_3, A_4, A_2, A_6, A_5, A_8, A_7]$.

$FWA(q_2) = (1.5259\alpha^2 + 9.7195\alpha + 13.493)/(1.12\alpha + 2.96)$.

$FWA(q_2) >= 4.56 + 1.29\alpha$ for $163/164 <= \alpha <= 1$.

$FWA(q_3) = (1.2937\alpha^2 + 9.1309\alpha + 14.3138)/(0.94\alpha + 3.14)$.

$FWA(q_3) <= 3.78 + 2.34\alpha$ for $163/164 <= \alpha <= 1$.

$FWA(q_3)$ is the minimum for $163/164 <= \alpha <= 1$.

Putting the pieces together, we have found that the minimum is

$(0.7537\alpha^2 + 9.2443\alpha + 14.7404)/(0.79\alpha + 3.29)$.   if $0 <= \alpha <= 0.19463^*$.

$(0.5215\alpha^2 + 8.6557\alpha + 15.5612)/(0.61\alpha + 3.47)$.   if $0.19463^* <= \alpha <= 0.93436^*$.

$(1.2937\alpha^2 + 9.1309\alpha + 14.3138)/(0.94\alpha + 3.14)$.   if $0.93436^* <= \alpha <= 1$

Next we calculate the maximum of the α-cuts of the fuzzy weighted average. There are 2 intersections of right-hand sides: A_2 and A_6 intersect at $136/245$ and A_4 and A_6 intersect at $61/171$..

Subinterval [0, 61/171]: Ordering $[A_1, A_3, A_6, A_4, A_2, A_5, A_8, A_7]$.

$FWA(q^7) = (-1.8193\alpha^2 + 4.4713\alpha + 22.0864)/(0.98\alpha + 3.1)$.

$FWA(q^7) <= 10 - 0.74\alpha$ for $0 <= \alpha <= 61/171$.

$FWA(q^6) = (-1.6269\alpha^2 + 1.6789\alpha + 24.6864)/(0.72\alpha + 3.36)$.

$FWA(q^6) <= 9.59 - 2.37\alpha$ for $0 <= \alpha <= 61/171$.

$FWA(q^5) = (-0.75\alpha^2 - 2.7463\alpha + 28.2347)/(0.35\alpha + 3.73)$.

$FWA(q^5) <= 8.63 - 2.51\alpha$ for $0 <= \alpha <= 61/171$.

$FWA(q^4) = (0.0783\alpha^2 - 6.4225\alpha + 31.0826)/(0.02\alpha + 4.06)$.

$FWA(q^4) <= 7.88 - 1.77\alpha$ for $0 <= \alpha <= 61/171$.

$FWA(q^3) = (0.2553\alpha^2 - 7.3875\alpha + 31.8706)/(-0.08\alpha + 4.16)$.

$FWA(q^3) >= 7.27 - 0.06\alpha$ for $0 <= \alpha <= 0.25062^*$.

$FWA(q^3)$ is the maximum for $0 <= \alpha <= 0.25062^*$.

$FWA(q^3) <= 7.27 - 0.06\alpha$ for $0.25062^* <= \alpha <= 61/171$.

$FWA(q^2) = (0.2835\alpha^2 - 10.8326\alpha + 35.2875)/(-0.55\alpha + 4.63)$.

$FWA(q^2) >= 7.17 - 1.32\alpha$ for $0.25062^* <= \alpha <= 61/171$.

$FWA(q^2)$ is the maximum for $0.25062^* <= \alpha <= 61/171$.

Subinterval [61/171, 136/245]: Ordering $[A_1, A_3, A_4, A_6, A_2, A_5, A_8, A_7]$.

$FWA(q^7) = (-1.8193\alpha^2 + 4.4713\alpha + 22.0864)/(0.98\alpha + 3.1)$.

$FWA(q^7) <= 10 - 0.74\alpha$ for $61/171 <= \alpha <= 136/245$.

$FWA(q^6) = (-1.6269\alpha^2 + 1.6789\alpha + 24.6864)/(0.72\alpha + 3.36)$.

$FWA(q^6) <= 9.59 - 2.37\alpha$ for $61/171 <= \alpha <= 136/245$.

$FWA(q^5) = (-0.75\alpha^2 - 2.7463\alpha + 28.2347)/(0.35\alpha + 3.73)$.

$FWA(q^5) <= 8.63 - 2.51\alpha$ for $61/171 <= \alpha <= 136/245$.

$FWA(q^4) = (0.0783\alpha^2 - 6.4225\alpha + 31.0826)/(0.02\alpha + 4.06)$.

$FWA(q^4) <= 7.27 - 0.06\alpha$ for $61/171 <= \alpha <= 136/245$.

$FWA(q^3) = (0.1065\alpha^2 - 9.8676\alpha + 34.4995)/(-0.45\alpha + 4.53)$.

$FWA(q^3) <= 7.88 - 1.77\alpha$ for $61/171 <= \alpha <= 136/245$.

$FWA(q^2) = (0.2835\alpha^2 - 10.8326\alpha + 35.2875)/(-0.55\alpha + 4.63)$.

$FWA(q^2) >= 7.17 - 1.32\alpha$ for $61/171 <= \alpha <= 136/245$.

$FWA(q^2)$ is the maximum for $61/171 <= \alpha <= 136/245$.

Subinterval [136/245,1]: Ordering $[A_1,A_3,A_4,A_2,A_6,A_5,A_8,A_7]$.

$FWA(q^7) = (-1.8193\alpha^2 + 4.4713\alpha + 22.0864)/(0.98\alpha + 3.1)$.

$FWA(q^7) \le 10 - 0.74\alpha$ for $136/245 \le \alpha \le 1$.

$FWA(q^6) = (-1.6269\alpha^2 + 1.6789\alpha + 24.6864)/(0.72\alpha + 3.36)$.

$FWA(q^6) \le 9.59 - 2.37\alpha$ for $136/245 \le \alpha \le 1$.

$FWA(q^5) = (-0.75\alpha^2 - 2.7463\alpha + 28.2347)/(0.35\alpha + 3.73)$.

$FWA(q^5) \le 7.27 - 0.06\alpha$ for $136/245 \le \alpha \le 1$.

$FWA(q^4) = (-0.7218\alpha^2 - 6.1914\alpha + 31.6516)/(-0.12\alpha + 4.2)$.

$FWA(q^4) \le 8.63 - 2.51\alpha$ for $136/245 \le \alpha \le 1$.

$FWA(q^3) = (0.1065\alpha^2 - 9.8676\alpha + 34.4995)/(-0.45\alpha + 4.53)$.

$FWA(q^3) \le 7.88 - 1.77\alpha$ for $136/245 \le \alpha \le 1$.

$FWA(q^2) = (0.2835\alpha^2 - 10.8326\alpha + 35.2875)/(-0.55\alpha + 4.63)$.

$FWA(q^2) \ge 7.17 - 1.32\alpha$ for $136/245 \le \alpha \le 1$.

$FWA(q^2)$ is the maximum for $136/245 \le \alpha \le 1$.

Putting the pieces together, we have found that the maximum is

$(0.2553\alpha^2 - 7.3875\alpha + 31.8706)/(-0.08\alpha + 4.16)$ if $0 \le \alpha \le 0.25062^*$,

$(0.2835\alpha^2 - 10.8326\alpha + 35.2875)/(-0.55\alpha + 4.63)$ if $0.25062^* \le \alpha \le 1$.

This leads to the following membership function for the fuzzy weighted average:

$\mu(x) = 0$ $\qquad\qquad$ if $x \le 4.480^*$

$\mu(x) = 0.524^*x - 6.133^* + 0.663^*\sqrt{0.6241x^2 - 4.687^*x + 41.018^*}$ $\qquad$ if $4.480^* \le x \le 4.811^*$

$\mu(x) = 0.585^*x - 8.299^* + 0.959^*\sqrt{0.3721x^2 - 3.322^*x + 42.460^*}$ $\qquad$ if $4.811^* \le x \le 5.966^*$

$\mu(x) = 0.363^*x - 3.529^* + 0.386^*\sqrt{0.8836x^2 - 0.917^*x + 9.302^*}$ $\qquad$ if $5.966^* \le x \le 6.063^*$

$\mu(x) = -0.970^*x + 19.105^* - 1.764^*\sqrt{0.3025x^2 - 6.665^*x + 77.329^*}$ $\qquad$ if $6.063^* \le x \le 7.255^*$

$\mu(x) = -0.157^*x + 14.468^* - 1.958^*\sqrt{0.0064x^2 + 3.066^*x + 22.029^*}$ $\qquad$ if $7.255^* \le x \le 7.661^*$

$\mu(x) = 0$ $\qquad\qquad$ if $x \ge 7.661^*$

5   Conclusion

A lot of research effort has been invested into the development of algorithms for the calculation of fuzzy weighted averages. Where the computational complexity of the algorithms improved in the course of the time, leading to the linear algorithm of Guu [6], the approach has always been to compute the α-cuts of the fuzzy weighted average for fixed value of α. As a consequence, one can only compute a finite number of values of the membership function of the fuzzy weighted average. In this paper we have presented an algorithm for the computation of the membership function of the fuzzy weighted average analytically, for triangular or trapeizodal weights and attributes. Our approach has been to generalise a simple, but not optimally efficient, single-α algorithm. Using our algorithm, one no longer needs to approximate the membership function from a finite number of values. The feasibility of our algorithm has been demonstrated by the explicit calculation of the exact membership functions of the fuzzy weighted averages of examples from the literature.

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