Rankings from Fuzzy Pairwise Comparisons

Pim van den Broek and Joost Noppen
Faculty of Electrical Engineering, Mathematics and Computer Science
University of Twente
P.O. Box 217, 7500 AE Enschede, The Netherlands
email: {pimvdb,noppen}@ewi.utwente.nl

Abstract

We propose a new method for deriving rankings from fuzzy pairwise comparisons. It is based on the observation that quantification of the uncertainty of the pairwise comparisons should be used to obtain a better crisp ranking, instead of a fuzzified version of the ranking obtained from crisp pairwise comparisons. With our method, a crisp ranking is obtained by solving a linear programming problem, when the fuzzy pairwise comparisons are fuzzy triangular numbers. Our method simplifies the recent method by Mikhailov.

1. Introduction

In multicriteria decision-making, the problem of deriving rankings from pairwise comparisons is discussed extensively ever since the introduction of the analytic hierarchy process (AHP) by Saaty [1]. Several authors presented methods for the case where the pairwise fuzzy comparisons are fuzzy numbers [2-6]. We will argue that all these methods have drawbacks, and present a new, simple method which does not suffer from these drawbacks.

A ranking w of order N is a sequence of N positive numbers $w_1,w_2,..,w_N$. The ranking w determines a ranking matrix A, with elements $A_{ij} = w_i/w_j$. A ranking matrix A satisfies

1. $A_{ij}>0$, for all $1<=i,j<=N$
2. $A_{ij} = 1/A_{ji}$, for all $1<=i,j<=N$;
   special case: $A_{ii} = 1$, for all $i<=i<=N$
3. $A_{ij}*A_{jk}=A_{ik}$, for all $1<=i,j,k<=N$

Two rankings are said to be equivalent if they have the same ranking matrix. It follows that two rankings v and w are equivalent if and only if they are equal up to a multiplicative constant: $v_i = a*w_i$, for all $1<=i<=N$, and each ranking is equivalent with a ranking w with $w_1+w_2+...+w_N = 1$.

If a matrix A satisfies the conditions 1,2 and 3 above, then it is the ranking matrix of some ranking w. This ranking w can be determined in a number of ways, for instance:

a. w = any column of A.
b. w_i is the N^{th} root of the product of row i. (So w is the multiplicative average of the columns of A.)
c. w is eigenvector of A with eigenvalue N.
d. w minimizes the sum of dist(A_{ij}, w_i/w_j), where dist is a distance function (metric).

The problem now is: given a matrix A which satisfies the constraints 1 and 2, but does not satisfy constraint 3, determine a reasonable ranking for A. Mathematically speaking, the problem is ill-defined, since "reasonable" is not defined. So, we are asked to give a model for a reasonable ranking. We can use a. or b. above, where b. seems to be more reasonable then a. We can use c., if we generalise it to: w is eigenvector of A with greatest eigenvalue. This is the approach taken by Saaty [1] in the AHP method; a variant is given by Cogger and Yu [7]. An overview of d. is given by Golany and Kress [8], for the distance functions least squares, weighted least squares, logarithmic least squares and logarithmic least absolute values.

Note that there is apparently some (unquantified) uncertainty in the matrix elements A_{ij}. Otherwise, condition 3 would not have been violated. So, w is uncertain, not only because our model cannot be validated, but also because of the uncertainty of the matrix elements A_{ij} from which it is derived.

Now consider the case where the matrix elements A_{ij} for i/=j are fuzzy numbers. Buckley [2] and

Wagenknecht and Hartmann [3] generalised method b, and Van Laarhoven and Pedrycz [4], De Boender, De Graan and Lootsma [5] and Mikhailov [6] generalized method d.

In [2-5], the authors compute the uncertainty of the ranking, given the uncertainty of the comparisons. In the next section we will argue that this is not the appropriate way to deal with this problem, and that the right way to approach fuzzy pairwise comparisons is to use the quantification of the uncertainty of the pairwise comparisons to obtain a better crisp ranking, as in [6].

2. Dealing with fuzzy estimations

We will consider a very simple example of fuzzy estimations. Consider the function *average*, which computes the average of two real numbers. A fuzzy version of this function may be defined with Zadeh's extension principle; for instance, given the fuzzy triangular numbers (10,20,30) and (24,26,28), their average will be (17,23,29).

Now suppose that x and y both are crisp estimations of a unknown number z. The best thing we can do to define z, is to let z be the average of x and y. Next suppose that the uncertainty in the estimations is quantified by letting x and y be fuzzy triangular numbers. For instance, x = (10,20,30) and y = (24,26,28). Is the best value for z the value (17,23,29), given by the fuzzified average? The answer must clearly be: no. Since the uncertainty in y is much less than the uncertainty in x, we expect z to be closer to 26 than to 20. A better approach is to let z be the value for which the minimum of the membership functions for x and y is maximal. This gives the value z = 25.

The same idea can be applied to the case of fuzzy pairwise comparisons. Also here, quantification of the uncertainty in the comparisons can be used to obtain a better crisp ranking, instead of a fuzzification of the ranking obtained from the crisp pairwise comparisons.

3. New approach

The problem in the case of crisp pairwise comparisons is that it is not possible, in general, to determine a ranking w such that $A_{ij} = w_i/w_j$, for all $1<=i,j<= N$. When A_{ij} is a fuzzy number, however, the condition $A_{ij} = w_i/w_j$ can be considered to be true to the degree $\mu_{ij}(w_i/w_j)$, where μ_{ij} is the membership function of A_{ij}. Then the condition that $A_{ij} = w_i/w_j$ for all $1<=i,j<= N$ is equal to the minimum of the set $\{\mu_{ij}(w_i/w_j) \mid 1<=i,j<= N\}$. Our proposed ranking is the ranking for which this minimum is as large as possible.

So, the first cornerstone of our approach is: the ranking w has the property that, for all $1<=i,j<= N$, w_i/w_j belongs to the α–cut of A_{ij}, where α is as large as possible.

The second cornerstone of our approach is the choice of fuzzy sets for modeling A_{ij}. It is common practice to use fuzzy triangular numbers for modeling A_{ij}. However, where the A_{ij} are defined on a multiplicative (geometric) scale, the piecewise linearity of the membership functions of fuzzy triangular numbers corresponds to an additive (linear) scale. Therefore, in our approach we will use fuzzy triangular numbers to model the logarithms of the A_{ij}. The fuzzy numbers A_{ij} themselves can be obtained by exponentiation of fuzzy triangular numbers via the extension principle. However, as it will turn out in the section on computation below, these exponentiation is not needed in actual calculations.

4. Comparison with Mikhailov's methods

Mikhailov's paper [6] contains two methods. In his first method, he considers α-cuts of the fuzzy numbers A_{ij}; for particular α he finds the ranking which is optimal with respect to tolerance parameters, which are provided by the decision-maker. These tolerance parameters describe how the matrix elements w_i/w_j are allowed to lie outside the α -cut of A_{ij}. The final ranking is then obtained by aggregating the rankings for a set of values of α.

We identify two drawbacks of this approach. First, the uncertainty in A_{ij} is quantified twice: by the fuzzy numbers A_{ij}, and by the tolerance parameters. Second, it is only a matter of taste how the aggregation of the results for different values of α should be done.

Comparing our method with Mikhailov's method we see that both methods seek solutions within α-cuts. Mikhailov determines a solution for some set of values of α. Since for α close to 1, the α–cuts will be too small to contain a solution, he needs some freedom to go outside the α–cuts, for which he introduces the tolerance parameters. Our method seeks a solution for a single value of α: the highest value for which the α–cuts contain a solution. So we need no tolerance parameters, and no aggregation of results for different values of α.

The difference between Mikhailov's second method and our method is the choice of membership functions: he uses fuzzy triangular numbers for the pairwise comparisons whereas we use fuzzy triangular numbers for the logarithms of the pairwise comparisons. This difference is crucial, since his choice leads to non-linear programs, whereas our choice leads to linear programs, as is shown in the next section.

5. Computation

In this section we will show that in order to calculate a ranking with our method, a linear programming problem has to be solved.

We define $v_i = \ln(w_i)$ and $B_{ij} = \ln(A_{ij})$ .Then the equation $A_{ij}=w_i/w_j$ becomes $B_{ij}=v_i-v_j$, $A_{ii}=1$ becomes $B_{ii}=0$, and $A_{ij}=1/A_{ji}$ becomes $B_{ij}=-B_{ji}$. Let the B_{ij} be fuzzy triangular numbers: $B_{ij} = (l_{ij},m_{ij},r_{ij})$. The α-cut of B_{ij} at α is the interval $[l_{ij} + \alpha(m_{ij}-l_{ij}),\ r_{ij} - \alpha(r_{ij}-m_{ij})]$. Without loss of generality we may take $v_1=0$ (i.e. $w_1=1$). We have an optimization problem with N variables: α and v_i with $2<=i<=N$. The expression to be optimized is just α. The constraints that should be satisfied are, for all $1<=i,j<=N$, that v_i-v_j belongs to the α-cut at α of B_{ij}, i.e.

$$l_{ij} + \alpha(m_{ij}-l_{ij}) <= v_i-v_j <= r_{ij} - \alpha(r_{ij}-m_{ij}). \quad (1)$$

For i=j the constraints are satisfied since $(l_{ii},m_{ii},r_{ii}) = (0,0,0)$. The constraint for i and j is satisfied if and only if the constraint for j and i is satisfied, due to the symmetry. Therefore, only the constraints with i>j need to be considered. So our problem is a linear programming problem with N variables and N^2-N linear constraints. In order to apply the simplex algorithm, the variables should satisfy positivity conditions. From eq. (1), with j=1, it follows, since $v_1=0$, that $v_i >= l_{i1} + \alpha(m_{i1}-l_{i1})$. With the variable transformation $u_i = v_i - l_{i1} - \alpha(m_{i1}-l_{i1})$ these N-1 constraints are just the positivity condition for the new variables u_i. In this form, the linear programming problem has N variables and $(N-1)^2$ constraints, and can be solved by the simplex algorithm.

6. Summary of the method

Step 1: Obtain triangular fuzzy numbers (l_{ij},m_{ij},r_{ij}) for the logarithms of the fuzzy pairwise comparisons A_{ij} for $1<=j<i<=N$.

Step 2: Solve the linear programming problem (using the simplex algorithm) with variables α and u_i ($2<=i<=N$) and the following constraints:

$$u_i-u_j <= r_{ij} - \alpha(r_{ij}-m_{ij}) - l_{i1} - \alpha(m_{i1}-l_{i1}) + l_{j1} + \alpha(m_{j1}-l_{j1}) \quad \text{for all } 2<=j<i<=N$$

$$u_i-u_j >= l_{ij} + \alpha(m_{ij}-l_{ij}) - l_{i1} - \alpha(m_{i1}-l_{i1}) + l_{j1} + \alpha(m_{j1}-l_{j1}) \quad \text{for all } 2<=j<i<=N$$

$$u_i <= r_{i1} - l_{i1} - \alpha(r_{i1}-l_{i1}) \quad \text{for all } 2<=i<=N$$

Step 3: $w_1 = 1$ and $w_i = \exp(u_i + l_{i1} + \alpha(m_{i1}-l_{i1}))$ for all $2<=i<=N$.

Step 4: normalize w.

7. Example

We consider the same example which has been shown in [2,4,5,6] where N=3 and the fuzzy pairwise comparisons are given by A_{21} = (2.5, 3, 3.5), A_{31} = (4, 5, 6) and A_{32} = (1.5 ,2, 2.5). Their logarithms are approximated by B_{21} = (0.916, 1.099, 1.253), B_{31} = (1.386, 1.609, 1.792) and B_{32} = (0.405, 0.693, 0.916).

The 4 constraints are:

$$u2 + 0.336\alpha <= 0.336$$
$$u3 + 0.405\alpha <= 0.405$$
$$-u2 + u3 + 0.264\alpha <= 0.446$$
$$u2 - u3 + 0.247\alpha <= 0.065$$

The solution, by means of the simplex algorithm is given by: $u_2 = 0$, $u_3 = 0.113$, $\alpha = 0.721$, which leads to the normalised ranking $w_1 = 0.11$, $w_2 = 0.31$ and $w_3 = 0.58$. This ranking is equal (in two decimal places) to the modal values of the fuzzy rankings obtained in [2,4,5] and to the result in [6]. When A_{32} is changed from (1.5, 2, 2.5) to (2,2,2), the result becomes $w_1 = 0.11$, $w_2 = 0.30$ and $w_3 = 0.59$, while the modal values of [2,4,5] would be unchanged.

8. References

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