On phase-locking of oscillators with delay coupling

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Summary. We consider two oscillators with delayed direct and velocity coupling. The oscillators have frequencies close or equal to 1:1 resonance. Due to the coupling the oscillations of the subsystems are in or out of phase. For these synchronized and anti-phase solutions, we use averaging for analytical stability results for small parameters. We also determine bifurcation curves of the delay system numerically. We identify regions in the parameter space (two coupling constants and the delay) where both solutions are stable or only one. For small parameters the averaging and numerical results are in good agreement. For larger values of the delay, we find multiple synchronized and anti-phase solutions. For small detuning we show that a minimal coupling value is needed to have almost synchronous or anti-phase behaviour.

Introduction

Systems of two coupled oscillators arise in a number of applications from physics to neuroscience. Typically, their interaction leads to phase-locking, although multi-frequency oscillations or chaos can also occur. A basic question is whether the phase-locking yields synchrony or anti-phase, solutions. Generally, this depends on the type and strength of coupling. Here we consider two nonlinear oscillators with delayed direct and velocity coupling given by

$$\begin{cases} \ddot{x}_1 + \omega_1^2 x_1 = \varepsilon\left(\dot{x}_1(\rho^2 - x_1^2 - \dot{x}_1^2) + \alpha x_2(t-\tau) + \beta\dot{x}_2(t-\tau)\right), \\ \ddot{x}_2 + \omega_2^2 x_2 = \varepsilon\left(\dot{x}_2(\rho^2 - x_2^2 - \dot{x}_2^2) + \alpha x_1(t-\tau) + \beta\dot{x}_1(t-\tau)\right). \end{cases} \tag{1}$$

In the absence of coupling ($\alpha = \beta = 0$), each system oscillates at its own frequency moving on a perfect circle. Turning on the coupling destroys leads to (almost) synchronous or anti-phase behavior, see Figure 1.

Averaging Analysis

We start by writing a Taylor series of the coupling terms w.r.t. the delay

$$\tilde{R}_i = R_i(t-\tau) = R_i(t) - \tau\dot{R}_i(t) + \ldots, \qquad \tilde{\theta}_i = \theta_i(t-\tau) = \theta_i(t) - \tau\dot{\theta}_i(t) + \ldots.$$

It can be shown that $\tilde{R}_i$ and $\tilde{\theta}_i$ are $O(\varepsilon)$. This key insight as described in [1] allows to approximate the infinite-dimensional dynamics by a finite dimensional system. We truncate the series to $O(\varepsilon)$ and this form then allows to use the method of averaging [2]. For this we introduce the following coordinates and $\omega_1 = 1, \omega_2 = 1 + \varepsilon\delta$:

$$x_i = R_i\cos(\omega_i t + \theta_i), \qquad \dot{x}_i = -R_i\omega_i\sin(\omega_i t + \theta_i), \qquad i = 1, 2. \tag{2}$$

Considering the difference of the phases $\phi = \theta_1 - \theta_2$ the averaged system can be written as

$$X : \begin{cases} \dot{R}_1 &= \varepsilon\left(R_1(R^2 - R_1^2) - \alpha R_2\sin(\phi-\tau) + \beta R_2\cos(\phi-\tau)\right), \\ \dot{R}_2 &= \varepsilon\left(R_2(R^2 - R_2^2) + \alpha R_1\sin(\phi-\tau) + \beta R_1\cos(\phi-\tau)\right), \\ \dot{\phi} &= \frac{1}{R_1R_2}\left(R_1^2\left(\alpha\cos(\phi-\tau) - \beta\sin(\phi-\tau)\right)\right) - R_2^2\left(\alpha\cos(\phi+\tau) + \beta\sin(\phi+\tau)\right)\big) + \delta. \end{cases} \tag{3}$$

Finally, we rescale time $t = \varepsilon\tilde{t}$. We then drop the ˜ and simply consider $\varepsilon = 1$. We note that averaging of two coupled van der Pol oscillators in a similar setting resulted in an equivalent reduced ODE after rescaling R, R_1, R_2 [1, 3]. We now consider the correspondence of fixed points of (3) with periodic orbits of system (1). For ODE's, the correspondence is

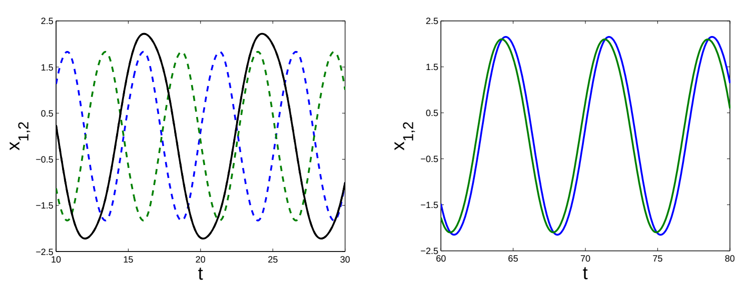

Figure 1: Typical solutions of system (1). The amplitudes $x_{1,2}$ of the two oscillators are shown. Left: $\omega_2 = 1$, $\alpha = \beta = 0.4$ The synchronized oscillation (black) has a larger amplitude. The anti-phase solution is shown with blue and green. Right: $\omega_2 = 1.01$, $\alpha = .2$,$\beta = 0.3$ near synchrony. Other parameters are $\rho = 2$, $\omega_1 = 1$, $\varepsilon = 1$ and $\tau = 1.0$.

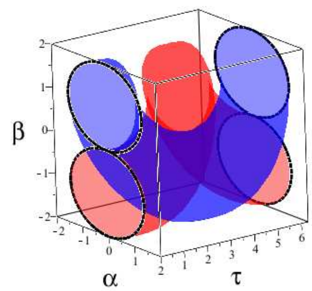

Figure 2: Stability boundaries for the symmetric (red) and asymmetric (blue) solutions, respectively. Stable solutions exist outside the corresponding ellipses.

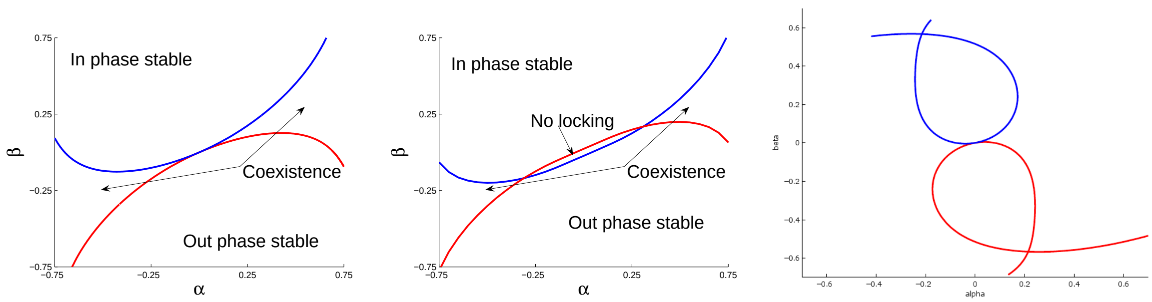

Figure 3: Bifurcation diagrams of system (1): Left ($\tau = .5, \omega_2 = 1$): Stability boundaries of the synchronized(red) and anti-phase(blue) oscillations. Middle ($\tau = .5, \omega_2 = 1.01$): Same as left, but with a region near the origin without locking due to the intrinsic frequency difference. Right: ($\tau = 6.5, \omega_2 = 1$): When the period and the delay are of similar magnitude, the system has multiple synchronized and anti-phase solutions. The bifurcating solution at the crossing of the curves differ in magnitude and period. Also notice that the size of the region with only one type of solution stable shrinks.

known, but for delays this is less trivial. For instance, including terms of $O(\varepsilon^2)$ would modify a second order equation to a third order, singularly perturbed equation and so on. For $\delta = 0$ the synchronized and anti-phase solutions satisfy $R_1 = R_2 = \rho$ and are parametrized by

$$(\phi, \rho_s) = \left(0, \sqrt{R^2 - \alpha \sin\tau + \beta\cos\tau}\right) \text{ and } (\phi, \rho_a) = \left(\pi, \sqrt{R^2 + \alpha\sin\tau - \beta\cos\tau}\right). \tag{4}$$

We now determine the stability boundary of these solutions. The solutions undergo a pitchfork bifurcation (without symmetry a saddle-node) when $\det(DX) = 0$ with DX the Jacobian evaluated at the solution. This yields the following equations

$$\frac{1}{2}\left(2\alpha\sin\tau - 2\beta\cos\tau \pm \frac{1}{2}R^2\right)^2 + (\alpha\cos\tau + \beta\sin\tau)^2 = \frac{1}{8}R^4, \tag{5}$$

where the $-$(resp. $+$) corresponds to the in- (resp. out-)phase stability boundary. We see that the two stability boundaries are given by ellipses that are tangent for $(\alpha, \beta) = (0, 0)$ and rotate around this point when changing τ, see Figure 2.

Numerical bifurcation analysis

We use the numerical bifurcation package KNUT [4, 5] to determine more precisely the bifurcations for system (1), see Figure 3. We set $\rho = 2$, $\omega_1 = 1$, $\varepsilon = 1$ and focus on relatively small values of α and β. We find limit point bifurcations of periodic orbits for both in- and out-phase solutions. For larger values of the coupling constants secondary bifurcations occur as well, but we omit these here. We conclude that for small values of τ the results of averaging and the numerical analysis are in good agreement. However, for larger τ we find that for $|\alpha, \beta| \approx .5$ two different solutions exist for both in-phase and out-phase oscillations. One has a period around 4 and the other has a period around 12.

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