Industrial Mathematics
Editors: Mohan C. Joshi et al

VARIATIONAL CHARACTERIZATION OF RESONANT STATES IN SOME INTEGRATED OPTICAL DEVICES

E. van Groesen

Applied Mathematics, University of Twente, The Netherlands, POBox 217, 7500 AE Enschede, The Netherlands

groesen@math.utwente.nl

Abstract Two examples from integrated optics are described that motivate the use of explicit variational characterizations for physical parameters that are relevant for the functioning of optical devices. For 1D optical gratings the boundary of the bandgaps, and for 2D square micro-resonators the resonant frequency, are formulated as eigenvalues of the governing Helmholtz equation. This requires to define the set on which the corresponding functional is optimised in an appropriate way, depending on the application.

1. Introduction

In modern optical telecommunication, all-optical devices are used to manipulate light for various purposes. These (nano- and micro-meter scale) devices exploit interference properties to manipulate the light that is transported to and from the device through waveguides. For instance, a grating acts like a physical mirror for an interval of wavelengths in the so called 'band gap', while a filter will select a specific wavelength from a broad spectrum of light. For the design and actual fabrication of such devices it is desirable to have direct characterizations for the critical parameters, such as the boundaries of the band gap and the filtered wavelength. Dependence of these critical values on material properties and geometric dimensions are desired as well.

We will show in this paper that variational formulations can lead to extremal characterizations of such critical values. The variational approach has another advantage that it is well suited for numerical discretizations, for instance it will lead in a standard way to consistent Finite Element procedures.

A major technical complication that has to be overcome is that such problems are typically modelled on unbounded domains. Although the device (and surrounding region where the changes in light are most essential) may be small, the presence of waveguides and the unavoidable radiation in unknown directions makes it difficult to 'confine' the problem, while this is desirable for mathematical analysis and numerical calculations.

We will consider in more detail the two examples of integrated optical devices mentioned above. The first device is a photonic crystal ([7]), a spatially periodic structure which can be impermeable for light in a certain range of wavelengths, the so called 'band-gap'. For the simplest case of a 1D crystal, then usually called 'grating', we will show that explicit variational methods can be used to characterize the boundaries of the bandgap. The second device is a so-called 'square micro-resonator' (see [1, 9, 10]), a square of high index material that can support various modes which may form a standing wave and let the square act like a cavity. Then a series of two squares can reroute light of one specific wavelength from one waveguide to another parallel waveguide, while other wavelengths remain practically unaffected. We will show that the wavelength of the rerouted light correspond to an eigenvalue of an eigenvalue problem, with Dirichlet conditions at a suitably rescaled square replacing the material interface conditions.

Both examples are planar structures, and we will assume the materials to be lossless (non-dissipative), nonmagnetic and linear. Then Mawell's equations reduce for monochromatic light to inhomogeneous Helmholtz equations, which uncouple into a set of two scalar equations for two polarizations. Restricting to TE-polarization, a scalar equation results for the spatially dependent part of the perpendicular component of the electromagnetic field. Writing $E(x,z;t) = u(x,z)exp(-i\omega t) + cc$ the equation for the (complex valued) spatial dependence is the Helmholtz equation

$$\left[\partial_x^2 + \partial_z^2 + \frac{\omega^2}{c^2} n^2\left(x,z\right)\right] u = 0, \qquad (1.1)$$

where ω is the frequency of the light, related to the (vacuum-) wavelength λ according to $\omega = \frac{2\pi}{\lambda} c$ with c the speed of light; $n(x,z)$ denotes the index of refraction. Different materials correspond to different values of the index of refraction, and the space-dependency of the index defines implicitly the optical device. The weak form of Helmholtz' equation is understood: at the interface, continuity of the field and its normal derivative have to be imposed. When using the variational formulation of Helmholtz' equation, the interface conditions arise as a natural consequence. Conditions at 'infinity' depend on the type of applications one is interested in.

2. Characterization of band gaps in gratings

The frist example is a 1D gratings. The index of refraction depends on only one variable, z, and changes periodically, say with period p :

$$\begin{aligned} \left[\partial_z^2 + \frac{\omega^2}{c^2} n^2(z)\right] u &= 0, \\ n(z+p) &= n(z). \end{aligned} \tag{2.1}$$

Often the case is considered that the index is piecewise constant. Such a grating is obtained for instance by positioning two materials of different constant index in a periodic way behind each other. Light will be partially reflected and partly transmitted at each interface, leading to summation of infinitely many contributions at each position. In these cases the transfer matrix of one period, and therefore the total solution of an infinite grating, can be written down explicitly and all bandgaps can be found, although the relevant equations are transcendental. Here a 'bandgap' is an interval of frequencies for which there is no light propagation through the structure. The method to be described below is more generally applicable, for any periodic index variation, and allows direct extension to nonlinear materials.

A first observation is that as a result of Floquet's theorem, or Bloch's theorem, it follows that any solution can be written like

$$u(z) = v(z, \omega) e^{ik(\omega)z}, \tag{2.2}$$

where $v(z, \omega)$ is a p-periodic function and the 'wave number' $k(\omega)$ can be real or complex. The intervals of values ω for which $k(\omega)$ is not real determine the band-gaps, since then no finite-amplitude solutions exist and no light propagation through the grating is possible. We will now characterize the boundaries of the band gaps.

The essential idea is that the 'boundary states', i.e., the states that correspond to the boundary of the band gap, satisfy the original equation (2.1) together with some (skew-) symmetry condition that will be detailed further on. Referring to (2.1), the variational principle

$$\delta \int \left[(\partial_z u)^2 - \frac{\omega^2}{c^2} n^2(z)\, u^2\right] dz = 0$$

leads to the correct equation, but is useless for the present purpose, since the value of ω is unknown and should be sought: our aim is to solve the *eigenvalue*