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Abstract

To prevent over-testing of the test-item during random vibration testing Scharton proposed and discussed the force limited random vibration testing (FLVT) in a number of publications, in which the factor $C^2$ is besides the random vibration specification, the total mass and the turn-over frequency of the load(test item), a very important parameter. A number of computational methods to estimate $C^2$ are described in the literature, i.e. the simple and the complex two degrees of freedom system, STDFS and CTDFS, respectively. Both the STDFS and the CTDFS describe in a very reduced (simplified) manner the load and the source (adjacent structure to test item transferring the excitation forces, i.e. spacecraft supporting an instrument).

The motivation of this work is to establish a method for the computation of a realistic value of $C^2$ to perform a representative random vibration test based on force limitation, when the adjacent structure (source) description is more or less unknown. Marchand formulated a conservative estimation of $C^2$ based on maximum modal effective mass and damping of the test item (load) , when no description of the supporting structure (source) is available [13].

Marchand discussed the formal description getting $C^2$, using the maximum PSD of the acceleration and maximum PSD of the force, both at the interface between load and source, in combination with the apparent mass and total mass of the the load. This method is very convenient to compute the factor $C^2$. However, finite element models are needed to compute the spectra of the PSD of both the acceleration and force at the interface between load and source.

Stevens presented the coupled systems modal approach (CSMA), where simplified asparagus patch models (parallel-oscillator representation) of load and source are connected, consisting of modal effective masses and the spring stiffnesses associated with the natural frequencies. When the random acceleration vibration specification is given the CMSA method is suitable to compute the value $C^2$.

When no mathematical model of the source can be made available, estimations of the value $C^2$ can be find in literature.

In this paper a probabilistic mathematical representation of the unknown source is proposed, such that the asparagus patch model of the source can be approximated. The computation of the value $C^2$ can be done in conjunction with the CMSA method, knowing the apparent mass of the load and the random acceleration specification at the interface between load and source, respectively.

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Strength & stiffness design rules for spacecraft, instrumentation, units, etc. will be practiced, as mentioned in ECSS Standards and Handbooks, Launch Vehicle User’s manuals, papers, books, etc. A probabilistic description of the design parameters is foreseen.

**Keywords:** Random vibration, Force limited vibration testing (FLVT), Coupled systems modal approach (CSMA), Probabilistic system.

## I. Introduction

The force limits are established to prevent over-testing of the test-article (load), because its dynamic behavior on the shaker table is different from its dynamic behavior when placed on the supporting structure (source).

In [17] the history, the actual status and application guidelines of the force limited vibration testing (FLVT) are discussed and 41 interesting references regarding the FLVT are provided.

During the FLVT both the random acceleration as well as the random force limits are specified, however, the random acceleration specification may be overruled by the random force limits.

The semi-empirical force-limit approach is a method to establish force-limits based on the extrapolation of interface force data for similar mounting structures, [16][17].

\[
\begin{align*}
W_{FF}(f) &= C^2 M_o^2 W_{AA}(f) & f \leq f_0, \\
W_{FF}(f) &= C^2 M_o^2 W_{AA}(f) \left(\frac{f_0}{f}\right)^n & f > f_0,
\end{align*}
\]

where \(W_{FF}(f)\) is the force spectral density, \(W_{AA}(f)\) is the acceleration spectral density, \(M_o\) is the total mass of the test item and \(C^2\) is a dimensionless constant which depends on the configuration. \(f\) (Hz) is the frequency and \(f_0\) is the natural frequency of the primary mode with a significant modal effective mass. The factor \(n\) can be estimated from the apparent mass of infinite systems. In general, \(n = 2\). \(C^2\) should not be selected without adequate justification [26].

Scharton et al revisited the force limiting vibration testing in a presentation [26] and reviewed the methods of estimation of \(C^2\) using the simple two degrees of freedom system (STDFS), Schweitzer’s method which tells us for lightly damped structures that \(C^2 = Q = 1/2\zeta\) (\(Q\) is the amplification factor and \(\zeta\) the damping ratio). The factor \(n\) can be estimated from the apparent mass of infinite systems. In addition to the presentation Scharton et al referenced 64 papers regarding force limiting.

Dharanipathi main conclusions in [6] are that the range of values of \(C^2\) is between 2 and 5, however, there are several cases where \(C^2 = 10 \cdots 17\), and that \(C^2\) does not depend on the damping in the structure.

In [28] Soucy et al recommend values for \(C^2\), however, based on limited number of flight data. It has been observed that in normal conditions \(C^2 = 2\) might be chosen for complete spacecraft or strut mounted heavier equipment. \(C^2 = 5\) might be considered for directly mounted lightweight test items.

Marchand derived an approximation of \(C_{max}^2\) in [13], given by the following expression

\[
C_{max}^2 \approx \left(\frac{M_{eff, max}}{2M_0^2}\right)^2,
\]

where \(M_{eff, max}\) is maximum modal effective mass of the load.

Based on the frequency shift of a two degrees of freedom system [24] Scharton developed two methods to establish the value \(C^2\); the simple two degrees of freedom system (STDFS) [16] and the complex two degrees of freedom system (CTDFS) [5].

Nagahama of JAXA presented in [15] a method to compute the force limits from envelopes of combinations of the apparent masses of source \(M_s\) and load \(M_l\), respectively,

\[
W_{FF}(f) = \left[\frac{M_s M_l - M_s M_l^0}{M_s + M_l^0}\right]^2 W_{AA}(f),
\]

Stevens presented a paper [32], to compute the force limits, based on the coupled system modal approach (CSMA). The coupled asparagus patch models of both source and load are
needed. These models can be extracted from finite element analysis models or apparent mass measurements. This CMSA method forms the core of this paper.

To compute $C^2$ with the STDFS, CTDFS, enveloping method JAXA or CMSA the dynamic characteristics of both source and load must be made available, simplified or more complex.

In general, the mathematical model (FEM, modal effective masses, ...) of the load is available, because the random vibration test will be conducted under the responsibility of contractor/subcontractor which is responsible for the design of the load as well. The mathematical description of the supporting structure (source) of the load is lacking. To apply the methods to obtain the value $C^2$ the dynamical properties of the source need to be known.

In this paper the replacement of the source by a probabilistic-source will be discussed. The mathematical modeling of the probabilistic source will be an asparagus patch model, consisting of a number of parallel placed lightly damped SDOF systems, with the modal effective masses as the discrete mass and the spring stiffnesses representing the undamped natural frequencies. To establish the probabilistic dynamic properties of the source well known design practices \cite{7, 8, 37} will be applied. The CMSA method \cite{32} is applied to compute maximum random accelerations and forces at the interface between load and source.

The Rosenblueth point estimated moments (PEM) will be applied \cite{19, 22} to introduce the probabilistic unknowns. It is assumed that the probability density functions of the unknown have uniform distributions.

### II. RANDOM VIBRATION TESTING $C^2$

#### VALUES FROM LITERATURE

Some characteristics of the random force limited vibration testing, the value $C^2$, the total mass of the source $M_s$, the total mass of the load $M_l$, the roll-off frequency $f_0$ and the exponent $n$ representing the apparent mass of the load are given in Table I (page 15). References are given too.

### III. FORCE LIMITS ANALYSIS METHOD

The semi-empirical force-limit vibration test (FLVT) approach has been established to prevent over-testing of a flexible test item when placed on the shaker table with a very high impedance compared to the impedance of the supporting structure of the test item. This (FLVT) test philosophy or method is described \cite{17}. The simple equations to compute the force limits $W_{FF}$ from the random acceleration test specification $W_{AA}$ are already given in \cite{1}.

Marchand provides in \cite{13} an equation to compute the value of $C^2$ in the interface between the source and the load, both consisting of MDOF systems. Considering that the maximum PSD of the interface force $W_{FF_{max}}$ and the maximum PSD of the interface acceleration $W_{AA_{max}}$, which need not to occur at the same frequency, the value of $C^2$ can be defined as

$$C^2 = \frac{W_{FF_{max}}}{M_o^2W_{AA_{max}}}$$

where $M_o$ is the total mass of the load.

### IV. COUPLED SYSTEM MODAL APPROACH METHOD

The CMSA method, proposed by Stevens in \cite{32}, is a method to compute the force limits for the random vibration testing of the load. The dynamic or apparent mass of the load, as well as the random acceleration test specification are required. The acceleration at the interface between load and source is illustrated in Fig. 1 page 13.

The apparent mass \cite{27, 36} at the interface of the load can be obtained by the following expression

$$M_l(f) = \sum_{i=1}^{n} m_{ill}(f_i) \left[ 1 + \left( \frac{f}{f_i} \right)^2 H \left( \frac{f}{f_i} \right) \right] - m_{rl}$$

where $m_{ill}$ is the modal effective mass and $m_{rl}$ is the residual mass of the load. $f_i$ are the natural frequencies of the clamped load at the
The frequency transfer function is given by

\[ H \left( \frac{f}{f_i} \right) = \frac{1}{1 - \left( \frac{f}{f_i} \right)^2 + 2j\zeta_i \left( \frac{f}{f_i} \right)}. \] (6)

The apparent mass \( M_i(f) \) will be used to compute the random interface loads \( W_{FF}(f) \) when the random interface acceleration spectrum \( W_{AA}(f) \) is provided

\[ W_{FF}(f) = |M_i(f)|^2 W_{AA}(f). \] (7)

The reduced asparagus patch models of both source and load are shown in Fig. 1. The spring stiffnesses and damper values are, respectively, given by \( k_{ii} = \omega_{ii}^2 m_{ii} \) and \( c_{ii} = 2\zeta_i \omega_{ii} m_{ii} \), where \( \omega_{ii}, i = 1, 2, \ldots, n \) are the natural frequencies of the load. \( \zeta_i \) is the modal damping ratio of mode \( i \). The notations for the source are similar.

The random acceleration vibration specification \( W_{AA}(f) \) at the interface between the source and the load is provided (specified). In general, this specification is an envelope that is based on data "smooths over" of peaks and valleys.

The process of deriving a random vibration specification is illustrated in Fig. 2. The black curve represents a hypothetical measurement of the acceleration at the interface of the source and the load. The vibration test specification is typically derived by averaging, enveloping the data. Unfortunately, the random acceleration notches at the load anti-resonance frequency, where the interface force is a maximum and the acceleration is a minimum, are disappeared by this smoothing process. The load is very responsive at the anti-resonance frequencies and acts as a dynamic absorber to reduce the input.

Eliminating the notch in the random acceleration input results in over-testing in conventional vibration tests by typically 10 dB to 20 dB [17].

To compute the parameter \( C^2 \) in (1), equation (4) is applied. Therefor we need to compute the random acceleration spectrum at the interface between the load and the source. That random acceleration spectrum is multiplied by the apparent mass of the load to obtain the random force spectrum at the interface. The mathematical models (parallel oscillators, Fig. 1) of the source and the load are represented by their modal effective masses and associated spring stiffness and damping and are coupled. The modal effective masses can be either calculated by a modal analysis with a fixed-free finite element model [36], or extracted from a measured apparent mass of the load, i.e. on a shaker table performing sinusoidal base-excitation [9, 27].

To calculate the maximum random force spectrum at the interface between source and load the following procedure is followed:

- Generate the mathematical models (Asparagus patch models) of both the source and load (Fig. 1).
- Compute or measure the apparent mass (dynamic mass) of the load, fixed at the interface between source and load.
- The random acceleration vibration specification to be applied to the load is given, i.e. the envelope acceleration spectrum as illustrated in Fig. 2.
- Define the random load spectrum to be applied subsequently at every oscillator of the source. This may be a unitary band-limited white noise spectrum or a unitary scaled random vibration spectrum.
- Perform for every subsequent loaded oscillator of the source a random acceleration response analysis and scale to the spectra such that the maximum acceleration at a certain excitation frequency is equal to the specified acceleration spectrum at that frequency. This is illustrated in Fig. 3. Multiply these scaled random acceleration spectra by the squared absolute value of the apparent mass spectra of the load. The composite random load spectrum \( W_{FF}(f) \) then represent the upper bound. This upper bound is divided by the square absolute value of the apparent mass spectra of the load to compute the associated upper bound interface random acceleration \( W_{AA}(f) \).
Apply (4) to compute $C^2$. $M_o$ is the rigid body mass of the load.

V. Definition (Availability) of Source and Load

To perform a random vibration test of the load the contractor needs the availability of a hardware (H/W) model of the load, i.e. the item to be tested on a shaker table. When the FLVT \cite{17} is planned the value of $C^2$ \cite{1} shall be obtained either by experience (data base) \cite{17} or applying the simple two degrees of freedom (STDFS) system and or the complex two degrees of freedom (CTDFS) system as described in \cite{29}. When modal characteristics of both source and load can be made available from FEA/FEM or measurements \cite{4} can be used \cite{13}. The CSMA method will be applied as illustrated in Fig. 1.

I. Load

I.1 Mathematical Model

We assume the availability of a mathematical description (finite element model) of the load. That means a complete description of the geometry, dimensions, material properties, mass distribution (structural and nonstructural mass) etc. \cite{7} is provided and the modal analysis is done in accordance to \cite{7}. An estimation of the modal damping ratio shall be done, in general, based on past experiences or measurements. The finite element model degrees of freedom at the interface between the load and source shall be fixed. The following modal data of the load is needed for the CMSA method:

- The undamped natural frequencies $f_{ii}, i = 1, 2, \cdots, n$
- The undamped vibration modes $\phi_{ii}, i = 1, 2, \cdots, n$
- The $6 \times 6$ modal effective mass matrices associated with the natural frequencies. About 90-95% of total mass matrix $M_o$ is covered by the sum of modal effective mass matrices $\sum_i M_{eff,i}$.

- The residual mass matrix $M_{res} = M_o - \sum_i M_{eff,i}$
- The reduced asparagus patch model of the load
- The apparent mass matrix $M_a$ of the load at the interface load/source

The mathematical representation of the load is either by a finite element model, and/or a modal effective/residual mass representation, or the (provided) apparent mass as illustrated in Fig. 4.

II. Source

Côté stated in his paper \cite{4} that the asparagus patch model of the source (common to the load); modal effective masses, natural frequencies, can be extracted from a finite element model, experiment or from experience. However, in this subsection we assume that the finite element model or experimental results cannot be made available, so the simplified model will be constructed using engineering design rules (i.e. ECSS). We will describe the experience needed to formulate the asparagus model of the source.

The dynamic characteristics of the load with respect to the interface between the load and the source are considered to be reference properties. The following properties are assumed to be known, such that we can build the asparagus model of the load.

- The total mass of the load $M_{tot}$ (kg)
- The undamped natural frequencies $f_{ii}, i = 1, 2, \cdots, n$ (Hz) assuming clamped conditions at the interface load/source
- The associated modal effective masses $m_{ii}, i = 1, 2, \cdots, n$ (kg) and the residual mass $m_{rl}$, in the three translational directions, respectively.
- The estimated or measured modal damping ratios $\zeta_{ii}, i = 1, 2, \cdots, n$
The apparent mass $M_l(f) \text{ (kg)}$ of the load in the three translational directions with respect to the interface.

The extraction of natural frequencies and associated modal effective masses is explained in [9, 27].

II.1 Design Parameters Source

In this section the unknown design parameters are discussed.

Total Mass

The total mass of the source $M_{os} \text{ (kg)}$ is associated with the distribution of the modal effective masses and residual mass and must be somehow made available by the project or estimated. In general, the mass of the source will be $M_{os} \geq M_{ol}$. This is not a strict requirement.

Natural Frequency Shift

To prevent dynamic coupling between the source and the load there must be a frequency shift with about a factor $\sqrt{2} \cdots 2$ between the natural frequencies with significant modal effective mass [11, 37]. The fundamental natural frequency of the load must be higher than the fundamental natural frequency of the source. If the load has a lower natural frequency then the natural frequency of the source a dynamic uncoupling between source and load will be achieved, however, high movements of the load will occur, which is not very likely.

First Approximation Modal Effective Mass

To get some feeling about the main modal effective mass value, the main modal effective mass will be calculated for simple systems assuming basic mode shapes.

A cantilevered beam is representing a clamped spacecraft. The mass per unit of length is $m$ and the length of the beam is $L$ (see Fig. 5). The total mass is $M_0 = mL$. The assumed bending mode $\Phi(x)$ (lateral direction) and the assumed longitudinal (launch) mode $\Psi(x)$ are given by

$$
\Phi(x) = 2 \left( \frac{x}{L} \right)^2 - 4 \left( \frac{x}{L} \right)^3 + \frac{1}{3} \left( \frac{x}{L} \right)^4, \\
\Psi(x) = x.
$$

A simply supported beam may be a representation of a fixed spacecraft. The assumed mode $\Phi(x)$ is taken to be

$$
\Phi(x) = \sin \left( \frac{\pi x}{L} \right),
$$

and a subsystem may be represented by a simply supported plate with edges $a, b$ and mass per unit of area $m$ we assume a vibration mode

$$
\Phi(x, y) = \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right).
$$

The simple structural representations of the cantilevered beam, simply supported beam and plate are illustrated in Fig. 5.

The corresponding modal effective masses are given in Table 2.

Modal Damping Ratio

The modal damping ratio $\zeta$ is equal for both the load and the source, however, is the same for all modes.

VI. Virtual Building of Asparagus Model of the Source

I. Total Mass

The total rigid body mass of the source $M_s$ shall be provided (i.e. by the prime contractor). If the $M_s$ can’t be made available the following total mass, with uniform distribution, of the source is assumed

$$
M_s = 0.1 \cdots 10 M_l
$$

When the mass of the source $M_s$ is known, the mean of the source mass is $\mu = M_s$ and the standard deviation $\sigma = 0$. 
II. Natural frequencies

When the lowest undamped natural frequency of the load is \( f_l \), the interface source/load fixed, the assumed undamped natural frequency of the source will vary between
\[
f_{1s} = \frac{f_1}{2} \cdots \frac{f_i}{\sqrt{2}}.
\]
(12)

This undamped natural frequency of the source is associated with a high modal effective mass \( m_{1s} \). The probability density function of the first natural frequency \( f_{1s} \) is uniform. The factor 2 is called by Steinberg [31] the reverse octave rule.

The following distribution of natural frequencies, with substantial modal effective mass, is (arbitrarily first estimation) defined:
\[
\begin{align*}
f_{2s} &= 2f_{1s}, \\
f_{3s} &= 3f_{1s}, \\
f_{4s} &= 6f_{1s}.
\end{align*}
\]
(13)

Force limits typically cover only the first three modes [12]. Therefore, it is usually adequate to specify the force limits only in the frequency regime encompassing the first few modes in each axis, which might be out to approximately 100 Hz for a large spacecraft, 500 Hz for an instrument, or 2000 Hz for a small component [17].

The User’s manuals [1, 30, 35] of the launch vehicles serviced by the European Company Arianespace [2] provide stiffness requirements for spacecraft launched with one of the launch vehicles ARIANE 5, Soyuz and VEGA.

III. Modal Effective masses

The theoretical and practical aspects of modal effective masses are discussed in detail in the ECSS Handbook ECSS-E-32-26A [8], in particular in chapter 5 of that handbook. The modal effective mass is the amount of mass that is represented by each undamped vibration mode, and the sum of the modal effective masses is equal the total mass of that structure [32, 56].

The first undamped natural frequency \( f_{1s} \) will be associated with the first significant modal effective mass \( m_{1s} \). The fundamental modal effective masses of simple systems is assumed to be a first approximation of modal effective mass of the source (section II.1). This modal effective will be assumed in the following mass range with a uniform probability distribution
\[
m_{1s} = 0.4 \cdots 0.6M_s.
\]
(14)

The residual mass is the sum of the modal effective masses excited outside the frequency range of interest and the residual mass \( m_{rs} \) will be assumed to be 5% of the total mass of the source, such that
\[
m_{rs} = 0.05M_s.
\]
(15)

Further \( \Delta m \) is the sum of the missing distribution of the modal effective mass and is defined by
\[
\Delta m = M_s - (m_{1s} + m_{rs}).
\]
(16)

The deterministic distribution (arbitrarily first estimation) of the modal effective \( m_{ks}(f_{ks}), k = 2, \cdots, 4 \) will be descending and is as follows:
\[
\begin{align*}
m_{2s} &= 0.5\Delta m, \\
m_{3s} &= 0.3\Delta m, \\
m_{4s} &= 0.2\Delta m.
\end{align*}
\]
(17)

IV. Modal Damping Ratio

We will assume a uniform distribution of the modal damping ratio \( \zeta = 0.1 \cdots 0.01 \).

V. Summary of Mean and Standard Deviation of Stochastic Variables

The probability density function of the stochastic variables \( M_s, f_{1s}, m_{1s} \) and \( \zeta \) are assumed to be uniform.

The summary of mean and standard deviation of the selected probabilistic variables, with a uniform distribution [3] is presented in Table [3].

\[ f(x) = \frac{1}{(b-a)}, a \leq x \leq b, f(x) = 0, \text{otherwise}, \mu = \frac{(a+b)}{2}, \sigma = |b-a|/(2\sqrt{3}). \]
VII. Experiment

A simple example problem with 8 DOFs \[10\] will be used to show the analysis results of \( C^2 \) compared to the real source with the probabilistic source. This example is illustrated in Fig. 6. We will calculate the value \( C^2 \) at the interface between nodes (masses) 3 and 4 (node 3 side) applying \[4\], by calculating the maximum PSD acceleration of node 3 and the maximum PSD of the interface force (PSD of force in spring between nodes 3 and 4). The total mass of the load (nodes 4 until 8) is \( M_0 = 31 \text{kg} \). The random load applied to node 1 is white noise with unit PSD \( W_{FF,1} = 1 \text{ N}^2/\text{Hz} \).

The response calculations are done using modal damping ratios \( \zeta = 0.01, 0.05, 0.10 \), the same for all modes. The corresponding values of \( C^2 \) are tabulated in the following table.

<table>
<thead>
<tr>
<th>modal damping ratio ( \zeta )</th>
<th>( C^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>2.77</td>
</tr>
<tr>
<td>0.05</td>
<td>2.72</td>
</tr>
<tr>
<td>0.10</td>
<td>2.54</td>
</tr>
</tbody>
</table>

I. Deterministic CMSA

The first three modes (modal effective mass and natural frequency) are taken from the Tables 4 and 5 to build the asparagus patch models of the source and load, respectively. The white noise (arbitrarily) random vibration specification, at the interface, is between 20-2000Hz, \( W_{AA} = 0.01 \text{ g}^2/\text{Hz} \). The white noise random force applied subsequently to \( m_{1s}, m_{2s}, m_{3s} \) is \( W_{FF} = 1 \text{ N}^2/\text{Hz} \). The modal damping ratio is constant for all modes, and applicable for the source and load is varying \( \zeta = 0.01, 0.05, 0.1 \). The calculated values for \( C^2 \) are tabulated in following table:

<table>
<thead>
<tr>
<th>modal damping ratio ( \zeta )</th>
<th>( C^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>3.24</td>
</tr>
<tr>
<td>0.05</td>
<td>3.25</td>
</tr>
<tr>
<td>0.10</td>
<td>3.21</td>
</tr>
</tbody>
</table>

II. STDFS & CTDFS

The \( C^2 \) values will be computed applying the STDFS and CTDFS simple models from \[17\].

The modal effective and residual masses of both source and load are listed in Tables 4 and 5. In case of the STDFS model Scharton suggested to use the residual masses of the source and load \[23\] (\( m_2/m_1 = 6.81/23.47 = 0.29 \)). For the CTDFS the following input parameter \( a_1 = 8.54, a_2 = 3.55 \) and \( M_2/M_1 = 0.29 \) are used. The \( C^2 \) values for \( \zeta = 0.01, 0.05, 0.1 \) are tabulated hereafter.

<table>
<thead>
<tr>
<th>modal damping ratio ( \zeta )</th>
<th>STDFS</th>
<th>CTDFS</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>5.28</td>
<td>4.42</td>
</tr>
<tr>
<td>0.05</td>
<td>5.71</td>
<td>9.11</td>
</tr>
<tr>
<td>0.10</td>
<td>5.86</td>
<td>11.70</td>
</tr>
</tbody>
</table>

The computed \( C^2 \) values by the CTDFS method are strongly dependent on the modal damping.

III. Stochastic CMSA, MCS

In this section a Monte Carlo Simulation (MCS) will be performed, where the same basic inputs are varied as mentioned in section VII-I. The stochastic variables \( M_s, f_1s, m_{1s} \) and \( \zeta \) are assumed to be uniform distributed and the range is given in the following list.

<table>
<thead>
<tr>
<th>Stochastic variable</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_s )</td>
<td>0.1 ( \cdots ) 10 ( M_l )</td>
</tr>
<tr>
<td>( f_1s )</td>
<td>0.5 ( \cdots ) 0.7071 ( f_1l )</td>
</tr>
<tr>
<td>( m_{1s} )</td>
<td>0.4 ( \cdots ) 0.5 ( M_s )</td>
</tr>
<tr>
<td>( \zeta )</td>
<td>0.01 ( \cdots ) 0.1</td>
</tr>
</tbody>
</table>

The number of samples of all 4 variables are varied simultaneously. The values of \( C^2 \) are presented in the following tabulation.

<table>
<thead>
<tr>
<th>( \neq ) Samples</th>
<th>( C^2_{\text{mean}} )</th>
<th>( C^2_{\text{std}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4.36</td>
<td>1.66</td>
</tr>
<tr>
<td>100</td>
<td>4.40</td>
<td>1.50</td>
</tr>
<tr>
<td>1000</td>
<td>4.35</td>
<td>1.49</td>
</tr>
<tr>
<td>10000</td>
<td>4.40</td>
<td>1.52</td>
</tr>
</tbody>
</table>

IV. Rosenblueth 2k+1 PEM CMSA

The Rosenblueth Point Estimates for probability moments \[19, 22\], the estimates of the mean and the variance of the value \( C^2 \) are computed in combination with the CMSA. The number of samples is 10 and the nature of samples is given in Table 6.
The mean of two point estimates \(Y_{kp}, Y_{km}\) is given by
\[
Y_k = \left| \frac{Y_{kp} + Y_{km}}{2} \right|, \quad k = 1, \cdots, 4, \quad (18)
\]
and the variance is \(V_k = \frac{Y_{kp} - Y_{km}}{Y_{kp} - Y_{km}}\), \(k = 1, \cdots, 4\). (19)

When the stochastic variables are statistically independent the following approximation of the mean \(\bar{Y} = \mu_Y\) and the variance \(V_Y = \sigma_Y / \mu_Y\) can be made \[22\]
\[
\bar{Y} = Y_0 = \prod_{k=1}^{4} Y_k \mu_Y, \quad (20)
\]
and
\[
1 + V_Y^2 = \prod_{k=1}^{4} (1 + V_k^2). \quad (21)
\]

The following estimates of mean and standard deviation are listed hereafter:

<table>
<thead>
<tr>
<th>Method</th>
<th># samples</th>
<th>(C_{mean}^2)</th>
<th>(C_{std}^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2k + 1 PEM</td>
<td>10</td>
<td>4.50</td>
<td>1.09</td>
</tr>
</tbody>
</table>

VIII. DISCUSSION

In this section the results of several methods to compute the value \(C^2\) are compared and discussed. Although this paper concentrates on the stochastic description of the unknown source with aid of a simple experimental 8 DOF dynamic system (Fig. 6), which is taken from a paper of Haille \[10\] to demonstrate probabilistic approach. The results of the computation of the value \(C^2\) will be discussed hereafter:

- The combination of the source and load is the 8 DOF dynamic system. The source consists of the discrete masses (nodes) 1-3. The load is build up from the discrete masses 4-8. Mass (node) 1 is excited by a unitary white noise. The random acceleration of mass 3 and the random interface force is extracted from the spring force between masses 3 and 4. The total mass of the load is 31kg. Equation \[4\] will result in the value \(C^2 \approx 2.7\) for low damping, however, the interface random acceleration and random interface force were not specified.

- The source and the load are converted into two asparagus patch models build up by parallel placed SDOF systems, expressing the modal effective masses and the corresponding natural frequencies. The not excited modal effective mass are represented by the residual mass. The asparagus patch models are coupled, see Fig. \[\text{[10]}\]. The modal effective masses of the deterministic source and load are presented in Tables \[4\] and \[5\]. The spring stiffness of the spring between masses 3 and 4 is doubled to get the right interface stiffnesses (equivalent spring stiffness of two springs in series with spring stiffness \(2k = k\)). The random acceleration vibration specification is specified at the interface between the source and the load and using the CSMA method the values of \(C^2\) are computed varying the modal damping ratio \(\zeta\). The mass of the load is 31kg. The calculated values, with a specified random acceleration specification, are \(C^2 \approx 3.2\), independent of the modal damping.

- As a reference the values of \(C^2\) are also calculated using the two-degrees-of-freedom STDFS and CTDFS methods. The mass of the load is 31kg. The residual masses are used for the STDFS method \[17\].

- The deterministic source is now replaced by a stochastic source described by 4 stochastic variables \(M_s, f_{1s}, m_{1s}, \zeta_s\), with a uniform distribution. The ranges are provided. The CMSA method is used to calculate random interface forces and accelerations. The modal damping applies for both the source and the load. The mass of the load is 31kg. Two probabilistic approaches were applied; the MCS and the Rosenblueth \(2k + 1\) PEM. Both
the MCS and PEM gave similar values for $C_2$, however, the number of samples using PEM is quite low compared to the MCS. The probabilistic approach results in slightly higher values for $C_2$, 4.5 compared to 3.2. The dynamic characteristics of deterministic and probabilistic source are not the same. The computed values of $C_2$ are within the range found in literature (Table 1).

In general, the probabilistic description of the source in combination with the Rosenblueth $2k + 1$ point estimates is quite satisfactory, but more data shall be collected.

IX. Conclusions and Recommendations

A first attempt has been made to describe the source in a probabilistic asparagus patch model, with a stochastic representation of the total mass, the modal effective masses, undamped natural frequencies, and damping. The stochastic variables have assumed uniform distribution within prescribed ranges. A simple 8 DOF dynamic system is used as an experiment to implement the probabilistic approach. The computed $C_2$ calculated with the CMSA methods with a deterministic load and stochastic source are satisfactory. The probabilistic asparagus patch model of the source is very convenient to describe very condensed models representing the dynamic properties of the source. Well known design rules are applied to build the probabilistic asparagus patch model of the source.

It is recommended to collect more data about dynamic properties from random vibration tests, published in the literature.

REFERENCES


MsC thesis in Mechanical Engineering, Ottawa-Carleton Institute for Mechanical and Aerospace Engineering, University of Ottawa, Ottawa, Canada


Asparagus patch models

Figure 1: Coupled system in parallel-oscillator representation

Figure 2: Dynamic absorber effect of load on interface acceleration
Figure 3: Scaling the random acceleration response

Figure 4: Mathematical representation of the load
Cantilevered beam

Simply supported beam

Simply supported plate

Figure 5: Simple representations of spacecraft and subsystems

Figure 6: 8 DOF simple problem [10]

Table 1

<table>
<thead>
<tr>
<th>Reference</th>
<th>C²</th>
<th>M₁ (kg)</th>
<th>M₂ (kg)</th>
<th>f₀ (Hz)</th>
<th>n</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>4</td>
<td>103</td>
<td></td>
<td>100</td>
<td></td>
<td>Space Shuttle flight</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
<td>14</td>
<td>8</td>
<td>500</td>
<td>2</td>
<td>Al panel + Box A</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
<td>5</td>
<td>8</td>
<td>250</td>
<td>4</td>
<td>Al panel + Box B</td>
</tr>
<tr>
<td>20</td>
<td>9</td>
<td>6</td>
<td></td>
<td>376</td>
<td>2</td>
<td>Mars Sample Return Container</td>
</tr>
<tr>
<td>23</td>
<td>9</td>
<td>200</td>
<td></td>
<td>65</td>
<td>2</td>
<td>JWST NIRSPEC</td>
</tr>
<tr>
<td>34</td>
<td>4</td>
<td>85</td>
<td></td>
<td>80-90</td>
<td>2</td>
<td>JWST MIRI</td>
</tr>
</tbody>
</table>
### Table 2: Modal effective mass of simple systems; beam, plate

<table>
<thead>
<tr>
<th>Structure</th>
<th>Assumed mode shape</th>
<th>$M_{eff}(mL)$ (kg)</th>
<th>$M_{eff}(M_o)$ (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cant. beam</td>
<td>$\Phi(x)$</td>
<td>$\frac{81}{10} mL = 0.62 mL$</td>
<td>$\frac{81}{10} M_o = 0.62 M_o$</td>
</tr>
<tr>
<td>Cant. beam</td>
<td>$\Psi(x)$</td>
<td>$\frac{4}{10} mL = 0.75 mL$</td>
<td>$\frac{4}{10} M_o = 0.75 M_o$</td>
</tr>
<tr>
<td>S.S. beam</td>
<td>$\Phi(x)$</td>
<td>$\frac{1}{10} mab = 0.81 mab$</td>
<td>$\frac{1}{10} M_o = 0.81 M_o$</td>
</tr>
<tr>
<td>S.S. plate</td>
<td>$\Phi(x, y)$</td>
<td>$\frac{64}{10} mab = 0.66 mab$</td>
<td>$\frac{64}{10} M_o = 0.66 M_o$</td>
</tr>
</tbody>
</table>

### Table 3: Mean and standard deviation stochastic variables

<table>
<thead>
<tr>
<th>Description</th>
<th>Symbol</th>
<th>Mean $\mu$</th>
<th>Standard deviation $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass (kg)</td>
<td>$M_s$</td>
<td>5.0500 $M_I$</td>
<td>2.8579 $M_I$</td>
</tr>
<tr>
<td>Natural frequency (Hz)</td>
<td>$f_{1s}$</td>
<td>0.6036 $f_I$</td>
<td>0.0598 $f_I$</td>
</tr>
<tr>
<td>Modal effective mass (kg)</td>
<td>$m_{1s}$</td>
<td>0.5000 $M_s$</td>
<td>0.0577 $M_s$</td>
</tr>
<tr>
<td>Modal damping ratio (-)</td>
<td>$\zeta$</td>
<td>0.055</td>
<td>0.026</td>
</tr>
</tbody>
</table>

### Table 4: Dynamic properties source (fixed between nodes 3 and 4)

<table>
<thead>
<tr>
<th>Mode #</th>
<th>Natural frequency (Hz)</th>
<th>Modal effective mass (kg)</th>
<th>Residual mass (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>22.57</td>
<td>200.5 ($m_{1s}$)</td>
<td>23.47</td>
</tr>
<tr>
<td>2</td>
<td>76.14</td>
<td>12.37 ($m_{2s}$)</td>
<td>11.10</td>
</tr>
<tr>
<td>3</td>
<td>141.2</td>
<td>11.10 ($m_{3s}$)</td>
<td>0.00</td>
</tr>
<tr>
<td>Total mass (kg)</td>
<td></td>
<td></td>
<td>224.0</td>
</tr>
</tbody>
</table>

### Table 5: Dynamic properties load (fixed between nodes 3 and 4)

<table>
<thead>
<tr>
<th>Mode #</th>
<th>Natural frequency (Hz)</th>
<th>Modal effective mass (kg)</th>
<th>Residual mass (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>81.25</td>
<td>24.19 ($m_{1l}$)</td>
<td>6.81</td>
</tr>
<tr>
<td>2</td>
<td>177.4</td>
<td>6.10 ($m_{2l}$)</td>
<td>0.71</td>
</tr>
<tr>
<td>3</td>
<td>246.6</td>
<td>0.71 ($m_{3l}$)</td>
<td>0.00</td>
</tr>
<tr>
<td>4</td>
<td>365.9</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>5</td>
<td>589.6</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Total mass (kg)</td>
<td></td>
<td></td>
<td>31.00</td>
</tr>
</tbody>
</table>

### Table 6: Number of samples

<table>
<thead>
<tr>
<th># Sample</th>
<th>$C^2$</th>
<th>$M_s$</th>
<th>$f_{1s}$</th>
<th>$m_{1s}$</th>
<th>$\zeta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$Y_0$</td>
<td>$\mu$</td>
<td>$\mu$</td>
<td>$\mu$</td>
<td>$\mu$</td>
</tr>
<tr>
<td>2</td>
<td>$Y_{1p}$</td>
<td>$\mu + \sigma$</td>
<td>$\mu$</td>
<td>$\mu$</td>
<td>$\mu$</td>
</tr>
<tr>
<td>3</td>
<td>$Y_{1m}$</td>
<td>$\mu - \sigma$</td>
<td>$\mu$</td>
<td>$\mu$</td>
<td>$\mu$</td>
</tr>
<tr>
<td>4</td>
<td>$Y_{2p}$</td>
<td>$\mu$</td>
<td>$\mu + \sigma$</td>
<td>$\mu$</td>
<td>$\mu$</td>
</tr>
<tr>
<td>5</td>
<td>$Y_{2m}$</td>
<td>$\mu$</td>
<td>$\mu - \sigma$</td>
<td>$\mu$</td>
<td>$\mu$</td>
</tr>
<tr>
<td>6</td>
<td>$Y_{3p}$</td>
<td>$\mu$</td>
<td>$\mu$</td>
<td>$\mu + \sigma$</td>
<td>$\mu$</td>
</tr>
<tr>
<td>7</td>
<td>$Y_{3m}$</td>
<td>$\mu$</td>
<td>$\mu$</td>
<td>$\mu - \sigma$</td>
<td>$\mu$</td>
</tr>
<tr>
<td>9</td>
<td>$Y_{4p}$</td>
<td>$\mu$</td>
<td>$\mu$</td>
<td>$\mu$</td>
<td>$\mu + \sigma$</td>
</tr>
<tr>
<td>10</td>
<td>$Y_{4m}$</td>
<td>$\mu$</td>
<td>$\mu$</td>
<td>$\mu$</td>
<td>$\mu - \sigma$</td>
</tr>
</tbody>
</table>