

Energy-conserving formulation of RLC-circuits with linear resistors

D. Eberard, B.M. Maschke and A.J. van der Schaft

Abstract—In this paper firstly, the dynamics of LC-circuits without excess elements is expressed in terms of contact systems encompassing in a single formulation the Hamiltonian formulation and the pseudo-gradient formulation called the Brayton-Moser equations. Indeed the contact formulation encompasses both the Hamiltonian system and its the adjoint variational system. Secondly we express the dynamics of RLC-circuits with linear resistors in the contact formalism by extending the state space to an associated space of Thermodynamic phase which again encompasses as well the Hamiltonian formulation with dissipation as the Brayton-Moser formulation.

I. INTRODUCTION

The dynamics of electrical circuits has been the subject of various analytical formulation including variational formulations, Lagrangian and Hamiltonian formulations and pseudo-gradient formulations [5] [2] [6] [14] [7] [13] [18] [23] [26]. In this paper we shall deal with the relation between the Hamiltonian formulation of LC- and RLC-circuits [2] [17] and the pseudo-gradient formulation (also called Brayton-Moser) formulation [6] [23] [14]. Therefore we shall use a formulation in terms of a contact vector field associated with the lift of Hamiltonian systems on some Thermodynamical Phase Space [9] [10]. We shall show that this formulation encompasses both representations and allows to relate them by considering the prolongation of a Hamiltonian system to its adjoint system.

II. LC-CIRCUIT STRUCTURE

In this section, we will recall briefly some basic definition of LC-circuits and the Brayton-Moser and Hamiltonian equations for connected LC-circuits without excess elements [17].

A. Constitutive and circuit equations of an LC-circuit

The energy storing elements of the circuits are the capacitors and inductors with constitutive relations defined as follows.

Definition 2.1 ([17], [16]): A capacitor (resp. an inductor) is defined by an energy variable $q_C \in \mathbb{R}$ denoting the electrical charge (resp. the total magnetic flux $\varphi_L \in \mathbb{R}$), an real function $H_C(q_C) \in \mathcal{C}^\infty(\mathbb{R})$ denoting the electrical

D. Eberard and B.M. are with Laboratory of Chemical Engineering and Automation, University of Lyon 1, France, eberard@lagep.univ-lyon1.fr and maschke@lagep.univ-lyon1.fr

A.J. van der Schaft is with Institute of Mathematics and Computer Science, University of Groningen, The Netherlands A.J.van.der.Schaft@math.rug.nl

energy stored in the capacitors (resp. the magnetic energy $H_L(\varphi_L) \in \mathcal{C}^\infty(\mathbb{R})$ stored in the inductors) and two port variables : the variation of charge, i.e. the current $i_C \in \mathbb{R}$ (resp. the variation of magnetic flux, i.e. the voltage $v_L \in \mathbb{R}$) and the co-energy variable $v_C \in \mathbb{R}$ denoting the voltage (resp. the current $i_L \in \mathbb{R}$) related by the constitutive relations

$$\begin{cases} i_C = \frac{dq_C}{dt} \\ v_C = \frac{\partial H_C}{\partial q_C}(q_C) \end{cases} \quad \text{and} \quad \begin{cases} v_L = \frac{d\varphi_L}{dt} \\ i_L = \frac{\partial H_L}{\partial \varphi_L}(\varphi_L) \end{cases} .$$

Furthermore we shall assume that no of the elements are degenerated, that is that there is a bi-univoque correspondance between the energy variables $x = (q_C, \varphi_L) \in \mathcal{N} = \mathbb{R}^{n_C} \times \mathbb{R}^{n_L}$ and the co-energy variables $x^* = (v_C, i_L) \in T^*\mathcal{N}$. In other words, the maps $\hat{v}_C : q_C \mapsto v_C$ and $\hat{i}_L : v_C \mapsto q_C$ are \mathcal{C}^∞ bijections, thus there exists $\hat{q}_C : v_C \mapsto q_C$ and $\hat{\varphi}_L : i_L \mapsto \varphi_L$ such that $\hat{v}_C \circ \hat{q}_C = \hat{q}_C \circ \hat{v}_C = \text{Id}$ and $\hat{i}_L \circ \hat{\varphi}_L = \hat{\varphi}_L \circ \hat{i}_L = \text{Id}$, see [14] for more details.

The inductors and capacitors are interconnected through a circuit which related the energy and co-energy variables by Kirchhoff's laws as follows:

$$\begin{pmatrix} i_C \\ v_L \end{pmatrix} = \begin{pmatrix} 0 & B^t \\ -B & 0 \end{pmatrix} \begin{pmatrix} v_C \\ i_L \end{pmatrix} = Jx^*, \quad (1)$$

where B is the fundamental matrix of cycles associated with the maximal tree given by the capacitors (by assumption that the circuit has no element in excess) [16], [14].

B. Hamiltonian formulation of the dynamics

The dynamics of LC-circuits have been formulated in various ways using variational, Lagrangian and Hamiltonian formulations (see the references in [17]). However the Hamiltonian formulation which are closest to the objects defining a circuit, use the energy variables as state variables and define the dynamics with the Hamiltonian being the total electro-magnetic energy and using a Poisson bracket associated with Kirchhoffs' laws (1) for circuits without elements in excess.

Remark 2.1: In the case of LC-circuits with elements in excess, the Hamiltonian formulation may be extended to implicit Hamiltonian systems defined with respect to Dirac structures associated to Kirchhoff's laws [25], [16] [2].

This Hamiltonian formulation is defined as follows. Define the Hamiltonian function

$$H(q_C, \varphi_L) = H_C(q_C) + H_L(\varphi_L) \quad (2)$$

representing the total energy, as the sum of the stored electric and magnetic energies in the capacitors and the inductors respectively given by

$$H_C(q_C) = \int_0^{q_C} \hat{v}_C(q) dq, \quad (3)$$

and

$$H_L(\varphi_L) = \int_0^{\varphi_L} \hat{i}_L(\varphi) d\varphi. \quad (4)$$

The dynamics is then given by the Hamiltonian equations

$$\dot{x} = J d_x H(x), \quad (5)$$

where d_x denotes the differentiation with respect to the variable x and the structure matrix J , defined equation (1), is defined by Kirchhoff's laws.

C. Brayton-Moser framework

The Brayton and Moser formulation is defined in a sort of dual way as a pseudo-gradient system [6] [23]. This system is defined with respect to a non-definite metric associated with the constitutive relations of the energy-storing elements and is generated by a function called Brayton-Moser potential which is associated with Kirchhoff's laws. The construction is the following.

Define the *co-energy function* $H^*(x^*)$ to be the x -Legendre transformation of $H(x)$ in the state space $T^*\mathcal{N}$

$$H^*(x^*) = x^{*t}x - H(x). \quad (6)$$

Recall that $x^* = (\partial H / \partial x)(x)$. Brayton and Moser [6] have shown that the dynamics of such a LC-circuit is given by

$$\mathcal{A}^*(x^*)\dot{x}^* = \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix} \frac{\partial P}{\partial x^*}(x^*), \quad (7)$$

where

$$\mathcal{A}^*(x^*) = \frac{\partial^2 H^*}{\partial x^{*2}}(x^*) = \begin{pmatrix} C(v_C) & 0 \\ 0 & L(i_L) \end{pmatrix},$$

with the capacitor and the inductor matrices given by

$$C(v_C) = \frac{\partial \hat{q}_C}{\partial v_C}(v_C) \quad \text{and} \quad L(i_L) = \frac{\partial \hat{\varphi}_L}{\partial i_L}(i_L),$$

and the *mixed-potential* $P(x^*)$ defined by

$$P(x^*) = i_L^t B v_C = P_C(x^*) = -P_L(x^*), \quad (8)$$

where $P_C = \hat{i}_C v_C$ (resp. $P_L = i_L^t \hat{v}_L$) denotes the power across the capacitors (resp. in the inductors).

Remark 2.2: The proof may be summarized as follows:

$$\begin{aligned} \mathcal{A}^*(x^*)\dot{x}^* &= \frac{\partial^2 H^*}{\partial x^{*2}}(x^*)\dot{x}^* \\ &= \frac{d}{dt} \left(\frac{\partial H^*}{\partial x^*}(x^*) \right) \\ &= \dot{x}, \quad \text{cause} \quad \frac{\partial H^*}{\partial x^*}(x^*) = x \\ &\stackrel{(5)}{=} J d_x H(x) \\ &= J x^*, \quad \text{cause} \quad \frac{\partial H}{\partial x}(x) = x^* \\ &\stackrel{(8)}{=} \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix} \frac{\partial P}{\partial x^*}(x^*) \end{aligned}$$

Brayton and Moser's formulation (7) may be interpreted as a pseudo-gradient system defined with respect to the pseudo-metric:

$$\mathcal{A}^{*-1}(x^*) \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix} = \begin{pmatrix} C(v_C)^{-1} & 0 \\ 0 & -L(i_L)^{-1} \end{pmatrix} \quad (9)$$

which is clearly associated with the constitutive relation of the energy-storing elements and is well-defined under the assumption that the elements are non-degenerated.

III. CONTACT FORMULATION

In this section, we present the formulation of the dynamics of LC-circuit as a contact vector field. We shall see that this formulation encompasses both the Hamiltonian formulation 5) and the Brayton-Moser formulation (7) by projecting the contact dynamics on the Legendre submanifolds generated by the energy or the co-energy function. Let us start with some definitions of contact geometry.

A. A very brief reminder on contact geometry

Roughly speaking, the contact structure is the analogue of the symplectic structure for odd-dimensional manifold [1] [15]. Contact geometry has been used in the frame of Thermodynamics in order to characterize the properties of thermodynamical systems [12] [4] [22]. In this frame the thermodynamic properties of a system are characterized as a *Legendre submanifold* of the Thermodynamic Phase space which may in general be defined by some cotangent bundle $\mathcal{T} = \mathbb{R} \times T^*\mathbb{R}^n$. Contact geometry has also been used to define the reversible and irreversible transformations of thermodynamical systems [19] [11] [9]. In the sequel we shall briefly recall the main geometrical objects needed and expressed in coordinates (for an intrinsic definition we refer to [15] and in the context of systems' theory to [10] [9]).

A *contact form* θ may be expressed in canonical coordinates $x^0, x^1, \dots, x^n, p_1, \dots, p_n$ of the Thermodynamic Phase Space \mathcal{T} , by

$$\theta = dx^0 - \sum_{i=1}^n p_i dx^i, \quad (10)$$

Then, \mathcal{T} endowed with this contact form θ is called a *contact manifold* and one may define Legendre submanifolds

as the solution of the Pfaffian equation associated with the contact structure.

Definition 3.1: A *Legendre submanifold* is a n -dimensional submanifold of \mathcal{T} , solution of the equation $\theta = 0$.

A Legendre submanifold may also be defined, in coordinates, as follows.

Theorem 3.1 ([3]): For a given set of canonical coordinates and any partition $I \cup J$ of the set of indices $\{1, \dots, n\}$ and for any differentiable function $F(x^I, p_J)$ of n variables, $i \in I, j \in J$, the formulas

$$x^0 = F - p_J \frac{\partial F}{\partial p_J}, \quad x^J = -\frac{\partial F}{\partial p_J}, \quad p_I = \frac{\partial F}{\partial x^I} \quad (11)$$

define a Legendre submanifold of \mathbb{R}^{2n+1} denoted \mathcal{L}_F . Conversely, every Legendre submanifold of \mathbb{R}^{2n+1} is defined in a neighborhood of every point by these formulas, for at least one of the 2^n possible choices of the subset I . For instance, each function $H \in C^\infty(\mathbb{R}^n)$ defines the following Legendre submanifold

$$\mathcal{L}_H = \{H(x), x, d_x H(x)\}, \quad (12)$$

also called the 1-jet of H . When H is defined by a thermodynamical potential, then the Legendre submanifold defines the thermodynamical properties of the system. For instance the thermodynamical potential may be chosen to be the internal energy of the thermodynamical system¹, see [11], [9].

With the contact structure, one may associate a class of vector fields called *contact vector fields*.

Definition 3.2: A *contact field* is a vector fields that preserves the contact structure.

In fact, there is an isomorphism between the class of contact fields and the space of smooth functions on \mathcal{T} . In a set of canonical coordinates (x^0, x, p) , a contact field X_f , generated by the function $f \in C^\infty(\mathcal{T})$, is written

$$\begin{aligned} X_f = & \left(f - \sum_{k=1}^n p_k \frac{\partial f}{\partial p_k} \right) \frac{\partial}{\partial x^0} + \frac{\partial f}{\partial x^0} \left(\sum_{k=1}^n p_k \frac{\partial}{\partial p_k} \right) \\ & + \sum_{k=1}^n \left(\frac{\partial f}{\partial x^k} \frac{\partial}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial}{\partial x^k} \right). \end{aligned} \quad (13)$$

Notice that the last term is precisely the same as the Hamiltonian field given by the canonical symplectic structure on the cotangent bundle $T^*\mathbb{R}^n$. Furthermore, when modelling the dynamics of a physical system, it appears that contact Hamiltonian has the dimension of *power* [11], [9].

Another key difference with symplectic geometry is that to ensure the conservation of energy² along trajectories of the contact field, we assume f identically zero on \mathcal{L}_H .

Theorem 3.2: Let (\mathcal{T}, θ) be a contact manifold and \mathcal{L} a Legendre submanifold. Then, for every smooth function f on \mathcal{T} , X_f is tangent to \mathcal{L} iff $\mathcal{L} \subset f^{-1}(0)$.

¹Legendre submanifolds have to be related with *Lagrange* submanifolds used in Analytical Mechanics to represent energy levels.

²Intrinsically satisfied in symplectic geometry.

B. Contact dynamics of LC-circuits

We shall now use the lift of the Hamiltonian systems to the Thermodynamic Phase Space according to [11], [9] in order to relate the Hamiltonian and Brayton-Moser formulation of LC-circuit dynamics.

The first step consist associated with the space of energy variables $\mathcal{N} = \mathbb{R}^{n_C} \times \mathbb{R}^{n_L} \ni (q_C, \varphi_L)$, the Thermodynamic Phase Space $\mathbb{R} \times T^*\mathcal{N} \ni (x^0, q_C, \varphi_L, p_C, p_L)$ where x^0 is the coordinate associated with the energy and p_C and p_L are the variables conjugated with the energy variables q_C and φ_L . The second step consist in considering the Legendre submanifold associated with the total electromagnetic energy $H(q_C, \varphi_L)$ given by (2), (3) and (4). It is then simply given by the definition of the co-energy variables and the energy:

$$\mathcal{L}_H = \{H(q_C, \varphi_L), q_C, \varphi_L, \hat{v}_C(q_C), \hat{i}_L(\varphi_L)\}, \quad (14)$$

In [11], [9] it has been shown that the Hamiltonian dynamics (5) may be lifted to the whole Thermodynamic Phase Space as the contact vector field generated by the Hamiltonian function:

$$K(x, p) = -p^t J d_x H(x), \quad (15)$$

This contact Hamiltonian satisfies the invariance condition of the theorem 3.2 by antisymmetry of J , hence the Legendre submanifold \mathcal{L}_H is invariant. It may be noticed firstly that the contact Hamiltonian K is bilinear in p and $d_x H$ and has the dimension of power as: $K(x, p) = -p^t J d_x H(x) = (p_C, p_L)^t \begin{pmatrix} i_C \\ v_L \end{pmatrix}$.

Furthermore this contact Hamiltonian can be related with the Brayton-Moser mixed-potential P in the following way. Define the *extended* or *virtual* power across capacitors \tilde{P}_C and in the inductors \tilde{P}_L as

$$\tilde{P}_C(p_C, i_L) = i_L^t B p_C \quad \text{and} \quad \tilde{P}_L(v_C, p_L) = -v_C^t B^t p_L, \quad (16)$$

where we split $p = (p_C, p_L)$. Then the contact Hamiltonian may be written as the sum of these Brayton-Moser extended potentials:

$$K = \tilde{P}_C + \tilde{P}_L$$

When one evaluates the contact Hamiltonian K on \mathcal{L}_H , the invariance condition in the theorem 3.2 is equivalent to the identity of the Brayton-Moser potential of the equation (8):

$$K|_{\mathcal{L}_H} = P_C(x^*) + P_L(x^*) \stackrel{(8)}{=} 0. \quad (17)$$

In the sequel we shall note that both the Hamiltonian formulation (5) and the Brayton-Moser formulation (7) are contained in the contact field formulation. In a first instance, consider the x -component of the contact field (13) restricted to the Legendre submanifold \mathcal{L}_H . It is precisely the Hamiltonian dynamics of the LC-circuit. Indeed, according to (13), the x -component of X_K is given by

$$\dot{x} = -\frac{\partial K}{\partial p}(x, p) = J d_x H(x). \quad (18)$$

In a second instance, consider the p -component of X_K , restricted to the Legendre submanifold \mathcal{L}_H . Since K is independent of x^0 , we have

$$\dot{p} = \frac{\partial K}{\partial x} = -p^t J \frac{\partial^2 H}{\partial x^2}(x) = ([\mathcal{A}^*(x^*)]^{-1})^t J p.$$

Recall that \mathcal{A}^* is symmetric and that

$$p|_{\mathcal{L}_H} = \frac{\partial H}{\partial x}(x) = x^*,$$

it follows

$$\dot{x}^* = [\mathcal{A}^*(x)]^{-1} J x^* = [\mathcal{A}^*(x^*)]^{-1} \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix} \frac{\partial P}{\partial x^*}(x^*),$$

where P is defined equation (8). These are precisely the Brayton-Moser equations (7).

Proposition 3.1: The contact vector field X_K generated by the contact Hamiltonian (15) both encompasses the Hamiltonian formulation (5) and Brayton-Moser formulation (7) of LC-circuits without excess elements by considering either the projection on the energy variables or on the co-energy variables of the restriction of the contact field to the Legendre submanifold \mathcal{L}_H .

This enlightens also the dual nature of both dynamics as the Brayton-Moser equations may be seen as the adjoint variational formulation of the Hamiltonian dynamics. Indeed, the adjoint system prolongs the dynamics of a system on the cotangent bundle and leads to lift a Hamiltonian system on a Thermodynamic Phase Space $\mathbb{R} \times T^*\mathcal{N}$ [9].

IV. CONSERVATIVE FORMULATION OF RLC-CIRCUITS

In this section, we shall show that the contact formalism allows to give an *energy-conserving* formulation of RLC-circuit with (linear) resistors. As stated in [8][24], we shall use a port-based approach to include energy-dissipating elements in the circuit. In other words, we may consider an RLC-circuit as a LC-circuit with ports which will be connected to the port variables of the resistors.

By extending the state space of the LC-circuit with an extra variable associated with the entropy, we are able to define an augmented total energy function which is conserved (according to the first law of Thermodynamic). Furthermore, using the previous remark on adjointness of Hamiltonian and Brayton-Moser equations, we will get both dissipative dynamics with the contact formulation.

A. Reminder on the Hamiltonian and Brayton-Moser dynamics of RLC-circuits

Let us first recall the Brayton-Moser and Hamiltonian equations for RLC-circuits with linear resistors. Denote by (v_{R_C}, i_{R_C}) and (v_{R_L}, i_{R_L}) the port variables of the resistive elements belonging to the tree and co-tree subnetworks respectively. According to [21] Lemma 1, there exists two matrices B_C and B_L such that

$$v_{R_C} = B_C v_C \quad \text{and} \quad i_{R_L} = B_L i_L.$$

which completes the matrix of fundamental loops (1) including the resistive ports. As we consider here *linear*

resistors, there exists two definite positive matrices R_C and R_L of the resistances of the energy dissipating elements. One then defines a *potential* P_D associated with the *dissipated power* in the resistors and given by

$$P_D(x^*) = \frac{1}{2} v_C^t \mathcal{R}_C^{-1} v_C - \frac{1}{2} i_L^t \mathcal{R}_L i_L, \quad (19)$$

where

$$\mathcal{R}_C^{-1} = B_C^t R_C^{-1} B_C \quad \text{and} \quad \mathcal{R}_L = B_L^t R_L B_L.$$

The Brayton-Moser potential is then augmented with the dissipative potential to a *total mixed-potential* [6] [14] as follows

$$P_{tot}(x^*) = P(x^*) - P_D(x^*), \quad (20)$$

where P is the *standard* mixed-potential function defined equation (8).

Remark 4.1: Notice that P_D defined equation (19) denotes the power given by the difference between the *voltage-controlled* resistors function (also called the resistors *co-content*) and the *current-controlled* resistors function (also called the resistors *content*).

The Brayton-Moser formulation of the dynamics of the RLC-circuit is then given by [6] in terms of the co-energy variables:

$$\mathcal{A}^*(x^*) \dot{x}^* = \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix} \frac{\partial P_{tot}}{\partial x^*}(x^*), \quad (21)$$

There exists also a Hamiltonian formulation of the dynamics of the RLC-circuit in terms of the energy variables. In this formulation the structure matrix is changed to a matrix composed of the sum of a skew-symmetric part corresponding precisely to the LC subcircuit and a symmetric matrix corresponding to the energy-dissipating elements:

$$R = \begin{pmatrix} \mathcal{R}_C^{-1} & 0 \\ 0 & \mathcal{R}_L \end{pmatrix} \quad (22)$$

The structure matrix $J - R$ corresponds to the definition of a bracket on the space of energy variables \mathcal{N} called Leibniz bracket [20]. The dynamics of the RLC-circuit is defined with respect to this Leibniz bracket and generated by the Hamiltonian H being the total electromagnetic energy (2) and leads to the following *dissipative* Hamiltonian system:

$$\dot{x} = (J - R) d_x H(x), \quad (23)$$

We shall now use the energy-conserving formulation of dissipative Hamiltonian systems, provided by the the contact formalism [10], to express (in the same way as in the LC case) both Hamiltonian and pseudo-gradient formulations of RLC-circuit dynamics.

B. Energy-conserving formulation of dissipative Hamiltonian systems

Let us briefly recall a result stated in [10]. Consider the case of an autonomous dissipative Hamiltonian system given by (23), where R denotes a symmetric positive definite matrix of dissipation. In order to determine the

contact Hamiltonian generating this dynamics, we first extend the base manifold as follows :

$$\mathcal{N}_e = \mathcal{N} \times \mathbb{R} \cong \mathbb{R}^{n+1},$$

where we denote by S an additional coordinate on \mathbb{R} associated with an abstract entropy. We then define the *extended* thermodynamic phase space

$$\mathcal{T}_e = \mathbb{R} \times T^*\mathcal{N}_e \ni (x^0, x, S, p, p_S), \quad (24)$$

where the conjugated variable to S is denoted by p_S .

Remark 4.2: The physical interpretation of the additional two coordinates (S, p_S) are they represent the extensive and intensive states of an infinite thermal reservoir (thermostat) representing the thermal environment of the circuit.

This extended thermodynamic phase space is endowed with a canonical contact form θ_e defined as the extension of the contact form θ defined equation (10) :

$$\theta_e = \theta - p_S dS = dx^0 - p dx - p_S dS.$$

We now define the *new* energy function

$$H_e(x, S) = H(x) + T_0 S,$$

where T_0 is a parameter standing for the constant virtual temperature of the environment, that generates the following Legendre submanifold (see (12))

$$\mathcal{L}_{H_e} = \left\{ H_e(x, S), x, S, \frac{\partial H_e}{\partial x} = \frac{\partial H}{\partial x}, \frac{\partial H_e}{\partial S} = T_0 \right\}$$

of the contact form θ_e . Notice that p_S restricted to \mathcal{L}_{H_e} is the temperature of the system denoted T . Then, we define the contact hamiltonian function

$$K_e = -p^t (J - R) \frac{\partial H}{\partial x} - \frac{p_S}{T_0} \frac{\partial H}{\partial x} R \frac{\partial H}{\partial x}. \quad (25)$$

By construction, this contact hamiltonian K_e vanishes on \mathcal{L}_{H_e} . Furthermore, it generates a contact vector fields giving the following dynamics when restricted to \mathcal{L}_{H_e} and projected no the (x, S) coordinates:

$$\begin{cases} \dot{x} = (J - R) dx H(x) \\ \dot{S} = \frac{1}{T_0} \frac{\partial H}{\partial x} (x) R \frac{\partial H}{\partial x} (x) \end{cases}. \quad (26)$$

We recognize the dynamics of a dissipative hamiltonian system together with the time variation of the entropy. Notice that this result agrees with the thermodynamic principles : the energy H_e is conserved and the entropy function is increasing.

C. Contact formulation of RLC-circuits

The previous results may be applied to the dynamics of RLC circuits. And let us first relate the contact Hamiltonian (25) to the Brayton and Moser potentials. Defining the extension of the dissipative potential P_D , cf (19), by the function \bar{P}_D on \mathcal{M}

$$\bar{P}_D(v_C, p_C, i_L, p_L) = \frac{1}{2} v_C^t \mathcal{R}_C^{-1} p_C - \frac{1}{2} i_L^t \mathcal{R}_L p_L, \quad (27)$$

we define moreover the extended potential associated with the dissipation as:

$$\tilde{P}_D = \bar{P}_D - \frac{p_S}{T_0} \bar{P}_D.$$

Then the contact Hamiltonian may be written as the sum of these Brayton-Moser extended potentials:

$$K = \tilde{P}_C + \tilde{P}_L + \tilde{P}_D. \quad (28)$$

Applying the previous result to an RLC-circuit, that is the contact Hamiltonian function K_e given in (25) with the dissipation matrix R defined in (22), leads to the Hamiltonian dynamics (23) together with the time variation of entropy (given in (26)).

We shall now see that the dynamics of the p variables restricted to the Legendre submanifold \mathcal{L}_{H_e} is precisely the Brayton-Moser dynamics (7) with the mixed-potential (20). Indeed, X_{K_e} provides

$$\dot{p} = \frac{\partial^2 H}{\partial x^2} (x) \left[(J + R) p - 2 \frac{p_S}{T_0} R \frac{\partial H}{\partial x} (x) \right].$$

Recall that $p|_{\mathcal{L}_{H_e}} = x^*$ and $p_S|_{\mathcal{L}_{H_e}} = T_0$, it follows obviously

$$\dot{p}|_{\mathcal{L}_{H_e}} = \dot{x}^* = \frac{\partial^2 H}{\partial x^2} (J - R) x^*.$$

Furthermore, as

$$R x^* = \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix} \frac{\partial P_D}{\partial x^*} (x^*),$$

we then obtain the Brayton-Moser equations:

$$\dot{p}|_{\mathcal{L}_{H_e}} = \frac{\partial K_e}{\partial x} (x, p) \iff (21).$$

In conclusion one again obtains that the contact formulation of RLC circuits on the extended thermodynamic space \mathcal{T}_e in (24) encompasses both the formulation of their dynamics as dissipative Hamiltonian systems and Brayton-Moser systems (pseudo-gradient systems).

Proposition 4.1: The contact vector field X_{K_e} generated by the contact Hamiltonian (28) both encompasses the dissipative Hamiltonian formulation (23) and Brayton-Moser formulation (21) of RLC-circuits without excess elements by considering either the projection on the energy variables or on the co-energy variables of the restriction of the contact field to the Legendre submanifold \mathcal{L}_{H_e} .

V. CONCLUSION

In this paper we have proposed a relation of the Hamiltonian formulation of the dynamics of LC- and RLC-circuit with the Brayton-Moser formulation by using its lift to an associated Thermodynamical Phase Space. We have shown that the dynamics on this extended space is generated by a contact Hamiltonian function which is defined using Brayton-Moser mixed potential associated with some virtual power associated as well to the Kirchhoff's equations as to the dissipation through the resistors. The total electromagnetic energy, on the other

hand, corresponds in this formulation to the definition of the Legendre submanifold associated with the constitutive relations of the energy storing elements (capacitors and inductors). In this way we have provided a frame where the interconnection structure (i.e. Kirchhoff's laws), the energy properties of the systems and the dissipation are defined by real valued functions. Using some general results presented in [9] [10], these results may be easily generalized to circuits with sources and in this way open the way to new perspectives to the interconnection, dissipation and energy shaping method for the stabilization of power systems.

ACKNOWLEDGEMENTS

This work was partially supported by the EU-project GeoPlex (www.geoplex.cc) EU-IST-2001-34166.

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