Composition of Infinite-Dimensional Linear Dirac-type Structures

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***Abstract*— In this paper, we define the Dirac structure and give some fundamental tools for its study. We then proceed by defining composition of "split Dirac structures". In the finite-dimensional case, composition of two Dirac structures always results in a new Dirac structure, but in the Hilbert-space setting this result no longer holds. Thus, the problem of finding necessary and sufficient conditions for the composition of two infinite-dimensional Dirac structures to itself be a Dirac structure arises very naturally. The main result of this paper provides these necessary and sufficient conditions. In addition, we give examples and relate composition of Dirac structures to the Redheffer star product of unitary operators.**

I. Introduction

In the Hamiltonian approach to systems theory, one takes an energy-conservation point of view to modelling physical systems, such as electrical networks or mechanical systems. There one considers the *power variables effort* and *flow*, the *power* being the *product* of the effort and the flow. The power variables are sometimes split into the *internal* flow/effort and the flow/effort at some *port to the external world*. Via these external ports one may connect a port-Hamiltonian system to another port-Hamiltonian system. For a thorough background on port-Hamiltonian systems, see [1] or [2].

The manifold on which the power variables of a Hamiltonian system lie, is called the *Dirac structure* (or interconnection structure) of the system. With the help of Dirac structures it is possible to formalise the concept of *interconnection of port-Hamiltonian systems*. The interconnection of two port-Hamiltonian systems is modelled by composition of their respective Dirac structures. [3]

In the finite-dimensional setting the composition of two Dirac structures is always a Dirac structure, see e.g. [4] or [1]. In this paper we present necessary and sufficient conditions for the composition of two infinite-dimensional Dirac structures to be a Dirac structure. In addition, we study in which cases the closure of the composition of two Dirac structures is again a Dirac structure. Partial solutions to these problems have earlier been given in [5].

We also interprete the composition results for a more general class of structures, namely the *maximally dissipative* ones. Dirac structures correspond to energy-conserving systems, i.e. systems that neither consume nor produce energy, and maximally dissipative structures correspond to passive systems, i.e. systems that are only assumed to lack internal sources of energy. Therefore, our result on composition covers the very large class of passive systems, of which the energy conservative systems form a subclass. For more information on passive systems, see [6] or [7].

In Sections II and III we give some basic results on Dirac structures and maximally dissipative structures, generalising selected parts of [5, Section 5]. Similar results, in a slightly different terminology, can be found in [8]. Thereafter, in Section IV, we proceed by studying the composition of Dirac structures. An explicit expression for a slightly generalised Redheffer star product (Section V) of unitary (or contractive) mappings is obtained as a by-product of our method for solving the composition problem.

II. Dirac-type structure basics

Let X and Y be Hilbert spaces, with inner products $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_Y$, respectively. By a *contractive* linear operator $T : X \supset \operatorname{dom} T \to Y$ we mean an operator satisfying $\langle Tx, Tx\rangle_Y \le \langle x, x\rangle_X$ for all $x \in \operatorname{dom} T$. In the case where $\operatorname{dom} T = X$, we say that T is *fully defined*. Otherwise T is *partially defined*.

If $\langle Tx, Tx\rangle_Y = \langle x, x\rangle_X$, then T is *isometric*. In the latter case, if moreover $\operatorname{dom} T = X$ and $\operatorname{ran} T = Y$, then T is *unitary*. For unitary operators, we have $T^*T = I_X$ and $TT^* = I_Y$.

Definition 2.1: Let $\mathcal{E}$ (the space of *efforts*) and $\mathcal{F}$ (the space of *flows*) be two isometrically isomorphic Hilbert spaces, with a bijective isometry $r_{\mathcal{E},\mathcal{F}} : \mathcal{E} \to \mathcal{F}$, $r_{\mathcal{E},\mathcal{F}}^{-1} = r_{\mathcal{F},\mathcal{E}} = r_{\mathcal{E},\mathcal{F}}^*$. Define the *bond space* $\mathcal{B}$ as the Cartesian product $\mathcal{B} = \left[\begin{smallmatrix}\mathcal{F}\\ \mathcal{E}\end{smallmatrix}\right]$ of $\mathcal{F}$ and $\mathcal{E}$. For the norm of $\mathcal{B}$, we choose the Cartesian Hilbert-space norm: $\left|\left[\begin{smallmatrix} f\\ e\end{smallmatrix}\right]\right|_{\mathcal{B}}^2 = \|f\|_{\mathcal{F}}^2 + \|e\|_{\mathcal{E}}^2$.

Denoting the inner products on $\mathcal{E}$ and $\mathcal{F}$ by $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{F}}$, respectively, we introduce the (indefinite) *power product* $[\cdot,\cdot]_{\mathcal{B}}$ on $\mathcal{B}$ by

$$\left[\begin{bmatrix} f^1\\ e^1\end{bmatrix}, \begin{bmatrix} f^2\\ e^2\end{bmatrix}\right]_{\mathcal{B}} := \langle f^1, r_{\mathcal{E},\mathcal{F}} e^2\rangle_{\mathcal{F}} + \langle e^1, r_{\mathcal{F},\mathcal{E}} f^2\rangle_{\mathcal{E}}. \tag{1}$$

For any subset $\mathcal{D} \subset \mathcal{B}$, we define the $[\cdot,\cdot]_{\mathcal{B}}$*-orthogonal companion* $\mathcal{D}^{[\perp]}$ of $\mathcal{D}$ by

$$\mathcal{D}^{[\perp]} := \{b \in \mathcal{B} \mid \forall d \in \mathcal{D} :\ [b, d]_{\mathcal{B}} = 0\}.$$

The subspace $\mathcal{D} \subset \mathcal{B}$ is a *Tellegen structure* (on $\mathcal{B}$) if $\mathcal{D} \subset \mathcal{D}^{[\perp]}$. It is a *Dirac structure* if, moreover, $\mathcal{D} = \mathcal{D}^{[\perp]}$.

If $\mathcal{E} = \mathcal{F} = \left[\begin{smallmatrix}\mathcal{F}_1\\ \mathcal{F}_2\end{smallmatrix}\right]$ and $r_{\mathcal{E},\mathcal{F}} = \left[\begin{smallmatrix} I_1 & 0\\ 0 & -I_2\end{smallmatrix}\right]$, then we say that the structure is *split*.

Frequently we take $\mathcal{E} = \mathcal{F}^*$ for the space of efforts, letting $\langle e, r_{\mathcal{F},\mathcal{E}} f\rangle_{\mathcal{E}}$ be given by $\langle e, r_{\mathcal{F},\mathcal{E}} f\rangle_{\mathcal{E}} = e(f)$, the usual duality pairing between $\mathcal{F}$ and $\mathcal{F}^*$.

It is possible to define a more general kind of split structure, where one chooses $r_{\mathcal{E},\mathcal{F}} = \begin{bmatrix} r_{\mathcal{E}_1,\mathcal{F}_1} & 0\\ 0 & -r_{\mathcal{E}_2,\mathcal{F}_2}\end{bmatrix}$.

All the computations we present in our slightly simplified setting remain valid in the more general case, but as one of our objectives with this paper is to present the composition criteria as clearly as possible, we choose to obtain simpler expressions.

Remark 2.2: The indefinite inner product (1) is non-degenerate, i.e. $\mathcal{B}^{[\perp]} = \{0\}$. In another terminology, the *indefinite-inner-product-space* language, our bond space $\mathcal{B}$, together with the indefinite power product $[\cdot,\cdot]_{\mathcal{B}}$, is a *Kreĭn space*. In this setting, Tellegen structures are called *isotropic (or neutral) subspaces* and Dirac structures are referred to as *Lagrangean subspaces*. [9]

We have, that $\mathcal{D}$ is a Tellegen structure if and only if $[d,d] = 0$ for all $d \in \mathcal{D}$. We relax this condition slightly in order to include another very important class of subspaces.

Definition 2.3: A subspace $\mathcal{D} \subset \mathcal{B}$ is a *dissipative structure* if all $d \in \mathcal{D}$ satisfy $[d,d]_{\mathcal{B}} \leq 0$. *Accumulative structures* are such that $[d,d]_{\mathcal{B}} \geq 0$ for all $d \in \mathcal{D}$. These subspaces are collectively called *semi-definite*.

The semi-definite structure $\mathcal{D}$ is *maximal* if it cannot be properly extended without the semi-definiteness being lost.

We have the following results connected to maximally semidefinite subspaces.

Theorem 2.4: Every maximally semi-definite subspace is closed (with respect to the Cartesian Hilbert-space norm).

The structure $\mathcal{D} \subset \mathcal{B}$ is maximally dissipative (accumulative) if and only if it is closed and $\mathcal{D}^{[\perp]}$ is accumulative (dissipative). Then also $\mathcal{D}^{[\perp]}$ is maximal.

A structure $\mathcal{D} \subset \mathcal{B}$ is a Dirac structure if and only if it is maximally dissipative and maximally accumulative.

By a *Dirac-type structure*, we mean a maximally dissipative or a Dirac structure. Maximally dissipative Tellegen structures are also somewhat interesting, as these correspond to energy-preserving systems. For a treatment of the connection between graphs of system nodes and Dirac-type structures, please see [8] (discrete-time systems) and [10] (continuous-time systems).

III. Scattering representation of dissipative structures

The scattering representation of a Dirac structure was developed in [5]. We now extend this important notion to any dissipative structure.

Let the Hilbert space $\mathcal{G}$, the *scattering variable space*, be isometrically isomorphic to $\mathcal{E}$ and $\mathcal{F}$, with isometric bijections $r_{\mathcal{E},\mathcal{G}}$ and $r_{\mathcal{F},\mathcal{G}}$, that satisfy $r_{\mathcal{E},\mathcal{G}} = r_{\mathcal{F},\mathcal{G}} r_{\mathcal{E},\mathcal{F}}$.

Considering a dissipative structure $\mathcal{D} \subset \mathcal{B}$, we define the *Cayley transform*

$$\begin{aligned} \mathcal{D}_{cayl} :&= \left\{ \begin{bmatrix} r_{\mathcal{E},\mathcal{G}} e + r_{\mathcal{F},\mathcal{G}} f \\ r_{\mathcal{E},\mathcal{G}} e - r_{\mathcal{F},\mathcal{G}} f \end{bmatrix} \mid \begin{bmatrix} f \\ e \end{bmatrix} \in \mathcal{D} \right\} \\ &= \begin{bmatrix} r_{\mathcal{F},\mathcal{G}} & r_{\mathcal{E},\mathcal{G}} \\ -r_{\mathcal{F},\mathcal{G}} & r_{\mathcal{E},\mathcal{G}} \end{bmatrix} \mathcal{D} \subset \begin{bmatrix} \mathcal{G} \\ \mathcal{G} \end{bmatrix}. \end{aligned}$$

It can be shown that $\mathcal{D}_{cayl}$ is the graph of some contractive operator $\mathcal{O}$, which we call the *scattering operator* of $\mathcal{D}$:

$$\mathcal{D}_{cayl} = \begin{bmatrix} \mathcal{O} \\ I_{\mathcal{G}} \end{bmatrix} \operatorname{dom} \mathcal{O}. \tag{2}$$

We see, that the domain and range of the scattering operator are given by $\operatorname{dom} \mathcal{O} = \begin{bmatrix} -r_{\mathcal{F},\mathcal{G}} & r_{\mathcal{E},\mathcal{G}} \end{bmatrix} \mathcal{D}$ and $\operatorname{ran} \mathcal{O} = \begin{bmatrix} r_{\mathcal{F},\mathcal{G}} & r_{\mathcal{E},\mathcal{G}} \end{bmatrix} \mathcal{D}$, respectively.

Conversely, if $\mathcal{D}_{cayl}$ is given by (2), where $\mathcal{O}$ is an arbitrary (possibly only partially defined) contraction on $\mathcal{G}$, then the *inverse Cayley transform*

$$\mathcal{D} := \begin{bmatrix} r_{\mathcal{G},\mathcal{F}} & -r_{\mathcal{G},\mathcal{F}} \\ r_{\mathcal{G},\mathcal{E}} & r_{\mathcal{G},\mathcal{E}} \end{bmatrix} \mathcal{D}_{cayl}$$

of the graph of $\mathcal{O}$ is a dissipative structure on $\mathcal{B} = \left[\begin{smallmatrix} \mathcal{F} \\ \mathcal{E} \end{smallmatrix} \right]$. The subspace $\mathcal{D} \subset \mathcal{B}$ is in general *not a graph space*.

Definition 3.1: For any dissipative $\mathcal{D} \subset \mathcal{B}$ (and $\mathcal{G}$ as above), we define the *scattering representation*

$$r_{\mathcal{E},\mathcal{G}} e + r_{\mathcal{F},\mathcal{G}} f = \mathcal{O}(r_{\mathcal{E},\mathcal{G}} e - r_{\mathcal{F},\mathcal{G}} f) \tag{3}$$

of $\mathcal{D}$. By this we mean, that $\left[\begin{smallmatrix} f \\ e \end{smallmatrix} \right] \in \mathcal{D}$ if and only if $r_{\mathcal{E},\mathcal{G}} e - r_{\mathcal{F},\mathcal{G}} f \in \operatorname{dom} \mathcal{O}$ and (3) holds.

The variables $s_+ := r_{\mathcal{E},\mathcal{G}} e + r_{\mathcal{F},\mathcal{G}} f$ and $s_- := r_{\mathcal{E},\mathcal{G}} e - r_{\mathcal{F},\mathcal{G}} f$ are referred to as *scattering variables.*

Note, that the scattering representation (3) is unique only after we fix $\mathcal{G}$ and $r_{\mathcal{F},\mathcal{G}}$. We also remark, that the scattering variables can be interpreted as an *incoming* and an *outgoing wave* of the structure, see [4].

The following theorem exposes a very important and useful connection between a dissipative structure $\mathcal{D}$ and its associated scattering operator $\mathcal{O}$.

Theorem 3.2: A subspace $\mathcal{D} \subset \mathcal{B}$ is dissipative if and only if there exists a (unique) contractive scattering operator $\mathcal{O} : \mathcal{G} \supset \operatorname{dom} \mathcal{O} \to \mathcal{G}$, such that $\left[\begin{smallmatrix} f \\ e \end{smallmatrix} \right] \in \mathcal{D} \iff$ (3) holds. Moreover, $\mathcal{D}$ is a Tellegen structure if and only if its scattering operator $\mathcal{O}$ is an isometry.

Recalling that $\operatorname{dom} \mathcal{O} = \begin{bmatrix} -r_{\mathcal{F},\mathcal{G}} & r_{\mathcal{E},\mathcal{G}} \end{bmatrix} \mathcal{D}$, we have that $\mathcal{D}$ is maximally dissipative if and only if $\operatorname{dom} \mathcal{O} = \mathcal{G}$.

Finally, $\mathcal{D}$ is a Dirac structure if and only if its scattering operator $\mathcal{O}$ is a unitary operator on $\mathcal{G}$, i.e. $\mathcal{O}$ is isometric with $\operatorname{dom} \mathcal{O} = \operatorname{ran} \mathcal{O} = \mathcal{G}$. All Dirac structures are maximal, owing to Theorem 2.4.

We conclude that there is a 1-1 correspondence between e.g. the set of unitary operators on $\mathcal{G}$ and the set of Dirac structures on $\left[\begin{smallmatrix} \mathcal{F} \\ \mathcal{E} \end{smallmatrix} \right]$. From this we see, that a Dirac structure is a skew-adjoint relation, which is essentially the same as a self-adjoint relation. Basic theory of linear relations is to be found in [11].

As a reinterpretation of Theorem 3.2 we have the following mixed characterisation of Dirac-type structures.

Corollary 3.3: The subspace $\mathcal{D} \subset \mathcal{B}$ is a Dirac structure if and only if the following (nonequivalent) conditions all hold.

1) $d \in \mathcal{D} \Longrightarrow [d,d]_{\mathcal{B}} = 0$, i.e. $\mathcal{D}$ is a Tellegen structure,
2) $\begin{bmatrix} -r_{\mathcal{F},\mathcal{G}} & r_{\mathcal{E},\mathcal{G}} \end{bmatrix} \mathcal{D} = \mathcal{G}$, i.e. the scattering variable $r_{\mathcal{E},\mathcal{G}} e - r_{\mathcal{F},\mathcal{G}} f$ is *free* and
3) $\begin{bmatrix} r_{\mathcal{F},\mathcal{G}} & r_{\mathcal{E},\mathcal{G}} \end{bmatrix} \mathcal{D} = \mathcal{G}$, i.e. the scattering variable $r_{\mathcal{E},\mathcal{G}} e + r_{\mathcal{F},\mathcal{G}} f$ is free.

We note, that $\mathcal{D}$ is maximally dissipative if and only if, in addition to condition 2, we have

1') $d \in \mathcal{D} \Longrightarrow [d,d]_{\mathcal{B}} \leq 0$.

In this case, condition 3 is irrelevant.

The following theorem describes a certain class of Tellegen structures that are "almost Dirac".

Theorem 3.4: Let $\mathcal{D} \subset \mathcal{B}$. The closure $\overline{\mathcal{D}}$ of $\mathcal{D}$ in $\mathcal{B}$ is a Dirac structure if and only if $\mathcal{D}$ is a Tellegen structure and $\operatorname{dom}\mathcal{O}$ and $\operatorname{ran}\mathcal{O}$ are dense in $\mathcal{G}$.

Letting $\mathcal{O}$ be the isometric scattering operator of $\mathcal{D}$, the unitary scattering operator $\overline{\mathcal{O}}$ of $\overline{\mathcal{D}}$ is then the unique continuous extension $\mathcal{O}|_{\mathcal{G}}$ of $\mathcal{O}$ to $\mathcal{G}$.

Similarly, $\overline{\mathcal{D}}$ is a maximally dissipative structure if and only if $\mathcal{D}$ is dissipative and $\operatorname{dom}\mathcal{O}$ is dense in $\mathcal{G}$.

Example 3.5: Let $H^1(0,1)$ denote the Sobolev space of absolutely continuous real-valued functions with (distributional) derivative in $L^2(0,1)$. By Theorem 3.6 of [12],

$$\mathcal{D} := \left\{ \begin{bmatrix} f_1 \\ f_2 \\ e_1 \\ e_2 \end{bmatrix} \in \begin{bmatrix} L^2(0,1) \\ \mathbb{R} \\ L^2(0,1) \\ \mathbb{R} \end{bmatrix} \mid e_1 \in H^1(0,1)\ , \right.$$
$$\left. f_1 = \frac{\mathrm{d}}{\mathrm{d}z} e_1, \begin{bmatrix} f_2 \\ e_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} e_1(1) - e_1(0) \\ e_1(1) + e_1(0) \end{bmatrix} \right\}.$$

is a split Dirac structure.

The split scattering variables are given by

$$\begin{bmatrix} e_1 + f_1 \\ e_2 - f_2 \end{bmatrix} = \begin{bmatrix} (I + \frac{\mathrm{d}}{\mathrm{d}z})e_1 \\ \sqrt{2}e_1(0) \end{bmatrix} \quad \text{and}$$
$$\begin{bmatrix} e_1 - f_1 \\ e_2 + f_2 \end{bmatrix} = \begin{bmatrix} (I - \frac{\mathrm{d}}{\mathrm{d}z})e_1 \\ \sqrt{2}e_1(1) \end{bmatrix}.$$

Thus, define $\varphi_a : L^2(0,1) \to \mathbb{R}$ by $\varphi_a e_1 := e_1(a)$, for $a \in \{0,1\}$. One can show that the operator $\left[\begin{smallmatrix} 1-\frac{\mathrm{d}}{\mathrm{d}z} \\ \sqrt{2}\varphi_1 \end{smallmatrix}\right]$: $H^1(0,1) \to \left[\begin{smallmatrix} L^2(0,1) \\ \mathbb{R} \end{smallmatrix}\right]$ is invertible, whereafter trivially

$$\mathcal{O} = \begin{bmatrix} I + \frac{\mathrm{d}}{\mathrm{d}z} \\ \sqrt{2}\varphi_0 \end{bmatrix} \begin{bmatrix} I - \frac{\mathrm{d}}{\mathrm{d}z} \\ \sqrt{2}\varphi_1 \end{bmatrix}^{-1}.$$

We skip the details for brevity, but in the end we obtain, that $\mathcal{O} = \left[\begin{smallmatrix} \mathcal{O}_{11} & \mathcal{O}_{12} \\ \mathcal{O}_{21} & \mathcal{O}_{22} \end{smallmatrix}\right]$ is given by (for $z \in [0,1]$):

$$\begin{aligned} (\mathcal{O}_{11}g_1)(z) &= -2\int_1^z \exp(z-\xi)g_1(\xi)\,\mathrm{d}\xi - g_1(z), \\ (\mathcal{O}_{12}g_2)(z) &= \sqrt{2}\exp(z-1)g_2, \\ \mathcal{O}_{21}g_1 &= \sqrt{2}\int_0^1 \exp(-\xi)g_1(\xi)\,\mathrm{d}\xi \quad \textit{and} \\ \mathcal{O}_{22} &= \exp(-1), \end{aligned}$$

where $\exp(\cdot)$ denotes the natural exponential function.

We note, that the scattering representation of even the simplest infinite-dimensional example is quite involved.

We end this section by giving the scattering representation of a general split dissipative structure $\mathcal{D} \subset \left[\begin{smallmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \\ \mathcal{F}_1 \\ \mathcal{F}_2 \end{smallmatrix}\right]$. We take also $\mathcal{G} = \left[\begin{smallmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{smallmatrix}\right]$ and proceed by choosing $r_{\mathcal{E},\mathcal{G}} = \left[\begin{smallmatrix} I_1 & 0 \\ 0 & I_2 \end{smallmatrix}\right]$, obtaining $r_{\mathcal{F},\mathcal{G}} = r_{\mathcal{E},\mathcal{G}} r_{\mathcal{F},\mathcal{E}} = \left[\begin{smallmatrix} I_1 & 0 \\ 0 & -I_2 \end{smallmatrix}\right]$.

The scattering representation of the split dissipative structure $\mathcal{D}$ then becomes

$$\begin{bmatrix} e_1 + f_1 \\ e_2 - f_2 \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{11} & \mathcal{O}_{12} \\ \mathcal{O}_{21} & \mathcal{O}_{22} \end{bmatrix} \begin{bmatrix} e_1 - f_1 \\ e_2 + f_2 \end{bmatrix},$$

for some contractive scattering operator $\left[\begin{smallmatrix} \mathcal{O}_{11} & \mathcal{O}_{12} \\ \mathcal{O}_{21} & \mathcal{O}_{22} \end{smallmatrix}\right]$, with domain given by $\operatorname{dom}\mathcal{O} = \left[\begin{smallmatrix} -I_1 & 0 & I_1 & 0 \\ 0 & I_2 & 0 & I_2 \end{smallmatrix}\right]\mathcal{D}$ and range given by $\operatorname{ran}\mathcal{O} = \left[\begin{smallmatrix} I_1 & 0 & I_1 & 0 \\ 0 & -I_2 & 0 & I_2 \end{smallmatrix}\right]\mathcal{D}$.

IV. Composition of Dirac-type structures

In this section we study the composition of split structures, taking the following definition as our starting point.

Definition 4.1: Let $\mathcal{D}_A \subset \left[\begin{smallmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \\ \mathcal{F}_1 \\ \mathcal{F}_2 \end{smallmatrix}\right]$ and $\mathcal{D}_B \subset \left[\begin{smallmatrix} \mathcal{F}_2 \\ \mathcal{F}_3 \\ \mathcal{F}_2 \\ \mathcal{F}_3 \end{smallmatrix}\right]$ be split Dirac-type structures.

By the *composition* $\mathcal{D}_A \circ \mathcal{D}_B$ of $\mathcal{D}_A$ and $\mathcal{D}_B$ we mean the subspace

$$\mathcal{D}_A \circ \mathcal{D}_B = \left\{ \begin{bmatrix} f_1 \\ f_3 \\ e_1 \\ e_3 \end{bmatrix} \mid \exists \begin{bmatrix} f_2 \\ e_2 \end{bmatrix} \in \begin{bmatrix} \mathcal{F}_2 \\ \mathcal{F}_2 \end{bmatrix} : \right.$$
$$\left. \begin{bmatrix} f_1 \\ f_2 \\ e_1 \\ e_2 \end{bmatrix} \in \mathcal{D}_A \ \wedge \ \begin{bmatrix} f_2 \\ f_3 \\ e_2 \\ e_3 \end{bmatrix} \in \mathcal{D}_B \right\} \subset \begin{bmatrix} \mathcal{F}_1 \\ \mathcal{F}_3 \\ \mathcal{F}_1 \\ \mathcal{F}_3 \end{bmatrix}.$$

Please see Fig. 1 for an illustration.

The composition is *regular* if $\left[\begin{smallmatrix} f_1 \\ f_3 \\ e_1 \\ e_3 \end{smallmatrix}\right] = 0$ implies that $\left[\begin{smallmatrix} f_2 \\ e_2 \end{smallmatrix}\right] = 0$, i.e. the external signals determine the internal signals uniquely.

This definition means, that we compose the two systems *fully and only* through $\mathcal{F}_2$, whereafter we disregard the internal variables. A direct implication is, that $\mathcal{D}_A \circ \mathcal{D}_B = \mathcal{P}(\mathcal{D}_B \circ \mathcal{D}_A)$, where $\mathcal{P}$ denotes a trivial permutation of the power variables.

We now proceed by studying composition from the scattering point of view. Let therefore $\mathcal{D}_A$ and $\mathcal{D}_B$ be split Dirac-type structures with split scattering representations

$$\begin{bmatrix} e_1 + f_1 \\ e_2^A - f_2^A \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{11}^A & \mathcal{O}_{12}^A \\ \mathcal{O}_{21}^A & \mathcal{O}_{22}^A \end{bmatrix} \begin{bmatrix} e_1 - f_1 \\ e_2^A + f_2^A \end{bmatrix} \quad \text{and}$$
$$\begin{bmatrix} e_2^B + f_2^B \\ e_3 - f_3 \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{22}^B & \mathcal{O}_{23}^B \\ \mathcal{O}_{32}^B & \mathcal{O}_{33}^B \end{bmatrix} \begin{bmatrix} e_2^B - f_2^B \\ e_3 + f_3 \end{bmatrix},$$

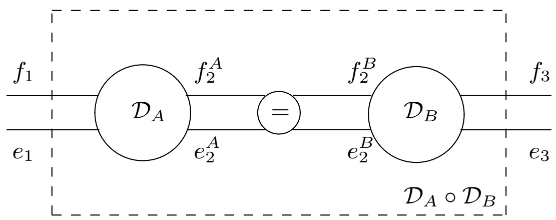

Fig. 1. A graphical interpretation of composition. We compose the structures $\mathcal{D}_A$ and $\mathcal{D}_B$ by applying "=", i.e. by setting $e_2^A = e_2^B$ and $f_2^A = f_2^B$.

respectively. Compose the structures by setting $e_2^A = e_2^B =: e_2$ and $f_2^A = f_2^B =: f_2$, or equivalently $e_2^A - f_2^A = e_2^B - f_2^B = e_2 - f_2$ and $e_2^A + f_2^A = e_2^B + f_2^B = e_2 + f_2$.

One can easily show, that the composition of two dissipative structures is again a dissipative structure, thus having a scattering representation:

$$\begin{bmatrix} e_1 + f_1 \\ e_3 - f_3 \end{bmatrix} = \tilde{\mathcal{O}} \begin{bmatrix} e_1 - f_1 \\ e_3 + f_3 \end{bmatrix},$$

for some contractive (scattering) operator $\tilde{\mathcal{O}}$. See Fig. 2.

In general, $\operatorname{dom} \tilde{\mathcal{O}} \neq \left[\begin{smallmatrix} \mathcal{F}_1 \\ \mathcal{F}_3 \end{smallmatrix}\right]$, but should we be able to choose the value of $\left[\begin{smallmatrix} e_1-f_1 \\ e_3+f_3 \end{smallmatrix}\right]$ arbitrarily, then $\left[\begin{smallmatrix} e_1-f_1 \\ e_3+f_3 \end{smallmatrix}\right]$ is said to be *free in the composition.*

We say that some choice of $\left[\begin{smallmatrix} e_1-f_1 \\ e_3+f_3 \end{smallmatrix}\right]$ is *consistent with* $\mathcal{D}_A \circ \mathcal{D}_B$ if there exists some internal waves e_2+f_2, e_2-f_2 and corresponding e_1+f_1, e_3-f_3, such that $\left[\begin{smallmatrix} f_1 \\ f_2 \\ e_1 \\ e_2 \end{smallmatrix}\right] \in \mathcal{D}_A$ and $\left[\begin{smallmatrix} f_2 \\ f_3 \\ e_2 \\ e_3 \end{smallmatrix}\right] \in \mathcal{D}_B$. We observe, that $\operatorname{dom} \tilde{\mathcal{O}}$ is the subspace of consistent choices of $\left[\begin{smallmatrix} e_1-f_1 \\ e_3+f_3 \end{smallmatrix}\right]$.

We now answer the question when the composition through $\mathcal{F}_2$ of two Dirac structures is again a Dirac structure. This is always the case if $\mathcal{F}_2$ is finite-dimensional, but we show that this is *not* the case for infinite-dimensional $\mathcal{F}_2$.

It turns out, that $\left[\begin{smallmatrix} e_1-f_1 \\ e_3+f_3 \end{smallmatrix}\right]$ is consistent with $\mathcal{D}_A \circ \mathcal{D}_B$ if and only if

$$\begin{bmatrix} \mathcal{O}_{21}^A & \mathcal{O}_{22}^A \mathcal{O}_{23}^B \end{bmatrix} \begin{bmatrix} e_1 - f_1 \\ e_3 + f_3 \end{bmatrix} \in \operatorname{ran}(\mathcal{O}_{22}^A \mathcal{O}_{22}^B - I_2).$$

The proof of the following theorem, is based on that insight.

Theorem 4.2: If $\mathcal{D}_A$ and $\mathcal{D}_B$ are both Tellegen structures, then we have the following results.

1) All elements of $\mathcal{D}_A \circ \mathcal{D}_B$ have zero power, i.e.
$$\forall \begin{bmatrix} f_1 \\ f_3 \\ e_1 \\ e_3 \end{bmatrix} \in \mathcal{D}_A \circ \mathcal{D}_B : \left[\begin{bmatrix} f_1 \\ f_3 \\ e_1 \\ e_3 \end{bmatrix}, \begin{bmatrix} f_1 \\ f_3 \\ e_1 \\ e_3 \end{bmatrix} \right]_{13} = 2\text{Re}\, \langle e_1, f_1 \rangle_1 - 2\text{Re}\, \langle e_3, f_3 \rangle_3 = 0.$$
In other words, $\mathcal{D}_A \circ \mathcal{D}_B$ is a split Tellegen structure.

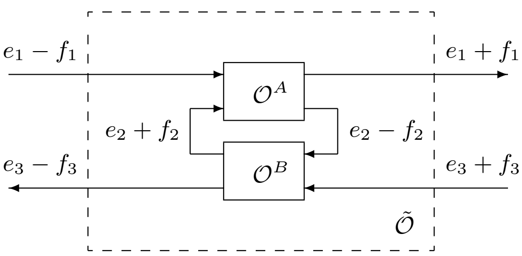

Fig. 2. Composition considered from a scattering point of view. Note, that there exists some internal flow/effort pair (f_2, e_2) if and only if there exists some internal waves $e_2 + f_2$ and $e_2 - f_2$.

2) The variable $\left[\begin{smallmatrix} e_1-f_1 \\ e_3+f_3 \end{smallmatrix}\right]$ is free in the composition of $\mathcal{D}_A$ and $\mathcal{D}_B$ if and only if the blocks of $\mathcal{O}^A$ and $\mathcal{O}^B$ satisfy
$$\operatorname{ran} \begin{bmatrix} \mathcal{O}_{21}^A & \mathcal{O}_{22}^A \mathcal{O}_{23}^B \end{bmatrix} \subset \operatorname{ran}(\mathcal{O}_{22}^A \mathcal{O}_{22}^B - I_2).$$
3) Denote the *bounded* adjoint of $\mathcal{O}_{22}^A$ by $\mathcal{O}_{22}^{A*}$ etc. Then the variable $\left[\begin{smallmatrix} e_1+f_1 \\ e_3-f_3 \end{smallmatrix}\right]$ is free in the composition if and only if
$$\operatorname{ran} \begin{bmatrix} \mathcal{O}_{22}^{B*} \mathcal{O}_{12}^{A*} & \mathcal{O}_{32}^{B*} \end{bmatrix} \subset \operatorname{ran}(\mathcal{O}_{22}^{B*} \mathcal{O}_{22}^{A*} - I_2).$$
4) The composition $\mathcal{D}_A \circ \mathcal{D}_B$ is a split Dirac structure if and only if the (nonequivalent) conditions 2 and 3 both hold.

If $\mathcal{D}_A$ and $\mathcal{D}_B$ are both dissipative, then $\mathcal{D}_A \circ \mathcal{D}_B$ is dissipative. Trivially $\mathcal{D}_A \circ \mathcal{D}_B$ is maximally dissipative if and only if condition 2 holds.

Theorem 4.2 was given in the case $\mathcal{O}^B = -I_2$ in [5]. We show the nonequivalence of conditions 2 and 3 of Theorem 4.2 in Example 4.6, which is also taken from [5].

We emphasise, that the composition of two dissipative structures is dissipative and the composition of two accumulative structures is an accumulative structure. Thus, in particular, the composition of two Tellegen structures is always a Tellegen structure. We refer to this property by saying, that the composition is *power conserving*.

Instead of computing the explicit scattering representations, one can also proceed as follows. One starts by composing any two compatible split structures, not even necessarily dissipative ones, whereafter one checks whether the composition is a Tellegen structure. If this is the case, then the next step is to check whether $\left[\begin{smallmatrix} e_1-f_1 \\ e_3+f_3 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} e_1+f_1 \\ e_3-f_3 \end{smallmatrix}\right]$ are free. Compare this procedure to Corollary 3.3. The maximally dissipative case is similar, but slightly simpler.

The following corollary explains why composition of two maximally dissipative/Dirac structures through a finite-dimensional space $\mathcal{F}_2$ always yields a new maximally dissipative/Dirac structure. It is a simple consequence of the fact, that all finite-dimensional spaces, in particular range spaces, are closed.

Corollary 4.3: Let $\mathcal{D}_A$ and $\mathcal{D}_B$ be split Dirac structures. If $\operatorname{ran} \mathcal{O}_{22}^A \mathcal{O}_{22}^B - I_2$ is closed in $\mathcal{F}_2$, then $\mathcal{D}_A \circ \mathcal{D}_B$ is a split Dirac structure.

The following corollary is also useful sometimes, because we know that computing a full scattering representation often is quite complicated. Computing only the feedthrough term $\mathcal{O}_{22}^A$ is a bit easier.

Corollary 4.4: If $\mathcal{D}_A$ is a split maximally dissipative/Dirac structure with $\|\mathcal{O}_{22}^A\| < 1$, then $\mathcal{D}_A \circ \mathcal{D}_B$ is a split maximally dissipative/Dirac structure whenever $\mathcal{D}_B$ is a split maximally dissipative/Dirac structure.

One may also wonder when the closure of a composition is a Dirac structure. At first glance, it might seem reasonable, that this could always be the case. However, in Example 4.6, we show that this is incorrect.

Theorem 4.5: Letting $\mathcal{D}_A$ and $\mathcal{D}_B$ be Dirac structures, the closure $\overline{\mathcal{D}_A \circ \mathcal{D}_B}$ in $\left[\begin{smallmatrix}\mathcal{F}_1\\ \mathcal{F}_3\end{smallmatrix}\right]$ of $\mathcal{D}_A \circ \mathcal{D}_B$ is a Dirac structure if and only if there exist dense subspaces $U, V \subset \left[\begin{smallmatrix}\mathcal{F}_1\\ \mathcal{F}_3\end{smallmatrix}\right]$, such that

1) $\operatorname{ran} \left[\begin{array}{cc} \mathcal{O}^B_{22}\mathcal{O}^A_{21} & \mathcal{O}^B_{23} \end{array}\right]\Big|_U \subset \operatorname{ran}(\mathcal{O}^B_{22}\mathcal{O}^A_{22} - I_2)$ and
2) $\operatorname{ran} \left[\begin{array}{cc} \mathcal{O}^{A*}_{12} & \mathcal{O}^{A*}_{22}\mathcal{O}^{B*}_{32} \end{array}\right]\Big|_V \subset \operatorname{ran}(\mathcal{O}^{A*}_{22}\mathcal{O}^{B*}_{22} - I_2)$.

We most frequently take $U = \operatorname{dom}\tilde{\mathcal{O}}$ and $V = \operatorname{ran}\tilde{\mathcal{O}}$. In any case, $U \subset \operatorname{dom}\tilde{\mathcal{O}}$ and $V \subset \operatorname{ran}\tilde{\mathcal{O}}$. Thus an equivalent statement is, that $\operatorname{dom}\tilde{\mathcal{O}}$ and $\operatorname{ran}\tilde{\mathcal{O}}$ both are dense in $\mathcal{G}$.

In the maximally dissipative case, condition 2 is irrelevant as usual.

The following example from [5] shows, that sometimes not even the closure of a composition is a Dirac structure. It also shows the nonequivalence of the conditions 2 and 3 in Theorem 4.2.

Example 4.6: Consider $\mathcal{F}_1 = l^2(-\mathbb{N}_0)$, $\mathcal{F}_2 = l^2(\mathbb{Z}_+)$ and $\mathcal{F}_3$ absent. We take $\mathcal{O}^A$ to be the (unitary) right shift on $l^2(\mathbb{Z})$, i.e. in particular $\mathcal{O}^A_{12} = 0$,

$$\mathcal{O}^A_{21} = \begin{bmatrix} \dots & 0 & 1 \\ \dots & 0 & 0 \\ \iddots & \vdots & \vdots \end{bmatrix} \quad \text{and}$$

$$\mathcal{O}^A_{22} = \begin{bmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Furthermore, we choose $\mathcal{O}^B_{22} = -I_2$, with $\mathcal{O}^B_{23}$ absent, so that $\mathcal{D}_B$ grounds the effort, while the flow is arbitrary. Then condition 1 of Theorem 4.5 becomes, that for some dense $U \subset \mathcal{F}_1$

$$\operatorname{ran} \begin{bmatrix} \dots & 0 & 1 \\ \dots & 0 & 0 \\ \iddots & \vdots & \vdots \end{bmatrix}\Bigg|_U \subset \operatorname{ran} \begin{bmatrix} 1 & 0 & 0 & \dots \\ 1 & 1 & 0 & \dots \\ 0 & 1 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Sadly, we have

$$\begin{bmatrix} \dots & 0 & 1 \\ \dots & 0 & 0 \\ \iddots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ u_{-1} \\ u_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots \\ 1 & 1 & 0 & \dots \\ 0 & 1 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \end{bmatrix}$$

if and only if $u_0 = w_1 = -w_2 = w_3 = \dots$ Moreover, $\left[\begin{smallmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \end{smallmatrix}\right] \in l^2(\mathbb{Z}_+)$ if and only if that $u_0 = w_i = 0$ for all $i \geq 1$.

The domain $\operatorname{dom}\tilde{\mathcal{O}} = \left\{ \left[\begin{smallmatrix} \vdots \\ u_{-1} \\ u_0 \end{smallmatrix}\right] \in l^2(-\mathbb{N}_0) \mid u_0 = 0 \right\}$ of $\tilde{\mathcal{O}}$ is not dense in $l^2(-\mathbb{N}_0)$, and since $U \subset \operatorname{dom}\tilde{\mathcal{O}}$, condition 1 of Theorem 4.5 is not satisfied. Not even $\overline{\mathcal{D}_A \circ \mathcal{D}_B}$ is a Dirac structure.

Since condition 2 of Theorem 4.2 implies condition 1 of Theorem 4.5, condition 2 of Theorem 4.2 is not satisfied either. For condition 3 of Theorem 4.2 we obtain

$$\operatorname{ran}\mathcal{O}^{B*}_{22}\mathcal{O}^{A*}_{12} = \{0\} \subset \operatorname{ran}(\mathcal{O}^{B*}_{22}\mathcal{O}^{A*}_{22} - I_2),$$

which is trivially true. Condition 3, but not condition 2, of Theorem 4.2 is satisfied, and we have shown that these two conditions are not equivalent.

Before continuing with the next section, we make the following final remark about Example 4.6. From the example, we see that the domain of the scattering operator $\tilde{\mathcal{O}}$ of $\mathcal{D}_A \circ \mathcal{D}_B$ is not necessarily dense in $\left[\begin{smallmatrix}\mathcal{F}_1\\ \mathcal{F}_3\end{smallmatrix}\right]$. In such cases, $\overline{\mathcal{D}_A \circ \mathcal{D}_B}$ cannot be even maximally dissipative.

V. THE REDHEFFER STAR PRODUCT

In a composed maximally dissipative structure, the (scattering) operator $\tilde{\mathcal{O}}$, that relates $\left[\begin{smallmatrix} e_1+f_1 \\ e_3-f_3 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} e_1-f_1 \\ e_3+f_3 \end{smallmatrix}\right]$, goes by the name the *Redheffer star product* of $\mathcal{O}^A$ and $\mathcal{O}^B$. See Fig. 2. For more information, consult [13, Chapter 10] and [14].

We provide an expression for $\mathcal{O}^A \star \mathcal{O}^B$, that is valid also in the *irregular composition* case, i.e. when possibly neither $\mathcal{O}^A_{22}\mathcal{O}^B_{22} - I_2$ nor $\mathcal{O}^{B*}_{22}\mathcal{O}^{A*}_{22} - I_2$ is injective. Also note, that we here only assume that $\mathcal{O}^A$ and $\mathcal{O}^B$ are fully defined contractions, and not e.g. unitary, as is often done.

Denote $\mathcal{F}_{2,1} := \ker(\mathcal{O}^A_{22}\mathcal{O}^B_{22} - I_2)^\perp$, let the (Hilbert-space) orthogonal projection onto $\mathcal{F}_{2,1}$ be denoted by $\mathcal{P}_1$ and let $\mathcal{I}_1 : \mathcal{F}_{2,1} \to \mathcal{F}_2$ be the injection operator $f_1 \to \left[\begin{smallmatrix} f_1 \\ 0 \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{F}_{2,1} \\ \mathcal{F}^\perp_{2,1} \end{smallmatrix}\right]$. Then, letting $(\mathcal{O}^A_{22}\mathcal{O}^B_{22} - I_2)_1 := \mathcal{P}_1(\mathcal{O}^A_{22}\mathcal{O}^B_{22} - I_2)\mathcal{I}_1$, $(\mathcal{O}^A_{22}\mathcal{O}^B_{22} - I_2)_1 \in \mathcal{L}(\mathcal{F}_{2,1})$ is injective, thus having a possibly unbounded inverse $(\mathcal{O}^A_{22}\mathcal{O}^B_{22} - I_2)^{-1}_1$ defined on its range.

Theorem 5.1: The scattering representation of $\mathcal{D}_A \circ \mathcal{D}_B$ is $\left[\begin{smallmatrix} e_1+f_1 \\ e_3-f_3 \end{smallmatrix}\right] = \tilde{\mathcal{O}} \left[\begin{smallmatrix} e_1-f_1 \\ e_3+f_3 \end{smallmatrix}\right]$, where

$$\operatorname{dom}\tilde{\mathcal{O}} = \left[\begin{array}{cc} -r_{\mathcal{F},\mathcal{G}} & r_{\mathcal{E},\mathcal{G}} \end{array}\right](\mathcal{D}_A \circ \mathcal{D}_B)$$

is the preimage of $\operatorname{ran}(\mathcal{O}^A_{22}\mathcal{O}^B_{22} - I_2)_1$ under the operator $\mathcal{P}_1 \left[\begin{array}{cc} \mathcal{O}^A_{21} & \mathcal{O}^A_{22}\mathcal{O}^B_{23} \end{array}\right]$, and

$$\tilde{\mathcal{O}} = \begin{bmatrix} \mathcal{O}^A_{11} & \mathcal{O}^A_{12}\mathcal{O}^B_{23} \\ 0 & \mathcal{O}^B_{33} \end{bmatrix} - \begin{bmatrix} \mathcal{O}^A_{12}\mathcal{O}^B_{22} \\ \mathcal{O}^B_{32} \end{bmatrix} \mathcal{I}_1 \times (\mathcal{O}^A_{22}\mathcal{O}^B_{22} - I_2)^{-1}_1 \mathcal{P}_1 \left[\begin{array}{cc} \mathcal{O}^A_{21} & \mathcal{O}^A_{22}\mathcal{O}^B_{23} \end{array}\right].$$

If $\mathcal{D}_A \circ \mathcal{D}_B$ is maximally dissipative, then we call $\tilde{\mathcal{O}}$ the *Redheffer star product* of $\mathcal{O}^A$ and $\mathcal{O}^B$ and write $\tilde{\mathcal{O}} = \mathcal{O}^A \star \mathcal{O}^B$.

We recall, that if $\overline{\mathcal{D}_A \circ \mathcal{D}_B}$ is a maximally dissipative structure, then $\operatorname{dom}\tilde{\mathcal{O}}$, is dense in $\left[\begin{smallmatrix}\mathcal{F}_1\\ \mathcal{F}_3\end{smallmatrix}\right]$. The scattering operator $\overline{\mathcal{O}}$ of $\overline{\mathcal{D}_A \circ \mathcal{D}_B}$ is then the unique continuous extension of $\tilde{\mathcal{O}}$ to $\left[\begin{smallmatrix}\mathcal{F}_1\\ \mathcal{F}_3\end{smallmatrix}\right]$.

VI. Final remarks

In the introduction we briefly mentioned that Dirac structures correspond to conservative systems, while maximally dissipative structures correspond to passive systems. For more details on passive systems, we refer to [6] and [7]. The connection to Dirac-type structures has been studied in [8] and [10]. We have not treated these issues in the present paper, but it is a matter of current research. The operator $\mathcal{O}_{22}$ has also been connected to the transfer function of the associated system.

Example 4.6 does not originate from a physical system. Indeed, the composition of two Dirac structures, originating from two physical systems, usually satisfies conditions 2 and 3 of Theorem 4.2. Some results on a "physical" interpretation of the conditions in this theorem have also been obtained.

VII. Acknowledgments

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While finishing the present paper, the authors became aware, that O. Iftime and A. Sandovici independently have obtained results on the composition of Dirac structures. Their results are similar to those presented in this paper.

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