LQR control for scalar finite and infinite platoons

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Keywords: linear quadratic control, spatially invariant systems, infinite-dimensional systems.

Abstract

In this paper we compare the behaviour of the LQR solution for a finite platoon model with its infinite version. We give examples where these are similar and some where they are quite different. For the scalar case we obtain sufficient conditions for the LQR solutions to be similar by relating the Toeplitz approximations to circulant approximating systems.

1 Introduction

In [9] a comparison was made between the LQR control of a long, finite platoon and of an infinite version (which is easier to analyse mathematically). For this particular example the infinite model reflected well the behaviour of the long finite platoon. Based on this and other examples they "argue that the infinite case is a useful paradigm to understand large platoons", but no theory to support this claim was given. In this paper we show by means of a counterexample that this claim is not true in general. More importantly, we give an insightful theoretical analysis of this paradigm for the scalar case.

The class of finite platoons of vehicles that we consider is given by

$$\begin{aligned}\dot{z}_r(t) &= \sum_{l=-N}^{N} a_l z_{r-l}(t) + \sum_{l=-N}^{N} b_l u_{r-l}(t), \\ y_r(t) &= \sum_{l=-N}^{N} c_l z_{r-l}(t), \quad -N \leq r \leq N,\end{aligned} \tag{1}$$

where only finitely many of the coefficients a_l, b_l, c_l are nonzero and z_k, y_k and u_k are set to zero for $|k| > N$. The above long platoons can also be written in a compact form $\Sigma(\mathbf{A}_N, \mathbf{B}_N, \mathbf{C}_N, 0)$

$$\begin{aligned}\dot{\mathbf{z}}_N(t) &= \mathbf{A}_N \mathbf{z}_N(t) + \mathbf{B}_N \mathbf{u}_N(t), \\ \mathbf{y}_N(t) &= \mathbf{C}_N \mathbf{z}_N(t), \ t \geq 0,\end{aligned} \tag{2}$$

where $\mathbf{z}_N(t) = \begin{bmatrix} z_{-N}(t) & z_{-N+1}(t) & \cdots & z_N(t) \end{bmatrix}^T$, $\mathbf{u}, \mathbf{y}$ are column vectors of size $2N+1$, and $\mathbf{A}_N, \mathbf{B}_N, \mathbf{C}_N$ are $(2N+1) \times (2N+1)$ banded Toeplitz matrices.
As $N \to \infty$ we arrive at the infinite-dimensional version which falls into the class of *spatially invariant systems* introduced in [1].

$$\dot{z}_r(t) = \sum_{l=-\infty}^{\infty} a_l z_{r-l}(t) + \sum_{l=-\infty}^{\infty} b_l u_{r-l}(t), \tag{3}$$

$$y_r(t) = \sum_{l=-\infty}^{\infty} c_l z_{r-l}(t), \quad r \in \mathbb{Z}, \quad t \geq 0, \tag{4}$$

where $a_l, b_l, c_l \in \mathbb{C}$ and $z_r(t)$, $u_r(t)$ and $y_r(t) \in \mathbb{C}$ are the state, the input and the output vectors, respectively, at time $t \geq 0$ and spatial point $r \in \mathbb{Z}$. As in [3, 4] we can formulate (3), (4) as a standard state linear system $\Sigma(A, B, C, 0)$

$$\begin{aligned}\dot{z}(t) &= (Az)(t) + (Bu)(t), \\ y(t) &= (Cz)(t), \ t \geq 0,\end{aligned} \tag{5}$$

with the state space Z, the input space U and the output space Y equal to $\ell_2(\mathbb{C}) = \{z \,|\, z = (z_r)_{r=-\infty}^{\infty}, z_r \in \mathbb{C}, \sum_{r=-\infty}^{\infty} |z_r|^2 < \infty\}$. A, B, C are convolution operators, e.g.,

$$((Ax)(t))_r = \sum_{l=-\infty}^{\infty} a_l x_{r-l}(t) = \sum_{l=-\infty}^{\infty} a_{r-l} x_l(t).$$

A more convenient representation is obtained we taking Fourier transforms $\check{x} = \mathfrak{F}x$, $A = \mathfrak{F}^{-1}\check{A}\mathfrak{F}$, so that

$$((\check{Ax})(t))(e^{j\theta}) = \check{A}(e^{j\theta})\check{x}(e^{j\theta}, t) = \left(\sum_{l=-\infty}^{\infty} a_l e^{j\theta l} \right) \check{x}(e^{j\theta}, t), \quad t \geq 0.$$

Note that our standing assumption is that only finitely many of the coefficients are nonzero which means that $\check{A}(e^{j\theta}), \check{B}(e^{j\theta}), \check{C}(e^{j\theta})$, are uniformly continuous in θ on $[0,2\pi]$ and $\check{A}, \check{B}, \check{C} \in \mathbf{L}_\infty(\partial\mathbb{D};\mathbb{C})$. Hence $\check{A}, \check{B}, \check{C}$ define bounded operators on $\mathbf{L}_2(\partial\mathbb{D};\mathbb{C})$. Now, $\ell_2(\mathbb{C})$ is isometrically isomorphic to $\mathbf{L}_2(\partial\mathbb{D};\mathbb{C})$ under the Fourier transform $\mathfrak{F}$ ($\|x\|_{\ell_2(\mathbb{C})} = \|\check{x}\|_{\mathbf{L}_2(\partial\mathbb{D},\mathbb{C})}$). Hence $\|A\| = \|\check{A}\|_\infty$, etc.
Taking Fourier transforms of the system equations (5), we obtain

$$\begin{aligned} \dot{\check{z}}(t) = \mathfrak{F}\dot{z}(t) &= \check{A}\check{z}(t) + \check{B}\check{u}(t), \\ \check{y}(t) = \mathfrak{F}y(t) &= \check{C}\check{z}(t), \ t \geq 0, \end{aligned} \tag{6}$$

where $\check{A} = \mathfrak{F}A\mathfrak{F}^{-1}$, $\check{B} = \mathfrak{F}B\mathfrak{F}^{-1}$ and $\check{C} = \mathfrak{F}C\mathfrak{F}^{-1}$ are multiplicative operators.
The state linear system $\Sigma(A,B,C,0)$ is isometrically isomorphic to the state linear system $\Sigma(\check{A},\check{B},\check{C},0)$ with the state space, input and output spaces $\mathbf{L}_2(\partial\mathbb{D};\mathbb{C})$. Their system theoretic properties are identical (see [2, Exercise 2.5]). For $\theta \in [0,2\pi]$ the system (6) can be written as

$$\begin{aligned} \dot{\check{z}}(e^{j\theta},t) &= \check{A}(e^{j\theta})\check{z}(e^{j\theta},t) + \check{B}(e^{j\theta})\check{u}(e^{j\theta},t) \\ \check{y}(e^{j\theta},t) &= \check{C}(e^{j\theta})\check{z}(e^{j\theta},t). \end{aligned} \tag{7}$$

The following example illustrates that the infinite platoon is not always a useful paradigm for the finite platoon.

Example 1.1 Let the positive parameter $\beta > 1$ be given. Consider the following finite platoon model of the form (1)

$$\begin{aligned} \dot{z}_r(t) &= z_r(t) + u_r(t) + \beta u_{r-1}(t), \quad -N+1 \leq r \leq N \\ \dot{z}_{-N}(t) &= z_{-N}(t) + u_{-N}(t), \\ y_r(t) &= z_r(t), \quad -N \leq r \leq N, \quad t \geq 0. \end{aligned}$$

which can be written in the compact form (2) with

$$\mathbf{A}_N = \mathbf{C}_N = I_{(2N+1)\times(2N+1)}, \quad \mathbf{B}_N = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ \beta & 1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \beta & 1 \end{bmatrix}.$$

The finite platoon is obviously stabilizable and detectable for all N. Factorize

$$\mathbf{B}_N\mathbf{B}_N^* = \begin{bmatrix} 1 & \beta & 0 & ... & 0 \\ \beta & 1+\beta^2 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \beta & 1+\beta^2 \end{bmatrix} = L_N \mathrm{diag}(\beta_k(N)) L_N^*,$$

where L_N is a unitary matrix. Then the solution $\mathbf{Q}_N$ to the corresponding control Riccati equation is readily calculated

$$\mathbf{Q}_N = L_N \mathrm{diag}\left(\frac{1+\sqrt{1+\beta_k(N)}}{\beta_k(N)}\right) L_N^*.$$

Hence

$$\|\mathbf{Q}_N\| = \max_{k=0,\ldots 2N} \frac{1+\sqrt{1+\beta_k(N)}}{\beta_k(N)},$$

which is achieved at the minimum value of $\beta_k(N)$. The closed-loop operator is given by

$$\mathbf{A}_N - \mathbf{B}_N\mathbf{B}_N^*\mathbf{Q}_N = L_N \mathrm{diag}\left(-\sqrt{1+\beta_k(N)}\right) L_N^*.$$

We claim that for $\beta > 1$ one eigenvalue approaches 0 as $N \to \infty$. It is readily verified that $\mathbf{B}_N\mathbf{B}_N^* v_N = w_N$, where

$$v_N = (-\beta^{-1}, \beta^{-2}, -\beta^{-3}, ..., \beta^{-2N}, -\beta^{-2N-1})^T, \quad w_N = (0,0,0,...,0,-\beta^{-2N-1})^T.$$

SInce $\beta > 1$, one eigenvalue becomes arbitrarily small as $N \to \infty$ which means that $\|\mathbf{Q}_N\| \to \infty$, and one eigenvalue of $\mathbf{A}_N - \mathbf{B}_N\mathbf{B}_N^*\mathbf{Q}_N$ approaches -1.
We show below that this behaviour is very different from that of the infinite platoon

$$\begin{aligned} \dot{z}_r(t) &= z_r(t) + u_r(t) + \beta u_{r-1}(t), \\ y_r(t) &= z_r(t), \quad r \in \mathbb{Z}, \quad t \geq 0. \end{aligned}$$

This system is isomorphic to version (7) with continuous operators $\check{A}(e^{j\theta}) = \check{C}(e^{j\theta}) = 1$, $\check{B}(e^{j\theta}) = 1 + \beta e^{-j\theta}$. It is clearly exponentially detectable and it is exponentially stabilizable, since the matrix $[\lambda - \check{A}(e^{j\theta}) : \check{B}(e^{j\theta})] = [\lambda - 1 : 1 + \beta e^{-j\theta}]$ has rank one for all $\lambda \in \overline{\mathbb{C}}_0^+$ and all $\theta \in [0, 2\pi]$ (see [3, 4]). The LQR Riccati equation $\check{A}^*\check{Q} + \check{Q}\check{A} - \check{Q}\check{B}\check{B}^*\check{Q} + \check{C}^*\check{C} = 0$ has the unique positive solution $\check{Q}(e^{j\theta}) = \frac{1+\sqrt{2+\beta^2+2\beta\cos\theta}}{1+\beta^2+2\beta\cos\theta}$. with norm

$$\|\check{Q}\| = \max_{0\leq\theta\leq 2\pi} \|\check{Q}(e^{j\theta})\| = \frac{1+\sqrt{1+(1-\beta^2)}}{(1-\beta)^2}.$$

The closed-loop operator is given by

$$\check{A}(e^{j\theta}) - \check{B}(e^{j\theta})\check{B}^*(e^{j\theta})\check{Q}(e^{j\theta}) = -\sqrt{2+\beta^2+2\beta\cos\theta}.$$

Since

$$\omega_\infty := \sup_{\theta\in[0,2\pi]} \sigma(\check{A}(e^{j\theta}) - \check{B}(e^{j\theta})\check{B}^*(e^{j\theta})\check{Q}(e^{j\theta})) = -\sqrt{2+\beta^2-2\beta},$$

the closed-loop system is is exponentially stable with stability margin $\sqrt{2+\beta^2-2\beta}$. Since $\beta > 1$, this is strictly larger than 1 and it increases with β. In contrast, for the finite platoon the stability margin converges to one as $N \to \infty$ for all $\beta > 1$.

The obvious conclusion from the above example is that the infinite-dimensional platoon is not always a useful paradigm for the large platoon as claimed in [9].

2 Main Results

The standard result on Riccati equations [2, Theorem 6.2.7], results on stabilizability and detectability of spatially invariant systems from[3, 4] and a continuity property from [11, Theorem 11.2.1] yield the following result.

Theorem 2.1 $\Sigma(A,B,C,0)$ *is exponentially stabilizable (detectable) if and only if* $(\check{A}(e^{j\theta}),\check{B}(e^{j\theta}),\check{C}(e^{j\theta}),0)$ *is stabilizable (detectable) for each* $\theta\in[0,2\pi]$. *If the above holds, then the control Riccati equation for (3), (4)*

$$A^*Q+QA-QBB^*Q+C^*C=0, \tag{8}$$

has a unique nonnegative solution Q and $A_Q=A-BB^*Q$ *generates an exponentially stable semigroup. Moreover, the control Riccati equation for (7)*

$$\check{A}^*\check{Q}+\check{Q}\check{A}-\check{Q}\check{B}\check{B}^*\check{Q}+\check{C}^*\check{C}=0, \tag{9}$$

has a unique nonnegative solution $\check{Q}\in\mathbf{L}_\infty(\partial\mathbb{D};\mathbb{C})$ *and* $\check{A}_Q=\check{A}-\check{B}\check{B}^*\check{Q}$ *generates an exponentially stable semigroup. Furthermore,* $\check{Q}(e^{j\theta})$ *is continuous in* θ *on* $[0,2\pi]$.

The problem of approximating solutions to operator Riccati equations has received much attention in the literature. However, the strongest convergence results (see [8]) are achieved only if the input and output spaces are finite-dimensional, which is never the case for spatially invariant systems. However, we can apply the theory in [10] applied to (8) with (2) as a sequence of approximating control systems.
Denote by $\pi^N:\mathbb{Z}=\ell_2\to\mathbb{C}^{2N+1}$ the natural projection with $i^N:\mathbb{C}^{2N+1}\to\ell_2$ the corresponding injection map: $\pi^N i^N=I_{2N+1}$. Denote $Z^N:=\mathbb{C}^{2N+1}$ with the induced inner product $<x,y>_N=<i^Nx,i^Ny>_{\ell_2}$. Then $\mathbf{A}_N,\mathbf{B}_N,\mathbf{C}_N$ are Toeplitz matrix representations of the maps $\pi^NA|_{Z^N}$, $\pi^NB|_{Z^N}$, $\pi^NC|_{Z^N}$, with Z^N as the state space, input space and output space. For simplicity of notation we use the same notation for the maps as for the matrices. Hence the finite platoon system $\Sigma(\mathbf{A}_N,\mathbf{B}_N,\mathbf{C}_N,0)$ converges strongly to the infinite-dimensional platoon system $\Sigma(A,B,C,0)$ in the following sense.

$$e^{At}z=\lim_{N\to\infty}i^Ne^{\mathbf{A}_Nt}\pi^Nz,\quad (e^{At})^*z=\lim_{N\to\infty}i^Ne^{\mathbf{A}_N^*t}\pi^Nz,\forall z\in\ell_2$$

uniformly on compact time intervals. Moreover, as $N\to\infty$

$$i^N\mathbf{B}_N\pi^Nu\to Bu,\quad i^N\mathbf{B}_N^*\pi^Nz\to B^*z,$$

$$i^N\mathbf{C}_N\pi^Nz\to Cz,\quad i^N\mathbf{C}_N^*\pi^Ny\to C^*y,\quad i^N\pi^Nz\to z \text{ for all } z,u,y\in\ell_2.$$

An application of [10, Theorem 1] together with [8] yield.

Theorem 2.2 *Consider the exponentially stabilizable and detectable state linear system (5)* $\Sigma(A,B,C,0)$ *and the sequence of finite-dimensional approximating systems*

(2) $\Sigma(\boldsymbol{A}_N,\boldsymbol{B}_N,\boldsymbol{C}_N,0)$. Suppose that $\Sigma(\boldsymbol{A}_N,\boldsymbol{B}_N,\boldsymbol{C}_N,0)$ are uniformly stabilizable and detectable, i.e., there exist $\boldsymbol{F}_N,\boldsymbol{L}_N \in \mathbb{C}^{(2N+1)\times(2N+1)}$, $F,L \in \mathcal{L}(\ell_2)$ such that

$$i^N \boldsymbol{F}_N \pi^N z \to Fz,\ i^N \boldsymbol{F}_N^* \pi^N z \to F^* z,\ i^N \boldsymbol{L}_N \pi^N z \to Lz,\ i^N \boldsymbol{L}_N^* \pi^N z \to Lz\ \ \forall z \in \ell_2,$$

$A+BF$ and $A+LC$ generate exponentially stable semigroups and there exist constants $M \geq 1, \beta > 0$ such that for all $N \in \mathbb{N}$

$$\|e^{(\boldsymbol{A}_N+\boldsymbol{B}_N\boldsymbol{F}_N)t}\| \leq Me^{-\beta t}, \quad \|e^{(\boldsymbol{A}_N+\boldsymbol{L}_N\boldsymbol{C}_N)t}\| \leq Me^{-\beta t}, t \geq 0.$$

If $Q \in \mathcal{L}(\ell_2)$ and $\boldsymbol{Q}_N \in \mathcal{L}(Z)$ denote the unique nonnegative solutions of their respective Riccati equations (8) and

$$\boldsymbol{A}_N^* \boldsymbol{Q}_N + \boldsymbol{Q}_N \boldsymbol{A}_N - \boldsymbol{Q}_N \boldsymbol{B}_N \boldsymbol{B}_N^* \boldsymbol{Q}_N + \boldsymbol{C}_N^* \boldsymbol{C}_N = 0, \tag{10}$$

then $\boldsymbol{Q}_N$ converges strongly to Q, i.e.,

$$Qz = \lim_{N\to\infty} i^N \boldsymbol{Q}_N \pi^N z, \quad \forall z \in \ell_2,$$

and consequently $\|\boldsymbol{Q}_N\|$is uniformly bounded in N. Denote $A_Q := A - BB^*Q$ and $\boldsymbol{A}_{Q_N} := \boldsymbol{A}_N - \boldsymbol{B}_N \boldsymbol{B}_N^* \boldsymbol{Q}_N$. Then $\boldsymbol{A}_{Q_N}$ converges strongly to A_Q, i.e.,

$$i^N e^{\boldsymbol{A}_{Q_N} t} \pi^N z \to e^{A_Q t} z, \quad \forall z \in \ell_2 \ \text{ as } \ N \to \infty \ \text{ uniformly on compact time intervals}.$$

Moreover, there exist positive constants $\overline{M}, \mu$ such that

$$\|e^{A_Q t}\| \leq \overline{M} e^{-\mu t}, \quad \|e^{\boldsymbol{A}_{Q_N} t}\| \leq \overline{M} e^{-\mu t} \ \text{ for all } \ t \geq 0. \tag{11}$$

We remark that the solutions $\mathbf{Q}_N$ of (10) not Toeplitz in general. It is easy to see that in our Example 1.1 the Toeplitz approximating system will be uniformly exponentially stable only if $\beta < 1$.

Although we have given conditions for the strong convergence of $e^{\mathbf{A}_{Q_N} t}$ to $e^{A_Q t}$, this says nothing about the convergence of the stability margins nor about the convergence of the closed-loop transfer functions in the $\mathbf{H}_\infty$- norm. In order to do this we examine the related *circular approximants* of $\check{A}, \check{B}, \check{C}$ of dimension $n = 2N+1$ denoted by $\tilde{A}_N, \tilde{B}_N, \tilde{C}_N$ (see the Appendix). For the proofs of the remaining theorems see [5].

Theorem 2.3 *Consider the exponentially stabilizable and detectable system $\Sigma(A,B,C,0)$ on the state-space ℓ_2 with Q the unique self-adjoint solution to the Riccati equation (8)*

1. *The following Riccati equation has a unique self-adjoint stabilizing solution $\tilde{Q}_N$ which is the circular approximant of $\check{Q}$*

$$\tilde{A}_N^* \tilde{Q}_N + \tilde{Q}_N \tilde{A}_N - \tilde{Q}_N \tilde{B}_N \tilde{B}_N^* \tilde{Q}_N + \tilde{C}_N^* \tilde{C}_N = 0. \tag{12}$$

$i^N \tilde{Q}_N \pi^N$ converges strongly to Q and $i^N \tilde{A}_{Q_N} \pi^N$ converges strongly to $A_Q = A - BB^*Q$ as $N \to \infty$, where $\tilde{A}_{Q_N} = \tilde{A}_N - \tilde{B}_N \tilde{B}_N^* \tilde{Q}_N$ is a contraction semigroup.

2. $\limsup_{N\to\infty}\|\tilde{Q}_N\| = \|\check{Q}\| = \|Q\|$.

3. $\|e^{\tilde{A}_{Q_N}t}\| \le e^{\omega_\infty t}$, *where* $\omega_\infty = \sup\{\mathrm{Re}(\lambda), \lambda \in \sigma(A - BB^*Q)\}$.

4. *The growth bound* ω_N *of* $e^{\tilde{A}_{Q_N}t}$ *satisfies* $\limsup_{N\to\infty}\omega_N = \omega_\infty$.

We illustrate this with the platoon from Example 1.1.

Example 2.4 The circulant approximating system $\Sigma(\tilde{A}_N, \tilde{B}_N, \tilde{C}_N, 0)$ has

$$\tilde{A}_N = I_{2N+1}, \quad \tilde{B}_N = \begin{bmatrix} 1 & 0 & 0 & \dots & \beta \\ \beta & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \beta & 1 \end{bmatrix}, \quad \tilde{C}_N = I_{2N+1}.$$

and it corresponds to the following (fictious) finite platoon model

$$\begin{aligned} \dot{z}_r(t) &= z_r(t) + u_r(t) + \beta u_{r-1}(t), \quad -N+1 \le r \le N \\ \dot{z}_{-N}(t) &= z_{-N}(t) + u_{-N}(t) + \beta u_N(t), \\ y_r(t) &= z_r(t), \quad -N \le r \le N, \quad t \ge 0. \end{aligned}$$

Using the properties of circulant matrices from the Appendix, we factorize

$$\tilde{B}_N\tilde{B}_N^* = \begin{bmatrix} 1+\beta^2 & \beta & 0 & \dots & \beta \\ \beta & 1+\beta^2 & \beta & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \beta & 0 & \dots & \beta & 1+\beta^2 \end{bmatrix} = U_N \mathrm{diag}(\mu_k(N)) U_N^*,$$

where the eigenvalues of $\tilde{B}_N\tilde{B}_N^*$ are $\mu_k(N) = 1+\beta^2+2\beta\cos\frac{2k\pi}{2N+1}, k = 0, \dots, 2N$ and the unitary matrix $U_N = \frac{1}{\sqrt{2N+1}}\left[e^{-\frac{2\pi jrs}{2N+1}}\right]_{r,s=0,\dots,2N}$. Hence we can derive the explicit solution to the corresponding circular Riccati equation

$$\tilde{Q}_N = U_N \mathrm{diag}\left(\frac{1+\sqrt{1+\mu_k(N)}}{\mu_k(N)}\right) U_N^*.$$

Hence

$$\|\tilde{Q}_N\| = \max_{k=0,..,2N} \frac{1+\sqrt{1+\mu_k(N)}}{\mu_k(N)} = \frac{1+\sqrt{2+\beta^2-2\beta\cos\frac{\pi}{2N+1}}}{1+\beta^2-2\beta\cos\frac{\pi}{2N+1}} \to \|Q\| \text{ as } N \to \infty.$$

The closed-loop operator is given by

$$\tilde{A}_{Q_N} = \tilde{A}_N - \tilde{B}_N\tilde{B}_N^*\tilde{Q}_N = U_N \mathrm{diag}\left(-\sqrt{2+\beta^2-2\beta\cos\frac{2k\pi}{2N+1}}\right) U_N^*.$$

The eigenvalues of $\tilde{A}_{Q_N}$ all lie in the spectrum of A_Q and the growth bounds of their semigroups $\omega_N = -\sqrt{2+\beta^2-2\beta\cos\frac{\pi}{2N+1}} \to \omega_0$ as $N \to \infty$.

The solutions $\tilde{Q}_N$ to the circulant Riccati equation (12) and the solutions $\mathbf{Q}_N$ to (10) are related.

Theorem 2.5 *Assume that* $\Sigma(A,B,C,0)$ *is stabilizable and detectable and* $\Sigma(\boldsymbol{A}_N,\boldsymbol{B}_N,\boldsymbol{C}_N)$ *is uniformly stabilizable and detectable. Then*

1. $i^N(\boldsymbol{Q}_N - \tilde{Q}_N)\pi^N$ *and* $i^N(\boldsymbol{A}_{Q_N} - \tilde{A}_{Q_N})\pi^N$ *converge strongly to* 0 *as* $N \to \infty$.
2. $|\boldsymbol{Q}_N - \tilde{Q}_N|_N \to 0, \quad |(\boldsymbol{A}_{Q_N} - \tilde{A}_{Q_N}|_N \to 0$ *as* $N \to 0$, *and* $\limsup_{N\to\infty} |\boldsymbol{Q}_N|_N \leq \|Q\|$.
3. *For sufficiently large N the growth bound of* $e^{\boldsymbol{A}_{Q_N}t}$ $\omega_N \leq \omega_\infty$. *Hence any* $\varepsilon > 0$ *there exists a* $M(N)$ *independent of* ε *such that*

$$\|e^{\boldsymbol{A}_{Q_N}t}\| \leq M(N)e^{(\omega_\infty+\varepsilon)t}.$$

4. *If* $|b_0| > \|\check{B} - b_0\|_\infty$, *and* $|c_0| > \|\check{C} - c_0\|_\infty$, *or if there exists a nonzero* δ *such that* $\lambda_{\min}(B_N B_N^*) \geq \delta^2$ *and* $\lambda_{\min}(C_N^* C_N) \geq \delta^2$, *then given* $\varepsilon > 0$, *there exists a positive* $M(\varepsilon)$ *such that for sufficiently large N there holds*

$$\|e^{\boldsymbol{A}_{Q_N}t}\| \leq M(\varepsilon)e^{(\omega_\infty+\varepsilon)t}.$$

5. *If* $\check{A}$ *has real values and* $\check{B} = b_0$, *then* $M(N) = 1$ *and we can take* $\varepsilon = 0$.. *If* $\check{A}$ *has real values and* $\check{C} = c_0$, *then M is independent of N and we can take* $\varepsilon = 0$.

The growth bound in Part 3 of the above theorem is the main result, but the weak part is the dependence of the gain factor $M(N)$ on N. We have shown that it is in fact independent of N under the conditions in parts 4 and 5. Although we believe that this will always be true, it seems difficult to prove for the general case.
Note that in [9] and other papers on platoons an alternative type of approximating Riccati equation of the following form were studied.

$$\mathbf{A}_N^*\overline{Q}_N + \overline{Q}_N\mathbf{A}_N - \overline{Q}^N\mathbf{B}_N\mathbf{B}_N^*\overline{Q}_N + (C^*C)_N \quad = \quad 0, \tag{13}$$

where $(C^*C)_N$ is the matrix representation of the map $\pi^N C^*C|_{Z^N}$ and $\pi^N : \mathbb{Z} = \ell_2 \to \mathbb{C}^{2N+1}$.
In [5] it is shown that they have similar convergence properties to the solutions of (10). As the following example shows, these are often easier to solve, since the term $(CC^*)_N$ is a Toeplitz matrix, whereas $\mathbf{C}_N^*\mathbf{C}_N$ is not.

Example 2.6 Consider the alternative Riccati equation (13) with $\mathbf{A}_N = \mathbf{B}_N = I_{2N+1}$ and

$$(C^*C)_N = \begin{bmatrix} 1+\kappa^2 & \kappa & 0 & \dots & 0 \\ \kappa & 1+\kappa^2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \kappa & 1+\kappa^2 \end{bmatrix} \neq \mathbf{C}_N^*\mathbf{C}_N.$$

We write

$$(C^*C)_N = (1+\kappa^2)I_N + \kappa T_N,$$

where the Toeplitz matrix

$$T_N = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} = V_N \text{diag}\left(2\cos\frac{(k+1)\pi}{2N+2}; k=0,1,\ldots,2N\right) V_N^*,$$

and V_N is a unitary matrix. Then the solution $\overline{Q}_N$ to (13) with these values is

$$\overline{Q}_N = V_N \text{diag}\left(1+\sqrt{1+\rho_k(N)}; k=0,1,\ldots,2N\right) V_N^*,$$

since $\rho_k(N) = 1+\kappa^2+2\kappa\cos\frac{(k+1)\pi}{2N+2}, k=0,1,\ldots 2N$. Hence

$$\|\overline{Q}_N\| = \max_{k=0,\ldots 2N}\left(1+\sqrt{1+\rho_k(N)}\right) = 1+\sqrt{2+\kappa^2+2\kappa\cos\frac{\pi}{2N+2}}.$$

The closed-loop operator is given by

$$\overline{A}_{Q_N} = \mathbf{A}_N - \mathbf{B}\mathbf{B}_N^*\overline{Q}_N = V_N \text{diag}\left(-\sqrt{1+\rho_k(N)}\right) V_N^*,$$

and its growth bound is $\omega_N = -\sqrt{2+\kappa^2-2\kappa\cos\frac{\pi}{2N+2}}$. For the corresponding infinite-dimensional problem we have $\check{Q}(e^{j\theta}) = \left(1+\sqrt{2+\kappa^2+2\kappa\cos\theta}\right)$ with norm $\|\check{Q}\| = 1+\sqrt{1+(1+\kappa)^2}$. The closed-loop operator is $\check{A}_Q = -\sqrt{2+\gamma^2+2\gamma\cos\theta}$ with $\omega_\infty = -\sqrt{1-2\kappa+\kappa^2}$. So the eigenvalues of $\overline{A}_{Q_N}$ all lie in the spectrum of A_Q and the growth bound converges to ω_∞ as $N\to\infty$. Moreover, $\|\overline{Q}_N\| \to \|\check{Q}\|$ as $N\to\infty$.

It is interesting to compare the above with the circular approximations. Following the approach in Example 2.4 we find the solution to (12) to be

$$\tilde{Q}_N = U_N \text{diag}\left(1+\sqrt{1+\mu_k(N)}\right) U_N^*,$$

where the unitary matrix U_N is as in Example 2.4 and $\mu_k(N) = 1+\kappa^2+2\kappa\cos\frac{2k\pi}{2N+1}, k = 0,\ldots,2N$. Hence

$$\|\tilde{Q}_N\| = \max_{k=0,..,2N}(1+\sqrt{1+\mu_k(N)}) = 1+\sqrt{2+\kappa^2+2\kappa} = \|Q\|.$$

The closed-loop operator is given by

$$\tilde{A}_{Q_N} = \tilde{A}_N - \tilde{B}_N\tilde{B}_N^*\tilde{Q}_N = U_N \text{diag}\left(-\sqrt{2+\kappa^2+2\kappa\cos\frac{\pi}{2N+1}}\right) U_N^*$$

and its growth bound is $\omega_N = -\sqrt{2+\kappa^2-2\kappa\cos\frac{\pi}{2N+1}}$. So the eigenvalues of $\tilde{A}_{Q_N}$ all lie in the spectrum of A_Q and the growth bounds of their semigroups converge to ω_∞ as $N\to\infty$

This example illustrates the fact that for the special case of only delays in C we obtain much nicer convergence results than in Theorem 2.5.

Lemma 2.7 *Suppose that $\check{A} = a_0, \check{B} = b_0 \neq 0, \check{C} \neq 0$. Denoting the solutions to (13) by $\overline{Q}_N$ and $\overline{A}_{Q_N} := \boldsymbol{A}_N - \boldsymbol{B}_N \boldsymbol{B}_N^* \overline{Q}_N$ we have*

1. $\lim_{N\to\infty} \|\overline{Q}^N\| = \|Q\|$.
2. *The growth bounds of $e^{\overline{A}_{Q_N} t}$ converge to those of $e^{A_Q t}$ and $\|e^{\overline{A}_{Q_N} t}\| \leq e^{\omega_\infty t}$.*
3. *If $\min_{0\leq\theta\leq 2\pi} |\check{C}(\theta)| > 0$, then $i^N \overline{Q}^N \pi^N$ converges strongly to Q as $N \to \infty$.*
4. *$\Sigma(\boldsymbol{A}_N, \boldsymbol{B}_N, \boldsymbol{C}_N, 0)$ is uniformly detectable, if and only if there exists a positive γ such that*
$$\langle \boldsymbol{C}_N z_N, \boldsymbol{C}_N z_N \rangle \geq \gamma \|z_N\|^2 \text{ for all } z_N \in Z^N.$$
If the above holds, then $i^N \boldsymbol{Q}_N \pi^N$ converges strongly to Q as $N \to \infty$ and $\limsup_{N\to\infty} \|\boldsymbol{Q}_N\| = \|Q\|$. Furthermore the growth bounds of $e^{\boldsymbol{A}_{Q_N} t}$ converge to those of $e^{A_Q t}$ and $\|e^{\boldsymbol{A}_{Q_N} t}\| \leq e^{\omega_\infty t}$.
5. *If $\Sigma(\boldsymbol{A}_N, \boldsymbol{B}_N, \boldsymbol{C}_N, 0)$ is not uniformly detectable, we still have*
$$\limsup_{N\to\infty} \|\boldsymbol{Q}_N\| \leq \|Q\|, \text{ and } \|e^{\boldsymbol{A}_{Q_N} t}\| \leq e^{-|a_0| t}.$$
6. *The feedback laws $u_N = -\boldsymbol{B}_N \tilde{Q}_N z_N$ and $u^N = -B^N T^N(Q) z^N$ stabilize with*
$$\|e^{(\boldsymbol{A}_N - \boldsymbol{B}_N \boldsymbol{B}_N^* \tilde{Q}_N)t}\| \leq e^{\omega_\infty t}, \quad \|e^{(\boldsymbol{A}_N - \boldsymbol{B}_N \boldsymbol{B}_N^* T^N(Q))t}\| \leq e^{\omega_\infty t}.$$

The above lemma shows that, whenever one has only delays in C, a good strategy is to use the feedback law $u_N = -\mathbf{B}_N \tilde{Q}_N z_N$, since $\tilde{Q}_N$ is easy to calculate.
The discretization of partial differential equations leads to systems with a self-adjoint A operator and constant B, C operators. For such systems we also obtain nice convergence results.

Corollary 2.8 *Suppose that $\check{A}$ is real, $\check{B} = b_0 \neq 0$, $\check{C} = c_0 \neq 0$. Then $\Sigma(\boldsymbol{A}_N, \boldsymbol{B}_N, \boldsymbol{C}_N, 0)$ is uniformly detectable and $i^N \boldsymbol{Q}_N \pi^N$ converges strongly to Q as $N \to \infty$ and $\lim_{N\to\infty} \|\boldsymbol{Q}_N\| = \|Q\|$. The growth bound of $e^{\boldsymbol{A}_{Q_N} t}$ converges to ω_∞ with $\|e^{\boldsymbol{A}_{Q_N} t}\| \leq e^{\omega_\infty t}$.*

References

[1] Bamieh B. and Paganini F. and Dahleh M.A. *Distributed control of spatially invariant systems*, IEEE Trans. Automatic Control **47**, (2002) 1091–1107.

[2] Curtain R.F. and Zwart H.J. *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer-Verlag, New York, (1995).

[3] Curtain R.F., Iftime O.V. and Zwart H.J. *System Theoretic Properties of Platoon-Type Systems* Proc. *47th IEEE Conference on Decision and Control*, Cancun, Mexico, (2008) 1442-1447.

[4] Curtain R.F., Iftime O.V. and Zwart H.J. *System theoretic properties of a class of spatially distributed systems* Automatica, accepted, (2008).

[5] Curtain R.F., Iftime O.V. and Zwart H.J. *A comparison of LQR control for finite and infinite platoons: the scalar case*, manuscript.

[6] Davies P.J. *Circulant matrices.* Wiley Interscience., New York, (1979).

[7] Gray R.M. *Toeplitz and Circulant matrices: A review.* Report 032, Stanford University Electronics Laboratory, Stanford California, (1971).

[8] Ito K. *Strong convergence and convergence rates of approximating solutions for algebraic Riccati equations in Hilbert spaces*, in *Distributed Parameter Systems*, Eds: W. Schappacher, F.Kappel and K.Kunisch. Springer Verlag, Berlin, (1987) 153–166.

[9] Jovanović M.R. and Bamieh B. *On the ill-posedness of certain vehicular platoon control problems*, IEEE Trans. Automatic Control, **50**, (2005) 1307–1321.

[10] Kappel F. and Salamon D. *An approximation theorem for the algebraic Riccati equation*, SIAM Journal on Control and Optimization, **28**, (1990) 1136–1147.

[11] Lancaster P. and Rodman L. *Algebraic Riccati Equations*, Oxford Science Publications, Clarendon Press, Oxford, UK, (1995).

3 Appendix: Toeplitz and Circulant matrices

In this section we summarize known results from Davis [6], Gray [7] on circulant approximations of Toeplitz operators with continuous scalar symbols

$$f(e^{j\theta}) = \sum_{l=-\infty}^{\infty} f_l e^{-jl\theta}.$$

We denote the infinite matrix representation of the Toeplitz operator by F and the *Toeplitz approximant* matrix of order n by $T^{(n)}(f)$

$$T^{(n)}(f) = \begin{bmatrix} f_0 & f_1 & f_2 & f_3 & \cdot & f_{n-1} \\ f_{-1} & f_0 & f_1 & f_2 & \cdot & f_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{-n+1} & f_{-n+2} & f_{-n+3} & f_{-n+4} & \cdot & f_0 \end{bmatrix}.$$

The spectrum of $T^{(n)}(f)$ can be very different from F, except in the self-adjoint case.

Lemma 3.1 *If f is real, denote by m_f and M_f the minimum and the maximum of f on $[0,2\pi]$, respectively. If the Toeplitiz approximants $T^{(n)}(f)$ have the eigenvalues $\lambda_k^{(n)}, k=1,...,n$. Then $m_f \leq \lambda_k^{(n)} \leq M_f$, and*

$$\lim_{n\to\infty} \max_k \lambda_k^{(n)} = M_f, \quad \lim_{n\to\infty} \min_k \lambda_k^{(n)} = m_f.$$

We define the *circulant approximant matrix of order n* by $C^{(n)}(f)$,

$$C^{(n)}(f) = \begin{bmatrix} c_0^{(n)} & c_1^{(n)} & c_2^{(n)} & c_3^{(n)} & \cdot & c_{n-1}^{(n)} \\ c_{n-1}^{(n)} & c_0^{(n)} & c_1^{(n)} & c_2^{(n)} & \cdot & c_{n-2}^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_1^{(n)} & c_2^{(n)} & c_3^{(n)} & c_4^{(n)} & \cdot & c_0^{(n)} \end{bmatrix},$$

where

$$c_k^{(n)} = \frac{1}{n}\sum_{l=0}^{n-1} f(e^{\frac{2\pi l}{n}\mathrm{j}})e^{\frac{-2\mathrm{j}\pi kl}{n}}.$$

Circulant approximant matrices have very nice properties

- $\|C^{(n)}(f)\| \leq \|F\|, \quad \lim_{n\to\infty} c_k^{(n)} = f_k.$
- $C^{(n)}(fg) = C^{(n)}(f)C^{(n)}(g); \quad C^{(n)}(f+g) = C^{(n)}(f) + C^{(n)}(g).$
- The eigenvalues of $C^{(n)}(f)$ are $\lambda_k^{(n)} = f(e^{\frac{2\pi k}{n}\mathrm{j}})$, $k = 0,1,...n-1$.
- $C^{(n)}(f) = U^{(n)}\mathrm{diag}(\lambda_k^{(n)})(U^{(n)})^*$, where $U_{rs}^{(n)} = \frac{1}{\sqrt{n}}\left[e^{-\frac{2\pi\mathrm{j}rs}{n}}\right]$, $r,s = 0,...,n-1$.

In addition to the matrix spectral or induced $\mathbf{L}_2$-norm denoted by $\|\cdot\|$, we introduce the following n-norm for square matrices of order n

$$|A|_n = \left(\frac{1}{n}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}|a_{kl}|^2\right)^{1/2} = \left(\frac{1}{n}\mathrm{trace}(A^*A)\right)^{1/2}.$$

This matrix norm has the following properties

- $|A|_n \leq \|A\|, \quad |AB|_n \leq |A|_n\|B\| \quad |AB|_n \leq \|A\||B|_n.$