MODULATION AND CODING FOR QUANTIZED CHANNELS

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ABSTRACT

We investigate reliable communication over quantized channels from an information theoretical point of view. People seldom consider the effect of quantization in conventional coded modulation systems since Analog-to-Digital (AD) converters used in these systems always have high resolution, e.g. 2/3 source bits are often quantized into 10/12 bits. However, AD converters with a high resolution are power consuming. In this paper, we present a scheme to design an optimum quantizer with low resolution which can be used to communicate over the quantized channel. Moreover, we show that reliable transmission over the Additive White Gaussian Noise (AWGN) channel at a rate of R bit/use is possible with $R+1$ or $R+2$ quantized bits.

Key words: quantized channels, quantization, coded modulation system, AD converter, AWGN channel

1. INTRODUCTION

In this paper we investigate reliable and bandwidth efficient communication over quantized channels. Quantized channels arise in practical communication systems where AD converters are used to sample analog signals corresponding to the transmitted data. Conventional coded modulation systems usually do not take quantization into account, which is reasonable since a large number of quantization levels are used. In this case the difference between the quantized and unquantized channel can be neglected. However, for e.g. mobile communication systems the power consumption at the receiver is proportional to the resolution of the AD converter. Hence it is of interest to lower the resolution of the AD converter.

In this paper we consider quantized channels from an information theoretical point of view. We investigate information theoretical limits of transmission over quantized channels. The main question we try to answer is as follows. If we wish to transmit reliably at a rate of R bit/use, do we actually require more than R quantization bits? In this paper we show that for the AWGN channel, reliable transmission at a rate of R bit/use is possible with $R+1$ or $R+2$ quantization bits without sacrificing transmission power.

The organization of the paper is as follows. First, we introduce the system model in section 2. Second, we investigate the effect of quantization from an information theoretical point of view in section 3. Furthermore, we present a scheme which can be used to communicate over the quantized channel in this section. In addition, we find out an optimum quantization scheme which makes the theoretical limits of the quantized AWGN channel as close to the limits of the unquantized AWGN channel as possible. We end with conclusions in section 4.

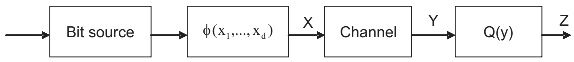

Figure 1: The system model

2. SYSTEM MODEL

We consider communication over the memoryless time-discrete AWGN channel which is defined by:

$$Y = X + N \tag{1}$$

where the channel input X is disturbed by additive noise N which has a Gaussian distribution with variance σ^2. To communicate over this channel we map a sequence of d bits $X_1, \ldots, X_d$ to a channel input symbol X:

$$X = \phi(X_1, \ldots, X_d) \tag{2}$$

where $\phi(\cdots)$ is defined as:

$$\phi : GF(2)^d \to \mathbb{R} \tag{3}$$

We refer to ϕ as the *modulation map* and it defines the signal constellation $\mathcal{S}$ and mapping from bits to constellation symbols:

$$\mathcal{S} = \{x \in \mathbb{R} : x = \phi(X_1, \ldots, X_d)\} \tag{4}$$

The energy expended per channel use is defined as the mathematical expectation of X^2:

$$E_s = E[X^2] \tag{5}$$

The channel output is quantized by a quantization map $\mathcal{Q}$ which is defined as:

$$\mathcal{Q} : \mathbb{R} \rightarrow A \tag{6}$$

where $A \subset \mathbb{N}$ and represents the output of the quantized channel. As we will see later it is not necessary to associate the elements of A with the real numbers. The elements just represent the channel outputs. A quantization map is surjective and $\mathcal{Q}$ also defines an inverse quantization map which is defined as the set function $\mathcal{Q}^{-1}$:

$$\mathcal{Q}^{-1}(i) = \{x \in \mathbb{R} : Q(x) = i, i \in A\} \tag{7}$$

We restrict ourselves to quantizers where each $\mathcal{Q}^{-1}(i)$ is of the form $\mathcal{I}_i = (a, b]$ for $a, b \in \mathbb{R}$. In this case $\{\mathcal{I}_i\}$ partitions $\mathbb{R}$ and the quantizer is defined by the set of intervals $\{\mathcal{I}_i\}$. The goal is to make $|A|$ as small as possible and still achieve a reasonable mutual information between the input bits and the quantized channel output. With these definitions the quantized channel is defined as:

$$Z = \mathcal{Q}(\phi(X_1, \ldots, X_d) + N) \tag{8}$$

where Z is the quantized channel output and takes values from A. Figure 1 shows an overview of the system model we have defined so far. In the next section we study this system from an information theoretical point of view and derive a scheme to communicate reliable over the quantized channel.

3. QUANTIZATION AND MUTUAL INFORMATION

In this section we study the effect of quantization from an information theoretical point of view. Moreover, in this section we consider the AWGN channel as an example where we restrict ourselves to uniform quantizers and conventional pulse amplitude modulation (PAM) signal constellations. Also, we consider the case of designing non-uniform quantizers for the communication over the AWGN channel with conventional PAM modulation. In later section we find out which is the optimum design of the quantization scheme.

We are interested in the mutual information between $(X_1, \ldots, X_d)$ and Z. First, note that the following sequence of random variables forms a Markov chain:

$$(X_1, \ldots, X_d) \rightarrow X \rightarrow Y \rightarrow Z \tag{9}$$

We can express $I((X_1, \ldots, X_d), X; Z)$ as:

$$\begin{aligned} I((X_1, \ldots, X_d), X; Z) &= \\ I(X; Z) + I((X_1, \ldots, X_d); Z|X) &= \\ I((X_1, \ldots, X_d); Z) + I(X; Z|(X_1, \ldots, X_d)) & \end{aligned} \tag{10}$$

where we have used the chain rule of mutual information. Since $(X_1, \ldots, X_d) \rightarrow X \rightarrow Z$ forms a Markov chain as well it follows that:

$$I((X_1, \ldots, X_d); Z|X) = 0 \tag{11}$$

Moreover, X is a function of $(X_1, \ldots, X_d)$ which implies that:

$$I(X; Z|(X_1, \ldots, X_d)) = 0 \tag{12}$$

With (10) we have the following equality:

$$I((X_1, \ldots, X_d); Z) = I(X; Z) \tag{13}$$

which shows that the mutual information of interest is fully defined by the signal constellation and the distribution of the quantized symbols. Next, we consider the mutual information between X and Z. From the chain rule of mutual information it follows that:

$$\begin{aligned} I((Y, Z); X) &= I(Y; X) + I(Z; X|Y) \\ &= I(X; Z) + I(Y; X|Z) \end{aligned} \tag{14}$$

Since $X \rightarrow Y \rightarrow Z$ forms a Markov chain $I(Z; X|Y) = 0$ from which follows that:

$$I(X; Z) = I(Y; X) - I(Y; X|Z) \tag{15}$$

Given a channel there exists a distribution which achieves the maximum value C of $I(Y; X)$. For the AWGN channel this distribution is the Gaussian distribution. A particular choice of modulation leads to a certain $I(Y; X)$. Moreover, when the output of the channel is quantized, the quantity of interest is $I(X; Z)$ which is upperbounded $I(X; Y)$ by (15). In this paper we assume the transmitted signal is PAM-modulated and consider the design of quantization schemes for the AWGN channel for which $I(X; Z)$ is as close to the capacity of the AWGN channel as possible. Furthermore, the number of quantization bits is only slightly larger than R, where R is the rate at which we transmit.

3.1. Uniform Quantization for AWGN with PAM Constellations

Now we consider uniform quantization for the AWGN channel where conventional PAM constellations are used. A PAM constellation with 2^d constellation symbols is defined as:

$$\begin{aligned} \mathcal{S}_{PAM2^d} = \{&-(2n-1), \ldots, -(2i-1), \ldots, \\ &-1, 1, \ldots, 2i-1, \ldots, 2n-1\} \end{aligned} \tag{16}$$

where $i = 1, 2, \ldots, n$ and $n = 2^{d-1}$. In this case the constellation symbols are selected with equal probability and E_s is given by:

$$E_s = \frac{4^d - 1}{3} \tag{17}$$

One can normalize these constellation such that $E_s = 1$. We define a uniform interval quantizer with 2^m levels and spacing q as an interval quantizer for which the set of quantization levels is given by:

$$\{\mathcal{I}_i : i = 1 \dots 2^m\} = \{(-\infty, -nq], \dots, (-iq, -(i-1)q], \dots, (-q, 0], (0, q], \dots, ((i-1)q, iq], \dots, (nq, +\infty)\} \quad (18)$$

where $i = 1, 2, \dots, n$ and $n = 2^{m-1} - 1$.
Given these definitions we can expressed the mutual information between the channel input and the quantized channel output as:

$$I(X;Z) = \sum_{x\in\mathcal{S}} \sum_{i\in A} P(Z=i, X=x) \log_2 \frac{P(Z=i|X=x)}{\sum_{x'\in\mathcal{S}} P(Z=i, X=x')} \quad (19)$$

where $P(Z = i, X = x)$ is given by:

$$P(Z=i, X=x) = P(Z=i|X=x)P(X=x) = 2^{-d}P(Z=i|X=x) \quad (20)$$

and $P(Z = i|X = x)$ can be computed as follows:

$$P(Z=i|X=x) = P(y \in \mathcal{Q}^{-1}(i)|X=x) = \int_{\mathcal{Q}^{-1}(i)} dy f_{Y|X}(y|x) \quad (21)$$

where $f_{Y|X}(y|x)$ is the channel transition probability density function (pdf).
Consider the case that E_s/σ^2 is equal to 5 dB and we transmit source bits over the AWGN channel using a PAM-4 constellation which is defined as $\mathcal{S} = \{-3, -1, 1, 3\}$. Suppose that we use a uniform interval quantizer to quantize the channel output into 2^m levels where m is equal to 2, 3 or 4. Figure 2 shows a plot of $I(X;Z)$ as a function of q for these values of m and the case where no quantization is used. The figure also shows the capacity of the AWGN channel. We observe that there is a loss compared to the case where no quantization is used. However, with a proper spacing q and an m slightly larger than d the loss in rate can be made small. Furthermore, one does not have to use a uniform interval quantizer. The quantization levels can be chosen in such a way to give a higher value of $I(X;Z)$.

3.2. Non-uniform Quantization for AWGN with PAM Constellations

Here we consider non-uniform quantization for the AWGN channel and conventional PAM constellations. In this case, we define a non-uniform interval quantizer with 2^m levels

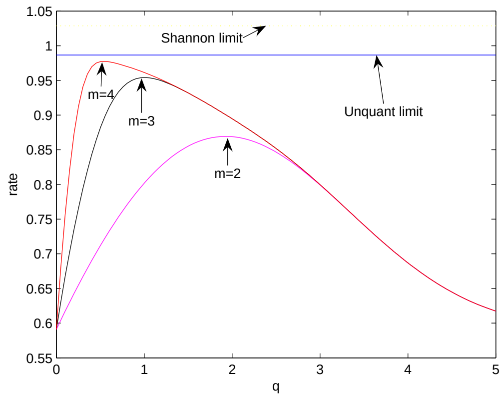

Figure 2: *Uniform quantized interval q with $d = 2$, PAM constellation and SNR=5dB*

and non-equal spacing as an interval quantizer for which the set of quantization levels is given by:

$$\{\mathcal{I}_i : i = 1 \dots 2^m\} = \{(-\infty, -q_n], \dots, (-q_i, -q_{i-1}], \dots, (-q_1, 0], (0, q_1], \dots, (q_{i-1}, q_i], \dots, (q_n, +\infty)\} \quad (22)$$

where $i = 1, 2, \dots, n$ and $n = 2^{m-1} - 1$.
From (19) $\sim$ (21), we see that $I(X;Z)$ determined by (22). In other words, we can optimize the $\{q_1, q_2, \dots, q_n\}$ which make $I(X;Z)$ as close to $I(X;Y)$ as possible. We can optimize the values of q_i by a numerical maximization of the mutual information between the channel input and the quantized channel output.
As an example, we assume that E_s/σ^2 is equal to 13 dB and we transmit bits over the AWGN channel using a PAM-8 constellation which is defined as $\mathcal{S} = \{-7, -5, -3, -1, 1, 3, 5, 7\}$. Suppose that we use a non-uniform interval quantizer to quantize the channel output into 2^m levels where m is equal to 4 or 5. The result of the optimization is as follows:

$$\{q_i\} = \{0.75, 1.58, 2.55, 3.52, 4.5, 5.44, 6.55\} \quad \text{for } m = 4 \quad (23)$$

$$\{q_i\} = \{0.55, 1.11, 1.51, 1.87, 2.28, 2.73, 3.42, 3.72, 4.4, 5.05, 5.39, 5.91, 6.47, 7.07, 7.34\} \quad \text{for } m = 5 \quad (24)$$

Figure 3 shows the quadrature signal constellation and optimum non-uniform quantization scheme which are generated by using each dimension independently.

3.3. Comparisons

In this section, we show the quantization effect from an information theoretical point of view and find out the optimum quantization scheme by comparing the uniform quantization scheme and the non-uniform quantization scheme. We map the source bits into a conventional PAM signal constellation symbol as defined in (16). We do the comparison in two situations as follows. First, we choose $d = 2$ and SNR = 10 dB, and optimize the set of quantization levels for the two schemes as described in section 3.1 and 3.1 when m is equal to 3 or 4. The optimum spacing q for the uniform quantization is:

$$q = \left\{ \begin{array}{ll} 1 & \text{for } m = 3 \\ 0.5 & \text{for } m = 4 \end{array} \right. \tag{25}$$

and the optimum set of quantization levels $\{q_i\}$ for the non-uniform quantization is:

$$\{q_i\} = \{0.64, 1.71, 2.37\} \quad \text{for } m = 3 \tag{26}$$

$$\{q_i\} = \{0.31, 0.79, 1.29, 1.75, 2.06, 2.38, 2.87\} \quad \text{for } m = 4 \tag{27}$$

For the second case, we choose $d = 3$ and SNR = 13 dB and find out the optimum set of quantization levels for each scheme for m = 4 or 5. The optimum results for the uniform quantization when $d = 3$ are:

$$q = \left\{ \begin{array}{ll} 1 & \text{for } m = 4 \\ 0.5 & \text{for } m = 5 \end{array} \right. \tag{28}$$

and the optimum set for the non-uniform quantization is shown in (23).

Suppose the channel in figure 1 is an AWGN channel. By using the quantization scheme we define in section 3.1 and 3.1, we calculate the theoretical channel capacity limits for the quantized channel, which are shown in figure 4. This figure shows the constrained capacity of the quantized channel $I(X;Z)$ as a function of the bit signal-to-noise ratio $\frac{E_b}{N_0}$ of the channel for each d, m and quantization scheme. The figure also shows the capacity limit of the AWGN channel and the Shannon limit.

Figure 4 shows the non-uniform quantization scheme works slightly better than the uniform one, especially in the lower m. In this figure, we observe that there is a loss in capacity limit and transmitting power due to the quantization, but the loss is small comparing to the unquantized channel. From an information theoretical point view, the rate difference between the quantized channel and the unquantized channel can be neglected for $m = d + 2$. So, for the AWGN channel reliable transmission at a rate of R bit/use requires at most $R + 2$ quantized bits.

4. CONCLUSIONS

We have investigated the quantized channel from an information theoretical point of view. We have shown an optimum quantized scheme which can be used in the reliable communication over the quantized channel. Furthermore, we have shown that $R + 2$ quantized bits is already enough for reliable communication at rate of R bit/use over the AWGN channel. Therefore, it is possible to lower the resolution of AD converter, which means the power consumption at the receiver can be lowered.

5. REFERENCES

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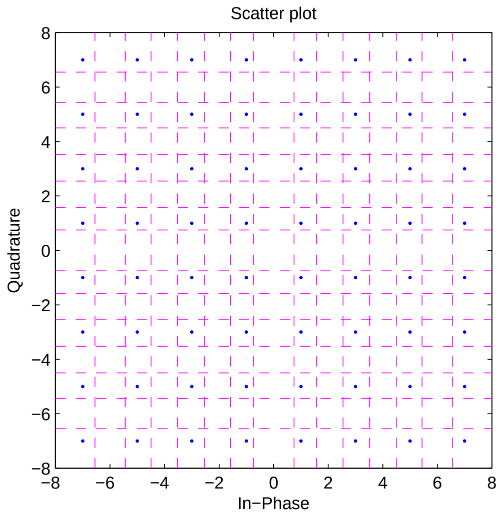
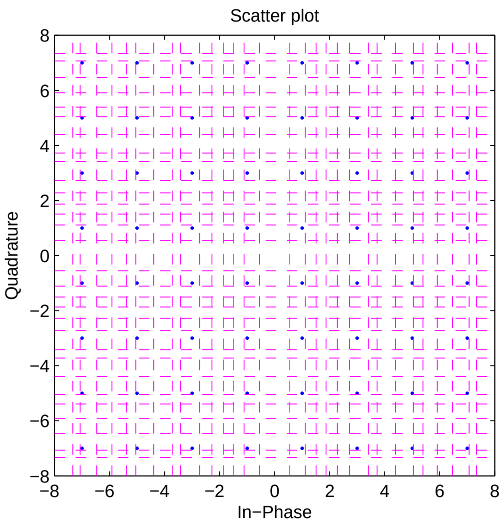
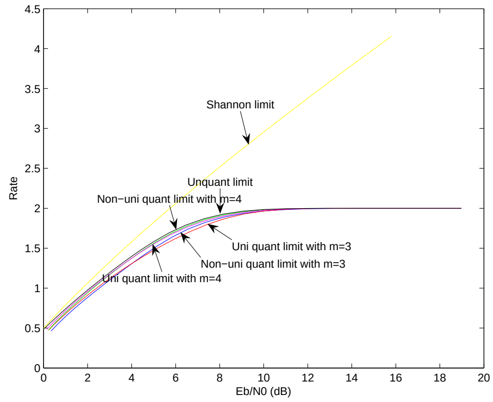


(a) $m = 4$

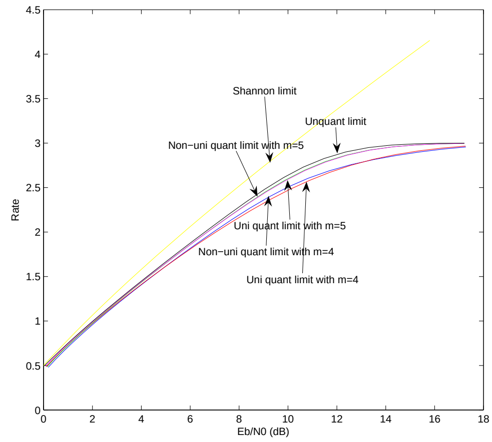


(b) $m = 5$

Figure 3: *The quadrature PAM signal constellation with 64 symbols and optimum non-uniform quantization with $d = 3$ and SNR = 13 dB.*



(a) d=2



(b) d=3

Figure 4: *The capacity limit of the quantized channel with PAM constellation*