On stable alluvial channels: a variational approach

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ABSTRACT

We consider the cross-sectional shape of alluvial channels, and assume that the time-scale of adjustment of this profile is much smaller than that of the longitudinal profile. We present a model based on a variational approach, which minimizes a "resistance-like" quantity. This quantity contains information about the sediment. Taking the water discharge as given, we determine the stage-discharge relationship of stable alluvial channels in fluvial environments; comparision with experimental data shows rather realistic results.

INTRODUCTION

We start from a depth-averaged shallow-water model for the water motion, and a sediment transport formula, which includes the downhill gravitational component and the steady-state Exner equation. In the case of a channel with fixed banks, this common model leads to a trivial equilibrium state: the water depth and the current velocity are constant across the channel. This result does not comply with the observed shapes of channels with alluvial banks. Moreover, the model does not include any physical mechanism which determines the width of the channel. Hence, we conclude the model must be incomplete. In fact, it lacks a mechanism which tends to carry the sediment on the banks uphill, thus balancing the downhill gravitational component.
Parker [1978] proposes to take the lateral dispersion of suspended sediment as this mechanism, and shows that this yields rather realistic stable cross-sectional shapes. In the case of predominant bed load transport, however, this mechanism cannot explain the channel formation. Other cross-stream transport mechanisms have to be taken into account, e.g. lateral dispersion of the transported sediment. Before digging further into this matter, we think it is appropriate to investigate what can be said about the cross-sectional shape without being specific about the transport model. To that end, we assume that part of the energy expenditure of the flow is used to maintain the shape of the cross-section, and this energy is related to some function of the flow velocity, and to the part of the wetted

perimeter, to which this velocity applies. Furthermore, we assume that in every part of the cross-section the square of depth-averaged velocity $u(y)$ is proportional to the local water depth $h(y)$ via Chézy's law $[u(y) = C\sqrt{i_0 h(y)}$, in which y is the cross-sectional horizontal coordinate, C the Chézy's factor, and i_0 the channel slope]. This means that any function of u can be expressed as a function of h. Hence, the "energy", spent to maintaining the channel shape can be written as

$$E_f(h,P) = \int_P f(h)\ dP \tag{1}$$

in which P denotes the wetted perimeter, and $f(h)$ is a yet undetermined function of $h(y)$.

In the next section, we present a model based on a variational approach of minimizing the "energy"-expenditure of the water flow (1) under the constraint of a given water discharge q

$$\underset{h}{Min}\left\{E_f(h,P)\ \middle|\ Q_w(h,P) = q\right\} \tag{2}$$

For arguments' sake, we shall now restrict ourselves to channels with a discrete cross-section.

DISCRETE CHANNEL SHAPES

In the case of steady flow, the quantity E_f in such channels can be approximated by

$$E_f(H_0, B_0) = f(H_0)B_0 + \alpha \int_0^{H_0} f(h)\ dh \tag{3}$$

in which H_0 is the water depth, and B_0 the width in the central part of the channel. The constant α depends on the general shape of the cross-section, but not on its dimensions. For example, in the case of a trapezoidal channel shape, $\alpha = 2/sin\gamma$, in which γ is the angle between the side-wall and the water surface, as represented in the situation sketch figure 1.

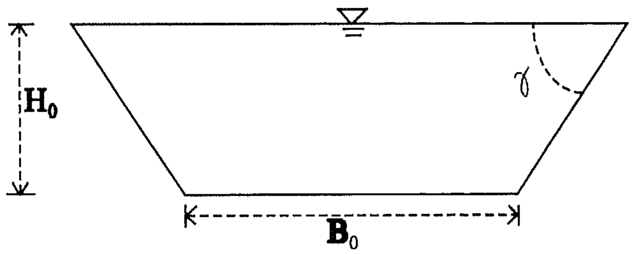

Figure 1. Situation sketch of the discrete channel shape

For this type of channels, the flow discharge Q_w can be expressed as

$$Q_w(H_0, B_0) = (1 + \frac{1}{tan\gamma}\frac{H_0}{B_0})H_0 B_0 u_0 \equiv q \tag{4}$$

in which u_0 is the flow velocity if the water depth equals H_0. Eliminating the flow velocity u_0 in favour of the depth H_0, using Chézy's law, leads to an expression for the width

$$B_0 = \frac{q}{C\sqrt{i_0}}H_0^{-\frac{3}{2}} - \frac{H_0}{tan\gamma} \tag{5}$$

Assuming that the system will tend to minimize the (still not specified) "energy" expenditure to maintain the cross-sectional shape (*cf. Lamberti*, [1988]), we use the generalized minimization principle (2), in which the width B_0 will be eliminated according to (5), so

$$\underset{H_0}{Min}\left\{G_q(H_0)\right\} \tag{6}$$

in which

$$G_q(H_0) = f(H_0)\left[\frac{2 - cos\gamma}{sin\gamma}H_0 + \frac{q}{C\sqrt{i_0}}H^{-\frac{3}{2}}\right] \tag{7}$$

This functional $G_q(H_0)$ wil have a minimum which depends on the undetermined sediment property function $f(H_0)$. For example, the effect of the choice $f(H_0) = H_0$ in $G_q(H_0)$ is shown in figure 2.

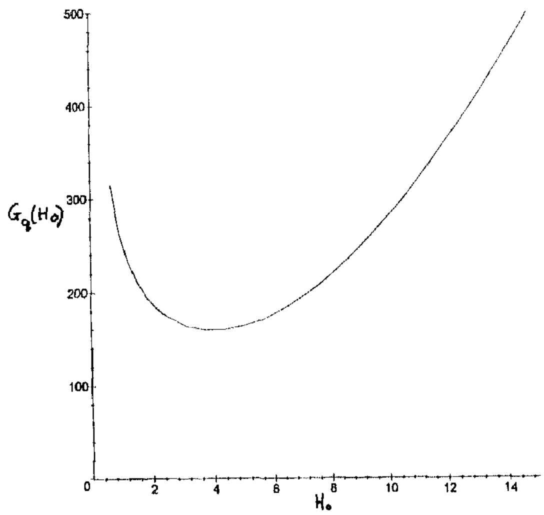

Figure 2. An example of the "energy" expenditure $G_q(H_0)$ as a function of the depth for the choice $f(H_0) = H_0$

In general, if $f(H_0)$ is a power-law function of H_0, say $f(H_0) = H_0^b$, the minimization boils down to

$$\frac{d}{dH_0}\left(\frac{2-cos\gamma}{sin\gamma}H_0^{b+1} + \frac{q}{C\sqrt{i_0}}H_0^{b-\frac{3}{2}}\right) = 0 \qquad (8)$$

whence,

$$H_0 = \left(\frac{(\frac{3}{2}-b)}{(b+1)}\,\frac{sin\gamma}{2-cos\gamma}\,\frac{q}{C\sqrt{i_0}}\right)^{\frac{2}{5}} \qquad (9)$$

So, the water depth H_0 according to this model is proportional to $q^{2/5}$ for all values of the exponent b within the range of $]-1, 3/2[$. This agrees with the q-H-relationships found in some natural rivers (*e.g. Jansen*, [1994], *Koch*, [1993]).

COMPARISION WITH DATA

Illustrations of the results from this approach are shown in figure 3 and table (1), which contains sediment charcteristics shown in *Wharton*, [1995]. In figure 3, we have compared the q-H-relationships of three natural rivers, the Benue (no.1), the Ganges (no.2), and the Jamuna (no.3) with the computed result (9), in which two variants are considered. The first (no.4 in the figure) represents a trapezoidal channel's cross-section as shown in figure 2, with an angle γ of thirty degrees, and the choice for the sediment property function $f(H_0)$ is H_0. The second variant (no.5) in figure 3 refers to a rectangular channel, in which $f(H_0) = H_0^{\frac{7}{5}}$.

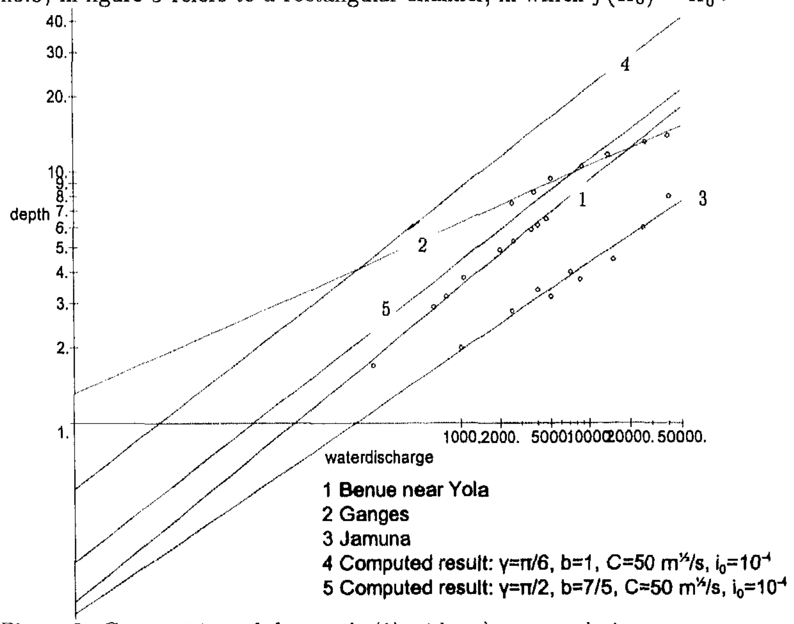

Figure 3. Comparision of the result (9) with some natural rivers

name river	β	α	p	reference
22 gravel-bed rivers, England and Wales	0.33	0.545		Nixon(1959)
7 river reaches of the Quesnel, Cariboo, Taseko, Cilko, and Thomson gravel rivers in south-central British Colombia	0.40	$0.166\ D_{90}^{-0.12}$		Kellerhals (1967)
23 gravel-bed rivers, UK	0.398	0.308		Paker (reported in Bray, 1982, p.543)
21 gravel-bed rivers, Alberta USA (single channel)	0.398	0.308		Paker (reported in Bray, 1982, p.543)
30 gravel-bed rivers, Alberta USA (braided)	0.331	0.292		Paker (reported in Bray, 1982, p.543)
66 sites on gravel-bed rivers, UK	0.38	$0.252 D_{50}^{-0.16}$		Hey (1982)
62 gravel-bed rivers, UK	0.37	$0.22 D_{50}^{-0.11}$		Hey and Thorne (1986)
classical regime equations for design of straight irrigation canals with sandy beds and sandy or cohesive banks	0.36 0.36	$1.21k$ ($R \leq 7$ ft) 0.93 k ($R > 7$ ft	2.0	Simons and Albertson (1963)
computed result (9)	$\frac{2}{5}$	$\frac{(\frac{3}{2}-b)}{(b+1)} \frac{sin\gamma}{2-cos\gamma} \frac{1}{C\sqrt{i_0}}$		this paper

Table 1: *The exponent β and coefficients α and p come from the q-H-relationship ($H = p + \alpha\ q^{\beta}$). In two of the above expressions the factor k depends on the sediment characteristics and varies from 0.23 in the case of a sand-bed and banks with heavy sediment load, to 0.52 for sand-bed channels; these data could be found in Wharton, [1995], together with the references (pp 337-341).*

DISCUSSION

In order to have an indication of the stage-discharge relationship in stable alluvial channels, we have used a variational principle. This is based on an expression for a "resistance-like" quantity (c.f. the energy expenditure) under the constraint of a given water discharge. In the expression for this energy expenditure, a function $f(H_0)$ appears, which depends on the sediment model. This factor multiplies the wetted perimeter to define the energy expenditure function E_f.

Various choices for this function $f(H_0)$ lead to similar results: figure 2 refers to a special case ($b = 1$) of the function $f(H_0) = H_0^b$ (*Berlamont*, [1995], *White*, [1982]). If the sediment property function obeys this power-law function, we obtain the result $H_0 \sim q^{\frac{2}{5}}$. This result is independent of the channel geometry, as shown in equation (9), which is confirmed by the data in figure 3 and table (1). Another result from the choice of the power-law function H_0^b for the sediment property function $f(h)$, is that we do not restrict ourselves in cross-sectional channel shapes: the exponent b allows us to derive the whole gamut of possible channel aspect ratios. Very wide and shallow shapes are found for $b \rightarrow \frac{3}{2}$, and very deep and narrow cross-sections are found for $b \rightarrow -1$.

In reality, the function $f(H_0)$ is not the only point where information on the properties of the sediment and its cross-sectional transport mechanism is introduced, also the bed roughness in the model depends on sediment properties (*c.f. Lamberti*, [1992]). Although we are aware of the limitations of our model, we think the results are encouraging and that this model provides scope for extension with more specific description of the transport mechanisms, in particular the lateral dispersion of the sediment.

REFERENCES

Berlamont, J., Modelling river morphological processes: a reflection. From: XXVIth IAHR Congress, seminar *Morphology of rivers, estuaries and coasts* London, 1995.

Jansen, P.Ph., Principles of river engineering: the nontidal alluvial channel. Delftse Uitgeversmij, Delft, The Netherlands, 1994.

Koch, F.G., Klaassen, G.J., Barua, D.K., FAP 24 River survey project: Morphological processes in the Bangladesh river system, a compilation of papers. Delft Hydraulics, 1993.

Lamberti, A., About extremal hypotheses and river regime, in *International Conference on River Regime*, edited by W.R. White. Hydraulic Research Limited, John Wiley & Sons. Ltd., 1988.

Lamberti, A., Dynamic and variational approaches to the river regime relation, in *Entropy and energy dissipation in water resources*, edited by V.P. Singh and M. Fiorentino. Kluwer Academic Publishers, The Netherlands, 507-525, 1992.

Parker, G., Self-formed rivers with equilibrium banks and mobile bed, 2 The gravel river. *J. Fluid Mech.*, **89**(1), 127-146, 1978.

Schielen, R., Doelman, A. and De Swart, H.E., On the nonlinear dynamics of free bars in straight channels. *J. Fluid Mech.*, **252**, 325-356, 1993.

Wharton, G., Information from channel geometry-discharge relations, in *Changing river channels*, edited by A. Gurnell and G. Petts. Wiley, Chichester, 325-346, 1995.

White, W.R., Bettes, R. and Paris, E., Analytical approach to river regime. *J. Hydr. Div.*, **108**(HY10), 1179-1193, 1982.