DEEP WATER PERIODIC WAVES AS HAMILTONIAN RELATIVE EQUILIBRIA

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Abstract

We use a recently derived KdV-type of equation for waves on deep water to study Stokes waves as relative equilibria. Special attention is given to investigate the cornered Stokes-120 degree wave as a singular solution in the class of smooth steady wave profiles.

Introduction

Surface waves on a layer of incompressible, inviscid fluid in irrotational motion, are described by Luke's variational principle. The formulation of the dynamics as a Hamiltonian system was found by Zakharov '68, Broer '74 and Miles '77. This Hamiltonian formulation is helpful to study basic properties of surface waves. The conservation of energy (which is the Hamiltonian H) is accompanied by conservation of horizontal momentum M when the case of a flat bottom or the case of infinitely deep water is considered. These two integrals are independent and Poisson commute. As is the case for any Hamiltonian system with additional integrals, relative equilibria (RE) can be considered; in this case these are the extremizers of the Hamiltonian on level sets of the momentum. An extremizer provides a surface profile η and corresponding velocity potential ϕ that satisfy the Lagrange multiplier rule: $\delta H = \mu \delta M$ for some multiplier μ. The action of the M flow is a shift in time at a constant velocity which is precisely the multiplier μ, provides a steadily propagating wave as dynamic solution of the Hamiltonian system.

Looking for the existence and shape of steady waves was an important motivation for research in the nineteenth century (wave of elevation) and continued in the twentieth century. The main motivation in the 1895 paper of Korteweg and de Vries was to investigate this existence matter. The existence and explicit formulation of steady finite energy solutions (later called solitons) and periodic (cnoidal) waves was their contribution to this issue, providing a positive answer about existence. All this in the approximation of the one-way directional KdV equation that they derived. Existence of periodic waves (for the full wave equation) has been investigated in the twentieth century by many scientists, among which Levi-Civita, Toland, while approximations of the periodic wave shapes using expansion methods (in amplitude, Fourier modes, etc,) have been derived to a high degree of accuracy, for instance by Rienecker & Fenton [1]. It is to be noted that as far as we are aware of, none of the approximations uses directly the Hamiltonian RE-description.

Among the steady periodic solutions, one is exceptional. It occurs on infinitely deep water, and, while the other profiles are smooth, this special one, referred to as S120 in the following, is a Stokes wave with a profile that has a corner of 120 degrees in the crest. It was shown recently by Rainey & Longuet-Higgens [2] that in a good approximation, S120 has actually the profile of a catenary, i.e. can be written as a hyperbolic cosine curve in between two successive crests. The appearance of a cornered profile indicates a certain singularity in the governing description, since any non-smoothness in an initial profile will be resolved by dispersion in regular dispersive nonlinear equations with differential operators.

In this paper we contribute to the steady wave issue on infinitely deep water from the point of view of considering these steady profiles as Hamiltonian RE. However, instead of the exact surface wave formulation, we use a KdV-type of equation, called the AB-equation, that was recently derived. This equation is exact up to and including quadratic nonlinear terms, and can be taken to have exact linear dispersive properties; the dispersion enters the nonlinear terms in a nontrivial way, correcting substantially inaccuracies in the nonlinear propagation properties. Moreover, this equation, unlike other KdV-type of equations, can also describe waves on infinitely deep water. The equation, and in particular the Hamiltonian and Momentum integrals, is given in section 2.

We formulate the RE-problem of this equation in section 3, and show in section 4 how close the RE are to the set of Rienecker & Fenton solutions, and, especially, how the cornered S120-wave can be approximated. All these results confirm that, at least for these steady solutions, the approximate Hamiltonian and Momentum must be rather accurate, providing relatively simple approximations of these integrals of the full surface wave problem. The explicit expresssion may make these integrals also useful for further advanced functional analytic investigations.

1 AB equation

The AB equation as derived in [3], specialised for the case of infinitely deep water, uses skew-symmetric and symmetric pseudo-differential operators A and B with symbols $\hat{A} := i.sign(k)\sqrt{|k|}$ and $\hat{B} = \sqrt{|k|}$. The Hamiltonian is given by

$$H_{AB}(\eta) = g\int \left[\eta^2 + \frac{1}{2}\eta\left\{(B\eta)^2 - (A\eta)^2\right\}\right] dx \quad (1)$$

and the horizontal Momentum integral by $M(\eta) = \sqrt{g}\int \eta B\eta$. The AB-equation is then given by the Hamiltonian system

$$\partial_t \eta = -\frac{A}{2\sqrt{g}}\delta H_{AB}(\eta), \quad (2)$$

which may also be written in the way of Benjamin [4] as $\partial_t \delta M(\eta) = -\partial_x \delta H(\eta)$; hence solutions have a constant center-of mass velocity. Note that the non-rational pseudo-differential operators A, B cannot be easily approximated with ordinary differential operators.

2 Periodic waves

In the following we consider periodic waves of fundamental period 2π and zero average $\int \eta dx = 0$. Taking the crest at $x = 0 mod(2\pi)$, we look for solutions of the RE equation in the form of a truncated Fourier series. With a an amplitude parameter, we look for solutions as

$$\eta_N = a\cos x + a^2\left[\Sigma_{k=2}^N \beta_k \cos(kx)\right].$$

Restricting the hamiltonian and momentum to this set, we arrive at functions H_N and M_N of $p = (a, \beta_2, ..., \beta_N)$ and the RE equation becomes

$$\nabla_p H_N = \mu_N \nabla_p M_N,$$

which corresponds to the Galerkin projected equation $\delta H = \mu\delta M$.

Splitting the Hamiltonian in a quadratic and cubic part, $H = H^{(2)} + H^{(3)}$ it is remarkable to observe that $H^{(3)}(\alpha\cos(kx)) = 0$, which implies that the energy for Fourier expansions as above does not contain terms of third order $H\left(a\cos x + a^2 v\right)/g = \frac{1}{2}a^2 + O\left(a^4\right)$. Also, since $\delta H^{(3)}(\alpha\cos(kx)) = \frac{1}{2}g\alpha^2\cos(2kx)$, the second order Stokes contribution appears immediately: $\delta H(a\cos x)/g = 2a\cos x + \frac{1}{2}a^2\cos(2x)$.

3 Illustrations

In Figs. 1,2 we show in the Momentum-Hamiltonian plane the curves parameterized by the first-order amplitude parameter a for various approximations of Stokes waves. The tangent at a point to a curve correspond to the propagation speed of the wave at that point, see Fig.3,4. The small deviation from a straight line (obtained for the first order mode only), indicates that increase in velocity with amplitude is rather small, in agreement with the vanishing of third order contribution in the Hamiltonian. The one-term approximation of the highest Stokes wave is indicated by a dot; for this solution $ak = 0.36$. Shown are the results for the Rienecker& Fenton (RF) solutions with 64 Fourier modes (solid line), and the AB relative equilibria with 2, 4 and 16 modes. The dot corresponds to the Rainey & Longuet-Higgins one-term approximation.

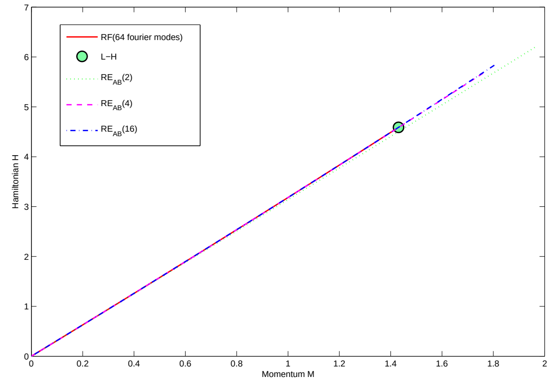

Figure 1: Values of Momentum and Hamiltonian for approximations of Stokes waves.

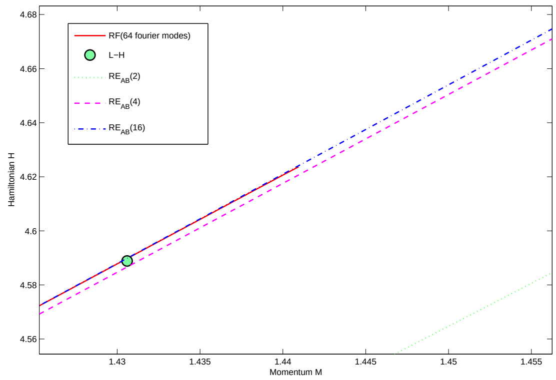

Figure 2: As Fig.1 zoomed in near the highest Stokes wave.

In Fig.5 the profiles are shown of the waves with the same momentum as the highest Stokes wave. Besides the cornered one-term approximation by Rainey & Longuet-Higgens, the Rienecker & Fenton 64-mode approximation and the 2 and 16-mode AB relative equilibria are plotted. Fig.6 shows a zoom-in near the crest.

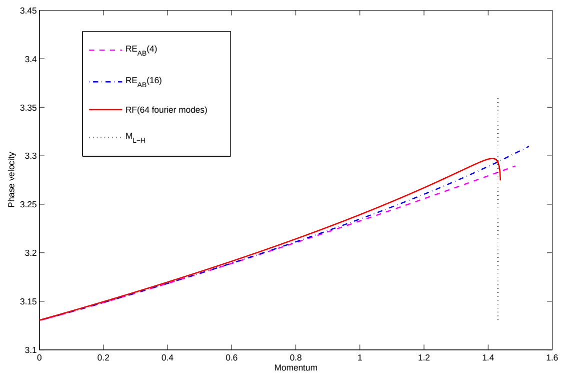

Figure 3: Plots of the phase speed corresponding to the curves in Fig.1.

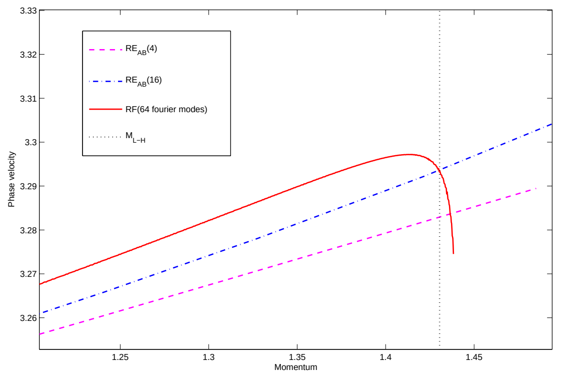

Figure 4: Zoom-in of Fig.3.

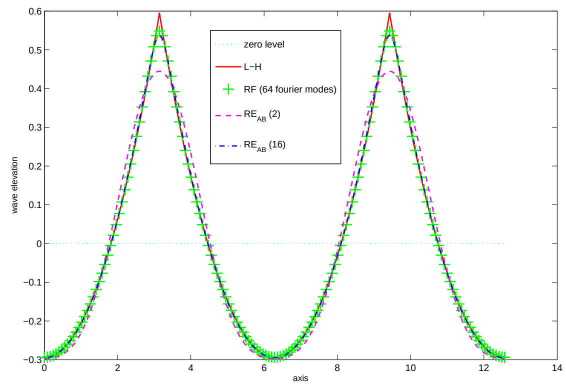

Figure 5: Approximations of the highest Stokes wave.

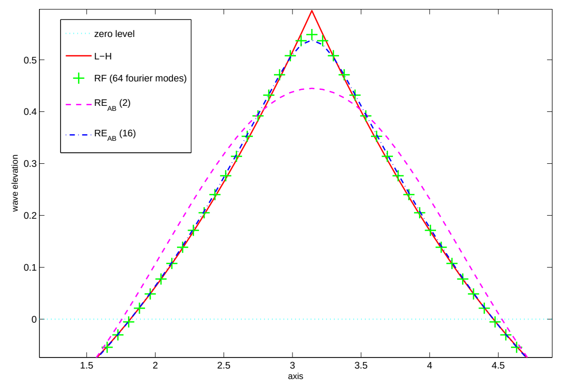

Figure 6: As Fig.5, zoome-in near the crest.

4 Numerical simulations

Calculations have been performed with a high-order (256 or 1024) pseudo-spectral implementation, for calculations over more than 100 wavelengths. As expected, all AB-RE travel virtually undisturbed at constant speed for RE's approximated with at least 6 modes. Also the highest Stokes wave travels with only a small breathing undisturbed in shape. The speeds of these solutions are equal: the tops remain in a small neighbourhood of eachother, with small periodic oscillations.

Acknowledgement

Use of the code of Gert Klopman to calculate the RF-solutions is acknowledged. This research is part of Netherlands Science Foundation NWO-STW, TWI-7216.

References

[1] M.M. Rienecker & J.D. Fenton, A Fourier Approximation Method for Steady Water Wave, Journal of Fluid Mechanics, 104(1981)119-137.

[2] R.C.T. Rainey & M.S. Longuet-Higgins, A close one-term approximation to the highest Stokes Wave on deep water, Ocean Engineering 33(2006) 2012-2024.

[3] E. van Groesen & Andonowati, Variational derivation of KdV-type of models for surface water waves, Physics Letters A, Feb 2007; http://dx.doi.org/10.1016/j.physleta.2007.02.031

[4] T.B. Benjamin, Lectures on nonlinear wave motion, in: Nonlinear Wave Motion, Lectures in Applied Mathematics, vol.15, AMS, Providence, 1974.