

# On approximate observability of strongly stable systems

Birgit Jacob\*      Hans Zwart†

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## Abstract

The Hautus test of Russel and Weiss [7] is sufficient for approximate observability if the system is exponentially stable. In this note we show by means of an example that this statement does not hold for strongly stable systems even if the system is modeled by a contraction semigroup.

## 1 Introduction and main result

We consider the abstract system

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad t \geq 0 \quad (1)$$

$$y(t) = Cx(t), \quad t \geq 0, \quad (2)$$

on a Hilbert space  $H$ . Here  $A$  is the infinitesimal generator of a  $C_0$ -semigroup  $(T(t))_{t \geq 0}$  and by the solution of (1) we mean  $x(t) = T(t)x_0$ , the weak solution. If  $C$  is a bounded linear operator from  $H$  to a second Hilbert space  $Y$ , then it is straightforward to see that  $y(\cdot)$  in (2) is well-defined, and continuous. However, in many PDE's, rewritten in the form (1)-(2),  $C$  is only an bounded

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\*Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, D-10623 Berlin, Germany, [jacob@math.tu-berlin.de](mailto:jacob@math.tu-berlin.de)

†University of Twente, Faculty of Electrical Engineering, Mathematics and Computer Science, Department of Applied Mathematics, P.O. Box 217, 7500 AE Enschede, The Netherlands, [h.j.zwart@math.utwente.nl](mailto:h.j.zwart@math.utwente.nl)

operator from  $D(A)$ , the domain of  $A$ , to  $Y$ , although the output is a well-defined (locally) square integrable function. In the following  $C$  will always be a bounded operator from  $D(A)$  to  $Y$ . If the output is square integrable on the time interval  $(0, \infty)$ , then  $C$  is called an *infinite-time admissible observation operator*, see Weiss [8] and Jacob and Partington [3]. Using the uniform boundedness theorem, we see that the observation operator  $C$  is infinite-time admissible if and only if there exists a constant  $L > 0$  such that

$$\int_0^\infty \|CT(t)x_0\|^2 dt \leq L\|x_0\|^2, \quad x_0 \in D(A). \quad (3)$$

Note that the first norm is in  $Y$ , whereas the second norm is in  $H$ . Assuming that the observation operator  $C$  is infinite-time admissible, (1)-(2) is said to be *approximately observable* if

$$\int_0^\infty \|CT(t)x_0\|^2 dt > 0, \quad x_0 \in D(A) \setminus \{0\}. \quad (4)$$

In Russell and Weiss [7] it is shown that a sufficient condition for approximate observability for exponentially stable systems is the following version of the Hautus test:

There exists a constant  $m > 0$  such that for every  $s \in \mathbb{C}_-$  and every  $x \in D(A)$ :

$$\|(sI - A)x\|^2 + |\operatorname{Re} s| \|Cx\|^2 \geq m|\operatorname{Re} s|^2 \|x\|^2, \quad (5)$$

Here  $\mathbb{C}_-$  denotes the open left half plane. We refer the reader to Russell and Weiss [7], and Jacob and Zwart [4, 5] for more information on this Hautus test. In this note we show that this results does not hold for strongly stable systems even if the operator  $C$  is bounded and  $A$  generates a contraction semigroup. Our main result is as follows.

**Theorem 1.1** *There exists a strongly stable contraction semigroup on a Hilbert space with generator  $A$  such that*

$$\|(sI - A)x\| \geq m|\operatorname{Re} s| \|x\|, \quad \operatorname{Re} s < 0, x \in D(A). \quad (6)$$

*In particular, the pair  $(A, 0)$  satisfies the Hautus test (5). Clearly the zero operator is infinite-time admissible for the semigroup, but the pair  $(A, 0)$  is not approximately observable.*

The proof of this theorem is given in Section 2.

## 2 Proof of the main result

In the following  $\mathbb{D}$  denotes the set  $\{\rho \in \mathbb{C} \mid |\rho| < 1\}$ .

**Lemma 2.1** *Let  $T \in \mathcal{L}(H)$  be an operator satisfying*

$$\|(\rho I - T)x\| \geq c(1 - |\rho|)\|x\|, \quad \rho \in \mathbb{D}, x \in H,$$

*for some constant  $c > 0$  independent of  $\rho$  and  $x$ . Then there exists a constant  $m > 0$  such that*

$$\|(\rho I - T)x\| \geq m \frac{1 - |\rho|}{1 - |\rho|} \|(I - T)x\|, \quad \rho \in \mathbb{D}, x \in H,$$

**Proof:** For  $\rho \in \mathbb{D}$  and  $x \in H$  we have

$$\begin{aligned} \|(I - T)x\| &\leq \|(\rho I - T)x\| + |1 - \rho| \|x\| \\ &\leq \|(\rho I - T)x\| + \frac{1}{c} \frac{|1 - \rho|}{1 - |\rho|} \|(\rho I - T)x\| \\ &\leq \left(1 + \frac{1}{c}\right) \frac{|1 - \rho|}{1 - |\rho|} \|(\rho I - T)x\| \end{aligned}$$

□

**Proposition 2.2** *There exists a contraction  $T \in \mathcal{L}(H)$  such that*

$$\|(\lambda I - T)x\| \geq \frac{1}{2}(1 - |\lambda|)\|x\|, \quad \lambda \in \mathbb{D}, x \in H,$$

and

$$\lim_{n \rightarrow \infty} \|T^n x\| = 0, \quad x \in H.$$

An even stronger version of this proposition can be found in Faddeev [2, Theorem 3]. We include a simplified proof which treats our situation.

**Proof:** As  $H$  we choose  $\ell^2(\mathbb{N})$ . We define  $T \in \mathcal{L}(\ell^2(\mathbb{N}))$  by

$$(Tx)_{n+1} := \mu_n x_n, \quad (Tx)_1 = 0, \quad x \in \ell^2(\mathbb{N}), n \in \mathbb{N},$$

where the sequence  $(\mu_n)_n$  will be defined later on. The operator  $T$  now satisfies for  $\lambda \in \mathbb{D}$ :

$$\|(T - \lambda I)x\|^2 = |\lambda|^2 \|x\|^2 + \sum_{j=1}^{\infty} |\mu_j|^2 |x_j|^2 - 2 \operatorname{Re} \left( \lambda \sum_{j=1}^{\infty} \mu_j x_j \overline{x_{j+1}} \right).$$

Using

$$|2\operatorname{Re} \lambda \mu_j x_j \overline{x_{j+1}}| \leq \beta_j |\lambda| \mu_j^2 |x_j|^2 + \beta_j^{-1} |\lambda| |x_{j+1}|^2$$

for  $j \in \mathbb{N}$ ,  $\beta_j > 0$ , we get

$$\begin{aligned} \|(T - \lambda I)x\|^2 &\geq \sum_{j=1}^{\infty} (|\lambda|^2 + \mu_j^2) |x_j|^2 - \sum_{j=1}^{\infty} (\beta_j |\lambda| \mu_j^2 |x_j|^2 + \beta_j^{-1} |\lambda| |x_{j+1}|^2) \\ &= \sum_{j=1}^{\infty} (|\lambda|^2 + \mu_j^2) |x_j|^2 - \sum_{j=1}^{\infty} \beta_j |\lambda| \mu_j^2 |x_j|^2 - \sum_{j=2}^{\infty} \beta_{j-1}^{-1} |\lambda| |x_j|^2 \\ &= \sum_{j=1}^{\infty} (|\lambda|^2 + \mu_j^2 - \beta_j |\lambda| \mu_j^2 - \beta_{j-1}^{-1} |\lambda|) |x_j|^2, \end{aligned}$$

where  $\beta_0^{-1} = 0$ . Choosing

$$\beta_j := \frac{j+1}{j}, \quad \beta_0^{-1} = 0 \quad \text{and} \quad \mu_j := \frac{j}{j+1}$$

we obtain

$$\|(T - \lambda I)x\|^2 \geq \sum_{j=1}^{\infty} \left( |\lambda|^2 + \frac{j^2}{(j+1)^2} - |\lambda| \left( \frac{j}{j+1} + \frac{j-1}{j} \right) \right) |x_j|^2.$$

It remains to show that

$$x^2 + \frac{j^2}{(j+1)^2} - x \frac{2j^2 - 1}{j^2 + j} \geq \frac{1}{4} (1 - x)^2 \quad (7)$$

for every  $j \in \mathbb{N}$  and every  $x \in [0, 1]$ . Equivalently, it is to show that

$$\frac{3}{4} x^2 + \left( \frac{j^2}{(j+1)^2} - \frac{1}{4} \right) - x \left( \frac{2j^2 - 1}{j^2 + j} - \frac{1}{2} \right) \geq 0 \quad (8)$$

for every  $j \in \mathbb{N}$  and every  $x \in [0, 1]$ . For  $j = 1$  inequality (8) reads  $x^2 \geq 0$  which it is of course true. Let now  $j \in \mathbb{N}$  with  $n \geq 2$ . We define  $\gamma_j := \frac{j^2}{(j+1)^2} - \frac{1}{4}$ ,  $\beta_j := -\left(\frac{2j^2-1}{j^2+j} - \frac{1}{2}\right)$  and  $\alpha_j := 3/4$ . Since  $\alpha_j > 0$ , it remains to show that the polynomial  $\alpha_j x^2 + \beta_j x + \gamma_j$  has no real root, and this is the case if  $\beta_j^2 - 4\alpha_j \gamma_j < 0$ . We have

$$\beta_j^2 - 4\alpha_j \gamma_j = \left( \frac{2j^2 - 1}{j^2 + j} - \frac{1}{2} \right)^2 - \frac{3j^2}{(j+1)^2} + \frac{3}{4} = \frac{1+j-2j^2}{(j+1)^2 j^2} < 0.$$

This shows (7) for every  $j \in \mathbb{N}$  and every  $x \in [0, 1]$ , and thus  $\|(\lambda I - T)x\| \geq \frac{1}{2}(1 - |\lambda|)\|x\|$  for  $\lambda \in \mathbb{D}$  and  $x \in H$ . Since  $|\mu_j| < 1$ ,  $j \in \mathbb{N}$ , it is easy to see that  $T$  is a contraction. Finally,  $\prod_{j=1}^{\infty} \mu_j = 0$  implies  $\lim_{n \rightarrow \infty} \|T^n x\| = 0$  for every  $x \in H$ .  $\square$

**Proof of Theorem 1.1:** Let  $T$  be the operator given by Proposition 2.2. Since  $T$  is power stable, we get that  $1 \notin \sigma_p(T)$ . By Lemma 2.1 there exists a constant  $m > 0$  such that

$$\|(\rho I - T)x\| \geq m \frac{1 - |\rho|}{|1 - \rho|} \|(I - T)x\|, \quad \rho \in \mathbb{D}, x \in H,$$

Let  $A : D(A) \subset H \rightarrow H$  be defined by

$$\begin{aligned} Ax &:= (T + I)(T - I)^{-1}x, \quad x \in D(A), \\ D(A) &:= R(T - I). \end{aligned}$$

In Sz.-Nagy and Foias [6, page 142] it is shown that  $A$  generates a strongly stable contraction semigroup. It remains to show that (6) holds. For  $x \in D(A)$  and  $s \in \mathbb{C}$  with  $\operatorname{Re} s < 0$  we have  $x = (T - I)y$  for some  $y \in H$ ,  $s = \frac{\rho + 1}{\rho - 1}$  for some  $\rho \in \mathbb{D}$ , and

$$\begin{aligned} \|(sI - A)x\| &= \left\| \frac{\rho + 1}{\rho - 1}(T - I)y - (T + I)y \right\| = \frac{2}{|1 - \rho|} \|(\rho I - T)y\| \\ &\geq \frac{2}{|1 - \rho|^2} m(1 - |\rho|) \|x\| \geq m \frac{1 - |\rho|^2}{|1 - \rho|^2} \|x\| = m |\operatorname{Re} s| \|x\|. \end{aligned}$$

$\square$

Hence we have constructed an example of a strongly stable contraction semigroup such that  $(A, 0)$  satisfies the Hautus test, whereas this system is (clearly) not approximately controllable. One might wonder whether a similar example is possible with a bounded  $A$ . In the following paragraph, we explain that this is not possible.

If  $A$  is bounded, then there exists a point in the left-half plane, which is in the resolvent set of  $A$ . Since for any  $s \in \mathbb{C}_-$ ,  $\|(sI - A)x\|^2 \geq m |\operatorname{Re}(s)|^2 \|x\|^2$ , this implies that  $\mathbb{C}_- \subset \rho(A)$ , and

$$\|(sI - A)^{-1}\| \leq \frac{\sqrt{m}}{|\operatorname{Re}(s)|}, \quad s \in \mathbb{C}_-.$$

By van Casteren [1] this implies that  $A$  is similar to a unitary group, and hence it cannot be strongly stable.

Note that the boundedness of  $A$  was only used to have a non-empty intersection of the left-half plane and the resolvent set. Thus the above reasoning still remains valid under the weaker assumption that  $\rho(A) \cap \mathbb{C}_- \neq \emptyset$ .

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