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A port-Hamiltonian approach to modeling and interconnections of canal systems

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Abstract— **We show how the port-Hamiltonian formulation of distributed parameter systems, which incorporates energy flow through the boundary of the spatial domain of the system, can be used to model networks of canals and study interconnections of such systems. We first formulate fluid flow with 1-d spatial variable whose dynamics are given by the well-known shallow water equations, with respect to a Stokes-Dirac structure, and then consider a slightly more complicated case where we have a modified (a non-constant) Stokes-Dirac structure. We also explore the existence of Casimir functions for such systems and highlight their implications on control of fluid dynamical systems.**

Keywords— **Shallow water equations, port-Hamiltonian systems, Dirac structures**

I. INTRODUCTION

In recent publications, see for e.g. [9], [8], the Hamiltonian formulation of distributed parameter systems has been successfully extended to incorporate boundary conditions corresponding to non-zero energy flow, by defining a Dirac structure on certain spaces of differential forms on the spatial domain and its boundary, based on the use of Stokes' theorem. This is essential from a control and interconnection point of view, since in many applications interaction of system with its environment takes place through the boundary of the system. This framework has been applied to model various kinds of systems from different domains, like telegraphers equations, fluid dynamical systems, Maxwell equations, flexible beams and so on. The results on interconnections of port-Hamiltonian systems have also been extended to the distributed parameter case, and so have been some energy shaping techniques for control of distributed parameter port-Hamiltonian systems.

In this paper we use this framework of distributed parameter port-Hamiltonian systems to model and study interconnections of canals described by the so called shallow-water equations. This is a case of a distributed parameter port-Hamiltonian system with 1-d spatial domain. The mass density and the velocity both are represented by a 1-form. We consider two different cases of fluid flow, first where we have only one velocity component. In this case we see that this system can be modeled with help of a constant Stokes- Dirac structure. Next we consider a slightly different case where we induce an additional velocity component in the z direction such that the mass density and the two velocity components are constant with respect to this additional coordinate. So, it can still be modeled as a system with $1-d$ spatial domain. In this case we need to define a modified Dirac structure (because of the additional velocity component) on the space of state variables. We study interconnection properties of canals modeled in this framework for both the cases and also see how the additional velocity component does not contribute to the power exchange through the boundary.

Finally we investigate the existence of Casimirs for both cases which give rise to some possibilities of passivity based control of fluid dynamical systems in the port-Hamiltonian framework. We see that in the first case the only conservation law is the mass balance, where as in the second case (the system with an additional velocity component) we have possibilities for more Casimir functions, in fact we have a whole class of Casimirs to choose from.

II. PORT-HAMILTONIAN FORMULATION OF THE SHALLOW-WATER EQUATIONS.

The dynamics of an open-channel canal can be described by the shallow water equations given by the following set of equations [5]

$$\partial_t \begin{bmatrix} \tilde{h} \\ \tilde{u} \end{bmatrix} + \begin{bmatrix} \tilde{u} & \tilde{h} \\ g & \tilde{u} \end{bmatrix} \partial_x \begin{bmatrix} \tilde{h} \\ \tilde{u} \end{bmatrix} = 0 \tag{1}$$

with $\tilde{h}(x,t)$ the height of the water level, $\tilde{u}(x,t)$ the water speed and g the acceleration due to gravity, with x being the spatial variable representing the length of the canal i.e., $x \in [0, L]$. The first equation expresses the mass-balance and the second equation comes from the momentum-balance. The total energy (Hamiltonian) is given by

$$\mathcal{H} = \frac{1}{2}\int_0^l [\tilde{h}\tilde{u}^2 + g\tilde{h}^2]dx \tag{2}$$

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A. Notations

We apply the differential geometric framework of differential forms on the spatial domain W of the system. The shallow water equations are a case of a distributed parameter system with a one-dimensional spatial domain and in this context it means that we distinguish between zero-forms (functions) and one forms defined on the interval representing the spatial domain of the canal. One forms are objects which can be integrated over every sub-interval of the interval where as zero-forms or functions can be evaluated at any points of the interval. If we consider a spatial coordinate x for the interval W, then a function is simply given by the values $f(x) \in \mathbb{R}$ for every coordinate value in x in the interval, while a one-form g is given as $\tilde{g}(x)dx$ for a certain density function g. We denote the set of zero forms and one-forms on W by $\Omega^0(W)$ and $\Omega^1(W)$ respectively. Given a coordinate x for the spatial domain we obtain by spatial differentiation of a function $f(x)$ the one-form $\omega := \frac{\mathrm{d}f}{dx}(x)dx$. In coordinate free language this is denoted as $\omega = \mathrm{d}f$, where d is called the exterior derivates mapping zero forms to one forms.We denote by $*$, the Hodge star operator mapping one forms to zero-forms, meaning that given a one-form g on W, the star operator converts the one form g to a function g, mathematically given as $*g(x) = \tilde{g}(x)$. Also denote by $\wedge$, the wedge product of two differential forms. Given a k-form ω_1 and an l-form ω_2, the wedge product $\omega_1 \wedge \omega_2$ is a $k + l$-form.

In case of shallow-water equations the energy variables are the height $h(x,t)$ and the velocity $u(x,t)$. The energy exchange of the system with the environment takes place through the boundary $\{0, L\}$ of the system.

The Stokes-Dirac structure corresponding to the 1-d fluid flow modeled by the shallow-water equations is defined as follows: The spatial domain $W \subset D \subset \mathbb{R}$ is represented by a 1-d manifold with point boundaries. The height of the water flow (representing the mass density) through the canal $h(x,t)$ is identified with a 1-form on W. Note that the integral of h over a subinterval denotes the total amount of water contained in that subinterval. Furthermore, assuming the existence of a *Riemannian metric* $<,>$ on W, we identify (by index raising w.r.t this Riemannian metric) the Eulerian vector field u on W with a 1-form. This leads to the consideration of the (linear) space of energy variables

$$X := \Omega^1(W) \times \Omega^1(W)$$

To identify the boundary variables we consider space of 0-forms, i.e., space of functions on ∂W, to represent both the boundary flow and the dynamic pressure at the boundary. We thus consider the space of boundary variables

$$\Omega^0(\partial W) \times \Omega^0(\partial W)$$

Proposition 1: Let $W \subset \mathbb{R}$ be a 1-dimensional manifold with boundary ∂W. Consider $V = \Omega^1(W) \times \Omega^1(W) \times \Omega^0(\partial W)$ and $V^* = \Omega^0(W) \times \Omega^0(W) \times \Omega^0(\partial W)$, together with the bilinear form

$$\begin{aligned} &<< (f_h^1, f_u^1, f_b^1, e_h^1, e_u^1, e_b^1), (f_h^2, f_u^2, f_b^2, e_h^2, e_u^2, e_b^2) >> \\ &:= \int_W (e_h^1 \wedge f_h^2 + e_h^2 \wedge f_h^1 + e_u^1 \wedge f_u^2 + e_u^2 \wedge f_u^1) \\ &+ \int_{\partial W} (e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1) \end{aligned} \tag{3}$$

with $f_h^i, f_v^i \in \Omega^1(W)$, $e_h^i, e_v^i, f_b^i, e_b^i \in \Omega^0(\partial W)$
Then, $D \subset V \times V^*$ defined as

$$\begin{aligned} D = \ & \{(f_h, f_u, f_b, e_h, e_u, e_b) \in V \times V^* \\ & f_h = \mathrm{d}e_u, f_u = \mathrm{d}e_h, f_b = e_h \mid_{\partial W}, e_b = -e_u \mid_{\partial W}\} \end{aligned} \tag{4}$$

where d is the exterior derivative (mapping $0-$forms into 1- forms), $|_{\partial W}$ denoting the restriction of 0-forms on W to 0-forms on the boundary ∂W, is a Dirac structure with respect to the bilinear form $<<,>>$ defined as above, that is $D = D^{\perp}$, where $\perp$ is with respect to (3). D is called a Stokes' Dirac structure. Note that in standard coordinate notation d would correspond to the spatial derivative, given by ∂_x
In terms of shallow-water equations the above terms would correspond to

$$\begin{aligned} f_h &= -\frac{\partial}{\partial t}h(x,t), e_h = \delta_h \mathcal{H} = \frac{1}{2}(*u)(*u) + g * h \\ f_u &= -\frac{\partial}{\partial t}u(x,t), e_u = \delta_u \mathcal{H} = *h * u \\ f_b &= \delta_u \mathcal{H} \mid_{\partial W}, e_b = -\delta_h \mathcal{H} \mid_{\partial W} \end{aligned} \tag{5}$$

with the Hamiltonian given as

$$\int_Z \mathcal{H} = \frac{1}{2}(*u)h(*u) + \frac{1}{2}g(*h)h$$

Substituting (5) into (4), we obtain the shallow water equations (1).

Proof: The proof follows the same arguments as in [9], making use of the Stokes' theorem and hence we omit the proof here. ∎

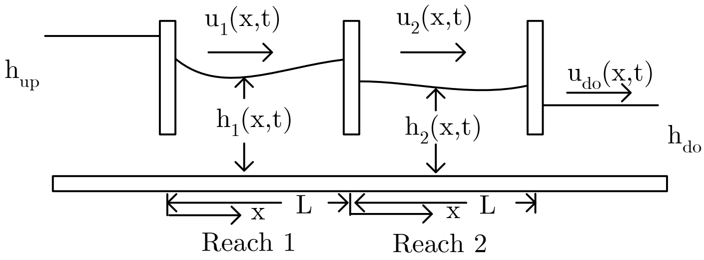

Fig. 1. Two canals in cascade

B. Energy Balance

Energy balance follows immediately from the power conserving property of the Stokes-Dirac structure, given by

$$\int_W (e_h \wedge f_h + e_u \wedge h_u) + \int_{\partial W} e_b \wedge f_b = 0$$

and hence

$$\begin{aligned}\frac{d}{dt}\mathcal{H} &= \int_{\partial W} e_b f_b \\ &= \check{h}\tilde{u}(\frac{1}{2}\tilde{u}^2 + g\tilde{h}) \mid_0^L \\ &= (\tilde{u}(\frac{1}{2}h\tilde{u}^2 + \frac{1}{2}g\tilde{h}^2)) \mid_0^L + (\tilde{u}(\frac{1}{2}g\tilde{h}^2)) \mid_0^L\end{aligned}$$

The first term in last line of the above expression for energy balance corresponds to the energy flux (the total energy times the velocity) through the boundary and the second term is the work done by the hydrostatic pressure given by pressure times the velocity.

C. Interconnections of canals modeled by shallow-water equations.

In this section we study the interconnection properties of two canals in cascade as shown in fig(1). The beds of the two canals are assumed to be horizontal and friction effects are neglected. Let $h_i(z,t), v_i(z,t)$[1] respectively be the height of the water level and water velocity at the $i-$th reach, $i = 1, 2$. We also assume

$$h_{up} > h_1 > h_2 > h_{do}$$

and that both reaches have the same length L
$h_i(x,t)$ is the water height in the i-th reach, with $i = 1, 2$

$$h_{i0} = h_i(0,t) \text{ and } h_{iL} = h_i(L,t)$$

$u_i(x,t)$ is the water velocity in the i-th reach

$$u_{i0} = u_i(0,t) \text{ and } u_{iL} = u_i(L,t)$$

h_{up} and h_{do} are the water heights of the left and right reservoirs respectively, as shown in the figure. The dynamics of each reach are given by the shallow-water equations (5)

$$\begin{bmatrix} f_{hi} \\ f_{ui} \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e_{hi} \\ e_{ui} \end{bmatrix} \tag{6}$$

together with the boundary conditions as above. The flows are assumed subcritical i.e. $u_i < \sqrt{gh_i}$. One end of each reach is coupled to a reservoir as shown in the figure. The interaction between the various subsystems takes place through the three gates. The interconnection constraints at each gate are given as follows:
Left gate:

$$\begin{aligned} h_{up} &= f_{b1,0} = \frac{u_{10}^2}{2g} + h_{10} \\ Q_0 &= e_{b1,0} = -h_{10}u_{10} \end{aligned} \tag{7}$$

[1]from now on we will often abuse the notation and simply write $h = \tilde{h}(x)dx)$ and similarly for other terms.

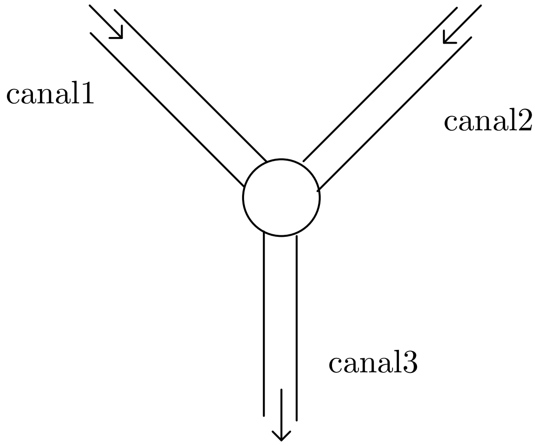

Fig. 2. Interconnections of cross canals

Intermediate gate:

$$f_{b1,L} = f_{b2,0} \iff \frac{u_{1L}^2}{2g} + h_{1L} = \frac{u_{20}^2}{2g} + h_{20}$$
$$e_{b1,L} = e_{b2,0} \iff h_{1L}u_{1L} = h_{20}u_{20} \tag{8}$$

Right gate:

$$f_{b2,L} = f_{do} \iff \frac{u_{2L}^2}{2g} + h_{2L} = \frac{u_{do}^2}{2g} + h_{do}$$
$$e_{b2,L} = e_{do} \iff -h_{2L}u_{2L} = h_{do}u_{do} \tag{9}$$

with $f_{bi,x}$, e_{bix} being the flows and effort variables at each end of the boundary if the i-th reach. It can easily be shown that the composed system is again a port-Hamiltonian system, with Dirac structure the composition of Dirac structures of the subsystems and Hamiltonian the sum of the Hamiltonians. Since the closed-loop system is again a port-Hamiltonian system, it easily ensures some desired properties and provides useful information for analysis and control of the closed-loop system by generating Casimirs for the closed-loop system.

Remark 2: In a similar way we can also look at more practical cases of interconnections of cross canals as shown in fig (2). If $(f_1, e_1), (f_2, e_2)$ and (f_3, e_3) are the boundary variables (the end where the three canals meet) of canal 1, 2 and 3 respectively, then the interconnection constraints would be as follows:

$$f_1 + f_2 = f_3$$
$$e_1 = e_2 = e_3$$

At steady state the boundary variables would be the same as those in the spatial domain. The first equation corresponds to the water flow or discharge at the junction and the second equation corresponds to the Bernoulli function.

III. Shallow water equations with an additional velocity component.

We consider a slightly different and more complicated case in which we consider an additional component of the velocity in the z direction as shown in the figure (3). In addition, we assume that the height h, the horizontal velocity u and the additional velocity component v do not depend on this additional coordinate and hence we can still model this as a 1-d fluid flow as shown below. The dynamics of the system are described by the following set of equations [5]

$$\partial_t \tilde{h} = -\partial_x(\tilde{h}\tilde{u})$$
$$\partial_t(\tilde{h}\tilde{u}) = -\partial_x(\tilde{h}\tilde{u}^2 + \frac{1}{2}g\tilde{h}^2)$$
$$\partial_t(\tilde{h}\tilde{v}) = -\partial_x(\tilde{h}\tilde{u}\tilde{v}) \tag{10}$$

with $\tilde{h}(x,t)$ the height of the water level, $\tilde{u}(x,t)$ the water velocity in the x direction and $\tilde{v}(x,t)$ the component of the velocity in the z direction with g the acceleration due to gravity. The first equation again corresponds to mass balance,

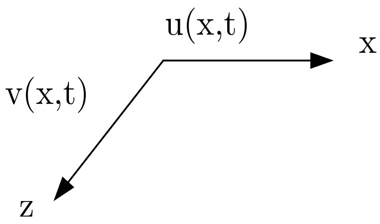

Fig. 3. The additional velocity component

while the second and third equations correspond to the momentum balance. The above set of equations can alternatively be written as

$$\begin{aligned}\partial_t\tilde{h} &= -\partial_x(\tilde{h}\tilde{u})\\ \partial_t\tilde{u} &= -\partial_x(\frac{1}{2}\tilde{u}^2+g\tilde{h})\\ \partial_t\tilde{v} &= -\tilde{u}\partial_x\tilde{v}\end{aligned} \tag{11}$$

In the port-Hamiltonian framework it is modeled as follows. The energy variables now are $h(x,t)$, $u(x,t)$ and $v(x,t)$, the Hamiltonian of the system is given by

$$\mathcal{H} = \int_Z \frac{1}{2}(\tilde{u}^2(x,t)+\tilde{v}^2(x,t)) + \frac{1}{2}g\tilde{h}^2 \tag{12}$$

and the variational derivatives are given by $\delta\mathcal{H} = [\frac{1}{2}\tilde{v}^2 + g\tilde{h} \;\; \tilde{h}\tilde{u} \;\; \tilde{h}\tilde{v}]^T$. As before the interaction of the system with the environment takes place through the boundary of the system $\{0, L\}$. The Stokes-Dirac structure corresponding to the shallow-water equations with an additional velocity component, and modeled as a 1-d fluid flow, is defined as follows: The spatial domain $W \subset D \subset \mathbb{R}$ as before is represented by a 1-d manifold with point boundaries. The height of the water flow through the canal $h(x,t)$ is identified with a 1-form on W and again assuming the existence of a *Riemannian metric* $<,>$ on W, we can identify (by index raising w.r.t this Riemannian metric) the Eulerian vector fields u and v on W with a 1-form. This leads to the consideration of the (linear) space of energy variables.

$$X := \Omega^1(W)\times\Omega^1(W)\times\Omega^1(W)$$

To identify the boundary variables we consider space of 0-forms, i.e., the space of functions on ∂W, to represent the boundary height ,the dynamic pressure and the additional velocity component at the boundary. We thus consider the space of boundary variables

$$\Omega^0(\partial W)\times\Omega^0(\partial W)\times\Omega^0(\partial W)$$

We will now define the Stokes-Dirac structure on $X\times\Omega^0(\partial W)$, (i.e., the space of energy variables and part of the space of the boundary variables) in the following way

Proposition 3: (Modified Stokes-Dirac structure) Let $W \subset \mathbb{R}$ be a 1 -dimensional manifold with boundary ∂W. Consider $V = X\times\Omega^0(\partial W) = \Omega^1(W)\times\Omega^1(W)\times\Omega^1(W)\times\Omega^0(\partial W)$, together with the bilinear form

$$\begin{aligned}&<< (f_h^1, f_u^1, f_v^1, f_b^1, e_h^1, e_u^1, e_v^1, e_b^1), (f_h^2, f_u^2, f_v^2, f_b^2, e_h^2, e_u^2, e_v^2, e_b^2) >>\\ &:= \int_W (e_h^1\wedge f_h^2 + e_h^2\wedge f_h^1 + e_u^1\wedge f_u^2 + e_u^2\wedge f_u^1 + e_v^1\wedge f_v^2 + e_v^2\wedge f_v^1)\\ &+ \int_{\partial W}(e_b^1\wedge f_b^2 + e_b^2\wedge f_b^1)\end{aligned} \tag{13}$$

where

$$\begin{aligned}f_h^i\in\Omega^1(W), f_u^i\in\Omega^1(W), f_v^i\in\Omega^1(W), f_b^i\in\Omega^0(\partial W)\\ e_h^i\in\Omega^0(W), e_u^i\in\Omega^0(W), e_v^i\in\Omega^0(W), e_b^i\in\Omega^0(W)\end{aligned}$$

Then $D\subset V\times V^*$ defined as

$$\begin{aligned}D = \{(f_h, f_u, f_v, f_b, e_h, e_u, e_v, e_b)\in V\times V^* \mid\\ \begin{bmatrix}f_h\\ f_u\\ f_v\end{bmatrix} = \begin{bmatrix}0 & \mathrm{d} & 0\\ \mathrm{d} & 0 & -\frac{1}{*h}\mathrm{d}(*v)\\ 0 & \frac{1}{*h}\mathrm{d}(*v) & 0\end{bmatrix}\begin{bmatrix}e_h\\ e_u\\ e_v\end{bmatrix};\end{aligned} \tag{14}$$

$$\begin{bmatrix} f_b \\ e_b \\ e'_v \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \frac{1}{*h} \end{bmatrix} \begin{bmatrix} e_u \mid_{\partial W} \\ e_h \mid_{\partial W} \\ e_v \mid_{\partial W} \end{bmatrix}$$

is a Dirac structure, that is $D = D^{\perp}$, where $\perp$ is with respect to (13).
In terms of shallow-water equations with an additional velocity component the above terms would correspond to

$$\begin{aligned} f_h &= -\frac{\partial}{\partial t} h(x,t), e_h = \delta_h \mathcal{H} = (\frac{1}{2}((*u)(*u) + (*v)(*v)) + g(*h)) \\ f_u &= -\frac{\partial}{\partial t} u(x,t), e_u = \delta_u \mathcal{H} = (*h)(*u) \\ f_v &= -\frac{\partial}{\partial t} v(x,t), e_v = \delta_v \mathcal{H} = (*h)(*v) \\ f_b &= \delta_u \mathcal{H} \mid_{\partial W}, e_b = -\delta_h \mathcal{H} \mid_{\partial W}, \\ e'_v &= \frac{1}{*h} \delta_v \mathcal{H} \mid_{\partial W} \end{aligned} \tag{15}$$

Substituting (15) into (14), we obtain the equations (11).

Proof: The proof is based on the skew symmetric term in the 3×3 matrix and also that the boundary variable e'_v in (14) does not contribute to the bilinear form (13) and also follows a procedure as in [9]. ■

Remark 4: The Dirac structure above is no more a constant Dirac structure as it depends on the energy variables h, u and v. Moreover, we will also see that of the three boundary variables f_b, e_b and e'_v, only f_b and e_b play a role in the power exchange through the boundary as will be seen in the expression for energy balance. We consider e'_v as the third boundary variable instead of $e_v \mid_{\partial W}$ because to study interconnections of such systems we would like to consider v as the boundary variable instead of hv at the boundary as will be shown later.

A. Energy Balance

It follows from the power conserving property of a Dirac structure that the modified Stokes-Dirac structure defined above has the property

$$\int_W (e_h \wedge f_h + e_u \wedge f_u + e_v \wedge f_v) + \int_{\partial W} e_b \wedge f_b = 0$$

and hence we can get the energy balance

$$\frac{d}{dt}\mathcal{H} = \int_{\partial W} e_b \wedge f_b$$

which can also be seen by the following

$$\begin{aligned} \frac{d}{dt}\mathcal{H} &= \int_W [\delta_h \mathcal{H} \wedge \frac{\partial h}{\partial t} + \delta_u \mathcal{H} \wedge \frac{\partial u}{\partial t} + \delta_v \mathcal{H} \wedge \frac{\partial v}{\partial t}] \\ &= -\int_W d[\delta_h \mathcal{H} \wedge \delta_u \mathcal{H}] \\ &= \int_{\partial W} \delta_h \mathcal{H} \wedge \delta_u \mathcal{H} \\ &= \int_{\partial W} e_b \wedge f_b \\ &= \check{h}\tilde{u}(\frac{1}{2}\tilde{u}^2 + g\tilde{h}) \mid_0^L \\ &= (\tilde{u}(\frac{1}{2}h\tilde{u}^2 + \frac{1}{2}g\tilde{h}^2)) \mid_0^L + (\tilde{u}(\frac{1}{2}g\tilde{h}^2)) \mid_0^L \end{aligned}$$

As in the previous case the first term in last line of the above expression for energy balance corresponds to the energy flux (the total energy times the velocity) through the boundary and the second term is the work done by the hydrostatic pressure given by pressure times the velocity. It is also seen that the boundary variables which contribute to the power at the boundary are f_b and e_b and the third boundary variable e'_v does not contribute to it.

B. Interconnections in this case

We again consider interconnections of canals as shown in fig (1), but now we have an additional velocity component $v(x,t)$ in the z direction.

$$v_{i0} = v(0,t), \text{ and } v_{iL} = v(L,t)$$

v_{up} and v_{do} are the velocity components in the z direction at the gates of the left and right reservoirs respectively. The constraints corresponding to f_b and e_b remain the same as the case discussed in the above section. The interconnection constraints due to the additional velocity component are accommodated as follows:

$$v_{up} = v_{10},\ v_{1L} = v_{20},\ v_{2L} = v_{do}$$

where

$$v_{ij} = e'_{vi,j},\ i = 1,2 \text{ and } j = \{0, L\}$$

Since we would like to equate $v(x,t)$ at the intermediate gates (boundaries if the system), it is more convenient to use $e'_v = \frac{1}{h}e_v\mid_{\partial W}$instead of $e_v = hv\mid_{\partial W}$ as the third boundary variable, which does not contribute to the power exchanged through the boundary of the system.

IV. Existence of Casimirs and control.

For the systems (4, 14) considered above one can infer conservation laws or Casimirs, which are independent of the Hamiltonian $\mathcal{H}$ of the system. We investigate such laws for both kinds of systems discussed above. What we see is that the only conservation law for the system described by (4) corresponds to the total mass. However, we also see that for system described by (14) there exist more Casimirs than just the total mass, in fact we have a whole class of Casimir functions to choose from.

For the system (14), it can be seen by the theory of achievable Casimirs [4], that any function $C : \Omega^1(W)\times\Omega^1(W)\times W \to \mathbb{R}$ which satisfies

$$\mathrm{d}(\delta_h C) = 0,\ \mathrm{d}(\delta_u C) = 0$$

In addition if $\delta_h C\mid_{\partial W} = 0$, and $\delta_u C\mid_{\partial W} = 0$ then we see that $\frac{dC}{dt} = 0$ along the trajectories of the system for any Hamiltonian $\mathcal{H}$. Then the only Casimir for the system is the total mass of the system given by $\int_W h$. It can then be easily verified that

$$\frac{d}{dt}\int_W h = \int_W \frac{\partial h}{\partial t} = -\int_W \mathrm{d}(\delta_u\mathcal{H}) = -\int_{\partial W}\delta_u H = \int_{\partial W} e_b$$

which corresponds to the mass balance.

Next we investigate as to what are the achievable Casimirs for the system whose dynamics are described by equations (14), with a modified Dirac structure. Applying the theory of achievable Casimirs [4], we see that any function C : $\Omega^1(W)\times\Omega^1\times\Omega^1(W)\times W\to\mathbb{R}$ is a Casimir function if it satisfies

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0 & \mathrm{d} & 0\\ \mathrm{d} & 0 & -\frac{1}{*h}\mathrm{d}*v\\ 0 & \frac{1}{*h}\mathrm{d}*v & 0\end{bmatrix}\begin{bmatrix}\delta_h C\\ \delta_u C\\ \delta_v C\end{bmatrix}$$

this is considering $\delta_h C\mid_{\partial W} = \delta_u\mid_{\partial W} = 0$. It follows from the first and the third rows of the above matrix that

$$\delta_u C = 0$$

meaning that the Casimir function does not depend on the u term, then to find all the Casimir functions we need to solve the equation given by the second row of the matrix i.e.,

$$\mathrm{d}\delta_h C = \frac{1}{h}\mathrm{d}(*v)\delta_v C$$

It can be shown that all the functions of the form given below are Casimirs for the system (see [7])

$$C = \int_W h.\phi(\frac{1}{*h}\mathrm{d}(*v))$$

for any function ϕ. We discuss here a few specific examples of Casimir functions:
Case 1: where $\phi(\frac{1}{h}\mathrm{d}(*v)) = 1$, we have $C = \int_W h$ which corresponds to mass conservation as in the above case
Case 2: $\phi(\frac{1}{*h}\mathrm{d}(*v)) = \frac{1}{*h}\mathrm{d}(*v)$, in which case $C = \int_W \mathrm{d}(*v)$ which is called *vorticity*
Case 3: $\phi(\frac{1}{*h}\mathrm{d}(*v)) = (\frac{1}{*h}\mathrm{d}(*v))^2$, and this corresponds to $C = \int_W \frac{1}{*h}(\mathrm{d}(*v))^2$, which is called *mass weighted potential enstrophy*.

The existence of Casimir gives rise to some possibilities for passivity based control of distributed parameter port-Hamiltonian systems by interconnection and energy shaping, see for eg [6]. A simple case could be to consider the stability of the interconnected system (4) or(14), for given gate openings and a given h_{up}, and h_{do} and consider stability of the forced equilibrium $\bar{h}_i(x,t), \bar{u}_i(x,t)$ (and also $\bar{v}_i(x,t)$ in case of the interconnected system as in fig (1)), $i = 1,2$.

We can then use the energy Casimir method for stability analysis of the closed-loop system, by using the following function as a candidate Lyapunov function

$$V := \mathcal{H}_{cl} + C$$

with H_{cl} the Hamiltonian system of the closed-loop system and C the corresponding Casimir function of the closed-loop system. The total system can be viewed as a plant-controller system in the following way: The plant system is the interconnection of the two canals in cascade and the reservoirs at both ends of the plant are viewed as the controller system. The height and velocity can be assumed to be fixed for the controller system, and hence no dynamics. The stability of the forced equilibrium can thus be analyzed for the interconnected plant controller system by generating Casimirs for the closed-loop system. A finite dimensional analysis of closed-loop system with a forced equilibrium can be found in [3]

V. CONCLUSIONS AND FUTURE WORK

We have shown how we can model water flow through canals using the port-Hamiltonian framework for distributed parameter systems, using a Dirac structure for a simple case and also a modified Dirac structure, for a slightly complicated case. We also study interconnections of various canals on this framework and also existence of Casimir functions which opens up a possibility for passivity based control of such systems.

Future work could certainly be on exploring passivity based control of fluid systems, by making use of the Casimir functions. Also a possibility could be to consider a higher dimensional spatial domain.

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