Boundary control for a class of dissipative differential operators including diffusion systems

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Abstract

In this paper we study a class of partial differential equations (PDE's), which includes Sturm-Liouville systems and diffusion equations. From this class of PDE's we define systems with control and observation through the boundary of the spatial domain. That is, we describe how to select boundary conditions, such that the resulting system has inputs and outputs acting through the boundary. Furthermore, these boundary conditions are chosen in a way that the resulting system has a nonincreasing energy.

1 Introduction

We study systems described by the following partial differential equation on the interval $z \in (a, b)$ and $t \geq 0$

$$\frac{\partial x}{\partial t}(t,z) = G_1 \frac{\partial}{\partial z}\left(S G_1^T \frac{\partial x}{\partial z}\right)(t,z) + P_1 \frac{\partial x}{\partial z}(t,z) + P_0 x(t,z), \quad x(0,z) = x_0(z), \tag{1a}$$

$$u(t) = \mathcal{B}x(t,z) \tag{1b}$$

$$y(t) = \mathcal{C}x(t,z), \tag{1c}$$

where S is a coercive operator on $L_2((a,b);\mathbb{R}^n)$ and G_i, P_i, $i = \{1,2\}$, are constant matrices of size $n \times m$, and $n \times n$, respectively. Furthermore they satisfy

$$P_0 = -P_0^T, \quad P_1 = P_1^T, \quad \text{and} \quad \begin{bmatrix} P_1 & G_1 \\ G_1^T & 0 \end{bmatrix} \text{ has full rank.} \tag{2}$$

$\mathcal{B}$ and $\mathcal{C}$ are linear boundary operators, i.e., they only depend on the values of $x(t,z)$ and $\frac{\partial x}{\partial z}(t,z)$ at the positions $z = a$ and $z = b$.

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Observe that Sturm-Liouville systems are a special class of this type of equations, choose $n = m = 1$. For more general n and m this class includes diffusion equations with control and observation through the boundary.

In order to determine a unique solution to the differential equation (1a), it is necessary to specify the value of the solution at the initial time, $x_0(z)$, and it is also necessary to impose conditions on the solution at the boundary. This is known as an initial boundary value problem (IBVP). In some cases appropriate boundary conditions can be found from physical considerations, but in other situations it may not be a trivial task. Here, we characterize those $\mathcal{B}$, and hence the inputs u, for which the differential operator in (1a) generates a contraction semigroup. Roughly speaking, this means that (1a)–(1b) has a unique mild solution and that the energy of the system is non-increasing for $u = 0$. It is also possible to choose $\mathcal{B}$ and $\mathcal{C}$ such that the energy of the system satisfies certain balance equation. We prove this by regarding the system (1) as the closed-loop of another dissipative system, for which all these properties were proved in [1].

In [1], the authors describe how to choose inputs and outputs for a class of systems related to skew-symmetric differential operators. This class includes a large group of hyperbolic PDE's like wave equations and some beam equations. As mentioned earlier, these results will be used to describe our class of systems. In other words, this means that we extend the results in [1] to include another class of systems.

Here, $H^N((a,b);\mathbb{R}^m)$ is the subspace

$$H^N((a,b);\mathbb{R}^m) = \left\{ x \in L_2((a,b);\mathbb{R}^m) \,\middle|\, \frac{\partial x}{\partial z}, \dots, \frac{\partial^N x}{\partial z^N} \in L_2((a,b);\mathbb{R}^m) \right\}. \tag{3}$$

A self-adjoint operator, L, is coercive[1] if there exists an $\epsilon > 0$ such that

$$\langle Lx, x\rangle \geq \epsilon \left\|x\right\|^2 > 0 \qquad \text{for all } x \in D(L). \tag{4}$$

2 Background

In the previous section we mentioned that we consider systems described by (1). The results presented in this paper not only hold for second order differential operators but also hold for higher order operators. Thus we shall consider systems of the form

$$\frac{\partial x}{\partial t}(t,z) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)x(t,z), \quad x(0,z) = x_0(z), \tag{5a}$$

$$u(t) = \mathcal{B}x(t,z), \quad z \in (a,b),\ t \geq 0 \tag{5b}$$

$$y(t) = \mathcal{C}x(t,z), \tag{5c}$$

where S is a coercive operator on $L_2((a,b);\mathbb{R}^n)$ and the differential operators $\mathcal{J}$ and $\mathcal{G}_R$ are given by

$$\mathcal{J}x = \sum_{i=0}^{N} P_i \frac{\partial^i x}{\partial z^i}, \quad \mathcal{G}_R x = \sum_{i=0}^{N} G_i \frac{\partial^i x}{\partial z^i}, \quad \mathcal{G}_R^* x = \sum_{i=0}^{N} (-1)^i G_i^T \frac{\partial^i x}{\partial z^i}, \tag{6}$$

[1] See Definition A.3.71 of [4].

with G_i, P_i, $i = \{1, 2, \ldots, N\}$, constant matrices of size $n \times m$, and $n \times n$, respectively. Furthermore, these matrices satisfy

$$P_i = (-1)^{i+1} P_i^T, \quad i = 1, 2, \ldots, N, \quad \text{and} \quad \begin{bmatrix} P_N & G_N \\ G_N^T & 0 \end{bmatrix} \text{ has full rank.} \tag{7}$$

Here, $\mathcal{G}_R^*$ is the formal adjoint of $\mathcal{G}_R$. Note that the assumption imposed on the matrices P_i means that $\mathcal{J}$ is formally skew symmetric. Also observe that if $N = 1$ and $G_0 = 0$, we obtain the class of systems described in (1). Note that now the boundary operators $\mathcal{B}$ and $\mathcal{C}$ are linear operators from $\{x \in H^N((a, b); \mathbb{R}^n) \,|\, S\mathcal{G}_R^* x \in H^N((a, b); \mathbb{R}^n)\}$ to $\mathbb{R}^{2nN}$ and they only depend on the values at the positions $z = a$ and $z = b$ of $x(t, z)$ and its derivatives up to an order of $N - 1$.

In the next subsection we give the precise definition of what we mean by a boundary control system (BCS).

2.1 Boundary control systems (BCS)

The class of BCS described here are based on [4, §3.3]. That is, BCS of the form

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}\, x(t), \quad x(0) = x_0, \\ u(t) &= \mathfrak{B}\, x(t), \end{aligned} \tag{8}$$

where $\mathfrak{A} : D(\mathfrak{A}) \subset X \ \rightarrow X$, $u(t) \in U$, a separable Hilbert space, and the boundary operator $\mathfrak{B} : D(\mathfrak{B}) \subset X \ \rightarrow U$ satisfying $D(\mathfrak{A}) \subset D(\mathfrak{B})$, and

Definition 2.1. The control system (8) is a boundary control system if the following hold:

a. The operator $A : D(A) \ \rightarrow X$ with $D(A) = D(\mathfrak{A}) \cap \ker(\mathfrak{B})$ and

$$A\, x = \mathfrak{A}\, x \quad \text{for } x \in D(A)$$

is the generator of a C_0-semigroup on X.

b. There exists a $B \in \mathcal{L}(U, X)$ such that for all $u \in U$, $Bu \in D(\mathfrak{A})$, the operator $\mathfrak{A}B$ is an element of $\mathcal{L}(U, X)$ and $\mathfrak{B}Bu = u$ for $u \in U$.

2.2 Relation with skew-symmetric operators

In order to deal with systems of the form (5) we consider the operator

$$\mathcal{J}_e = \begin{bmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{bmatrix}, \tag{9}$$

where $\mathcal{J}$, $\mathcal{G}_R$, and $\mathcal{G}_R^*$ are given by (6)–(7). The reason of studying this operator will become clear in the next section. Meanwhile, notice that if we define

$$\begin{pmatrix} f \\ f_p \end{pmatrix} = \mathcal{J}_e \begin{pmatrix} e \\ e_p \end{pmatrix} = \begin{bmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{bmatrix} \begin{pmatrix} e \\ e_p \end{pmatrix}$$

and let $e_p = Sf_p$ with S a coercive operator on $L_2((a,b);\mathbb{R}^n)$, then we obtain

$$f = \mathcal{J}e - \mathcal{G}_R S \mathcal{G}_R^* e,$$

which is the same operator that appears in (5a). This same idea of feedback will be used to prove the main results in the next section. Next, we prove that the operator $\mathcal{J}_e$ is formally skew-symmetric, which means that we can use the results given in [1] and [3] to formulate BCS of the form (5).

Proposition 2.2. *The operator $\mathcal{J}_e$ defined by (9), (6), and (7) is formally skew-symmetric and can be written as:*

$$\mathcal{J}_e \begin{pmatrix} e \\ e_r \end{pmatrix} = \sum_{i=0}^{N} \begin{bmatrix} P_i & G_i \\ (-1)^{(i+1)} G_i^T & 0 \end{bmatrix} \frac{\partial^i}{\partial z^i} \begin{pmatrix} e \\ e_r \end{pmatrix}. \tag{10}$$

with

$$\widetilde{P}_i = \begin{bmatrix} P_i & G_i \\ (-1)^{(i+1)} G_i^T & 0 \end{bmatrix} = (-1)^{i+1} \begin{bmatrix} P_i & G_i \\ (-1)^{i+1} G_i^T & 0 \end{bmatrix}^T = (-1)^{i+1} \widetilde{P}_i^T. \tag{11}$$

Proof. That $\mathcal{J}_e$ is formally skew-symmetric follows from

$$\begin{aligned}
\langle \mathcal{J}_e x^1, x^2 \rangle &= \left\langle \begin{bmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{bmatrix} \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix}, \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} \right\rangle = \langle \mathcal{J} x_1^1 + \mathcal{G}_R x_2^1, x_1^2 \rangle + \langle -\mathcal{G}_R^* x_1^1, x_2^2 \rangle \\
&= \langle x_1^1, -\mathcal{J} x_1^2 \rangle + \langle x_2^1, \mathcal{G}_R^* x_1^2 \rangle + \langle x_1^1, -\mathcal{G}_R x_2^2 \rangle \\
&= \left\langle \begin{pmatrix} x_1^1 \\ x_2^1 \end{pmatrix}, - \begin{bmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{bmatrix} \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix} \right\rangle \\
&= \langle x^1, -\mathcal{J}_e x^2 \rangle .
\end{aligned}$$

Using (6) into (9) we can see that $\mathcal{J}_e$ can be rewritten as

$$\begin{aligned}
\mathcal{J}_e \begin{pmatrix} e \\ e_r \end{pmatrix} &= \sum_{i=0}^{N} \begin{bmatrix} P_i & G_i \\ -(-1)^i G_i^T & 0 \end{bmatrix} \frac{\partial^i}{\partial z^i} \begin{pmatrix} e \\ e_r \end{pmatrix} \\
&= \sum_{i=0}^{N} \begin{bmatrix} P_i & G_i \\ (-1)^{(i+1)} G_i^T & 0 \end{bmatrix} \frac{\partial^i}{\partial z^i} \begin{pmatrix} e \\ e_r \end{pmatrix}.
\end{aligned}$$

Equation (11) follows easily from (7). □

In [1], the authors have parameterized the boundary conditions for which a formally skew-symmetric operator generates a contraction semigroup. We will use their result to proved a similar result for our class of systems. In the remainder of this section we state some results, which are collected from [1].

Theorem 2.3. *Let $\mathcal{J}_e$ be a skew symmetric operator defined by (9), and let $H^N((a,b);\mathbb{R}^{2n})$ denote the Sobolev space of N times differentiable functions on the interval (a,b). Then for any* *two functions* $e_{e,i} = \binom{e_1}{e_r} \in H^N((a,b);\mathbb{R}^{2n})$, $i \in \{1,2\}$ *we have that*

$$\int_a^b e_{e,1}^T(z)(\mathcal{J}_e e_{e,2})(z) + e_{e,2}^T(z)(\mathcal{J}_e e_{e,1})(z)dz = \tag{12}$$

$$\left[\begin{pmatrix} e_{e,1}^T(z), & \cdots & \frac{d^{N-1}e_{e,1}^T}{dz^{N-1}}(z) \end{pmatrix} Q \begin{pmatrix} e_{e,2}(z) \\ \vdots \\ \frac{d^{N-1}e_{e,2}}{dz^{N-1}}(z) \end{pmatrix}\right]_a^b,$$

where

$$Q = \begin{pmatrix} \widetilde{P}_1 & \widetilde{P}_2 & \widetilde{P}_3 & \cdots & \widetilde{P}_{N-1} & \widetilde{P}_N \\ -\widetilde{P}_2 & -\widetilde{P}_3 & -\widetilde{P}_4 & \cdots & \widetilde{P}_N & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ (-1)^{N-1}\widetilde{P}_N & 0 & \cdots & \cdots & & 0 \end{pmatrix}, \tag{13}$$

with $\widetilde{P}_i$ *given by (11). Furthermore,* Q *is a nonsingular symmetric matrix.*

Definition 2.4. The matrix Q_{ext} in $\mathbb{R}^{4nN\times 4nN}$ associated with the differential operator $\mathcal{J}_e$ is defined by:

$$Q_{\text{ext}} = \begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix}. \tag{14}$$

Lemma 2.5. *The matrix* R_{ext} *defined as*

$$R_{\text{ext}} = \frac{1}{\sqrt{2}} \begin{bmatrix} Q & -Q \\ I & I \end{bmatrix} \tag{15}$$

is invertible, and satisfies

$$\begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix} = R_{\text{ext}}^T \Sigma R_{\text{ext}}, \tag{16}$$

where

$$\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{17}$$

All possible matrices R *which satisfies (16) are given by the formula*

$$R = U R_{\text{ext}},$$

with U *satisfying* $U^T \Sigma U = \Sigma$.

Definition 2.6. The boundary port variables associated with the differential operator $\mathcal{J}_e$ are the vectors $e_{e,\partial}, f_{e,\partial} \in \mathbb{R}^{2nN}$, defined by

$$\begin{pmatrix} f_{e,\partial} \\ e_{e,\partial} \end{pmatrix} = R_{\text{ext}} \begin{pmatrix} e(b) \\ e_r(b) \\ \vdots \\ \frac{d^{N-1}e}{dz^{N-1}}(b) \\ \frac{d^{N-1}e_r}{dz^{N-1}}(b) \\ e(a) \\ e_r(a) \\ \vdots \\ \frac{d^{N-1}e}{dz^{N-1}}(a) \\ \frac{d^{N-1}e_r}{dz^{N-1}}(a) \end{pmatrix}, \tag{18}$$

where R_{ext} is defined by (15).

Following Theorem 4.2 of [1] we immediately obtain the following result.

Theorem 2.7. *Let W be a $2nN \times 4nN$ matrix. If W has full rank and satisfies $W\Sigma W^T \geq 0$, where Σ is defined in (17), then the system*

$$\frac{\partial x}{\partial t}(t) = \mathcal{J}_e x(t)$$

with input

$$u(t) = W \begin{pmatrix} f_{e,\partial}(t) \\ e_{e,\partial}(t) \end{pmatrix}$$

is a boundary control system. Furthermore, the operator $A_{\text{ext}} = \mathcal{J}_e$ with domain

$$D(A_{\text{ext}}) = \left\{ \begin{pmatrix} e \\ e_r \end{pmatrix} \in \begin{pmatrix} H^N((a,b),\mathbb{R}^n) \\ H^N((a,b),\mathbb{R}^n) \end{pmatrix} \middle| \begin{pmatrix} f_{e,\partial} \\ e_{e,\partial} \end{pmatrix} \in \ker W \right\}, \tag{19}$$

generates a contraction semigroup.

Let $\tilde{W}$ be a full rank matrix of size $2nN \times 4nN$ with $\left(\begin{smallmatrix} W \\ \tilde{W} \end{smallmatrix}\right)$ invertible. If we define the linear mapping $\mathcal{C} : H^N((a,b),\mathbb{R}^{2n}) \to \mathbb{R}^{2nN}$ as,

$$\mathcal{C}x(t) := \tilde{W} \begin{pmatrix} f_{e,\partial}(t) \\ e_{e,\partial}(t) \end{pmatrix} \tag{20}$$

and the output as

$$y(t) = \mathcal{C}x(t), \tag{21}$$

then for $u \in C^2((0,\infty);\mathbb{R}^{2nN})$, $x(0) \in H^N((a,b),\mathbb{R}^{2n})$, and $\mathcal{B}x(0) = u(0)$ the following balance equation is satisfied:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = \frac{1}{2} \begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_{W,\tilde{W}} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, \tag{22}$$

where

$$P_{W,\tilde{W}}^{-1} = \begin{bmatrix} W\Sigma W^T & W\Sigma\tilde{W}^T \\ \tilde{W}\Sigma W^T & \tilde{W}\Sigma\tilde{W}^T \end{bmatrix}. \tag{23}$$

Furthermore, we have that the matrix $\left(\begin{smallmatrix} W\Sigma W^T & W\Sigma\tilde{W}^T \\ \tilde{W}\Sigma W^T & \tilde{W}\Sigma\tilde{W}^T \end{smallmatrix}\right)$ is invertible if and only if $\left(\begin{smallmatrix} W \\ \tilde{W} \end{smallmatrix}\right)$ is invertible.

3 Main results

The following lemma will turn out to be essential.

Lemma 3.1. *Let P be a nonnegative bounded operator and let M be the generator of a contraction semigroup. Then, $M - P$ also generates a contraction semigroup.*

Proof. We use Corollary 2.2 of [4], which states that a densely defined operator is the infinitesimal generator of a contraction semigroup if and only if $\langle J_W e, e\rangle \leq 0$ on $D(J_W)$ and $\langle J_W^* e, e\rangle \leq 0$ on $D(J_W^*)$.

Since M generates a contraction semigroup, we know that it satisfies these properties. Also, since P is bounded we get that $D(M-P) = D(M)$. Using this and the fact that P is nonnegative we obtain for any $x \in D(M-P)$

$$\langle (M-P)x, x\rangle = \langle Mx, x\rangle - \langle Px, x\rangle \leq \langle Mx, x\rangle \leq 0.$$

Using the same idea we can also prove that $\langle (M-P)^* x, x\rangle \leq 0$ on $D((M-P)^*) = D(M^*)$. □

Before stating the main result, first observe that if we define

$$\begin{pmatrix} f \\ f_p \end{pmatrix} = \mathcal{J}_e \begin{pmatrix} e \\ e_p \end{pmatrix} = \begin{bmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{bmatrix} \begin{pmatrix} e \\ e_p \end{pmatrix}$$

and let $e_p = S f_p$ with S a coercive operator on $L_2((a,b);\mathbb{R}^n)$, we obtain

$$f = \mathcal{J}e - \mathcal{G}_R S \mathcal{G}_R^* e,$$

which is the same operator that defines our class of systems. This idea of feedback will be used to prove the next theorem. In order to simplify the notation we introduce the following definition.

Definition 3.2. The boundary port variables associated with the differential operator $(\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)$ are the vectors $g_{f,\partial}, g_{e,\partial} \in \mathbb{R}^{2nN}$, defined by

$$\begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix} = R_{\text{ext}} \begin{pmatrix} e(b) \\ (-S\mathcal{G}_R^* e)(b) \\ \vdots \\ \frac{d^{N-1}e}{dz^{N-1}}(b) \\ \frac{d^{N-1}(-S\mathcal{G}_R^* e)}{dz^{N-1}}(b) \\ e(a) \\ (-S\mathcal{G}_R^* e)(a) \\ \vdots \\ \frac{d^{N-1}(-S\mathcal{G}_R^* e)}{dz^{N-1}}(a) \end{pmatrix}, \tag{24}$$

where R_{ext} is defined by (15).

Remark 3.3. Observe that (24) is the same as (18) whenever $e_e = \begin{pmatrix} e \\ -S\mathcal{G}_R^* e \end{pmatrix}$.

Theorem 3.4. *Consider the operator $A = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)$ with domain*

$$D(A) = \left\{ e \in H^N((a,b);\mathbb{R}^n) \,\middle|\, S\mathcal{G}_R^* e \in H^N((a,b);\mathbb{R}^n), \begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix} \in \ker W \right\}. \tag{25}$$

If W has full rank and satisfies $W \Sigma W^T \geq 0$, then A generates a contraction semigroup.

Proof. As mentioned before, the proof is based on a feedback argument on the operator $\mathcal{J}_e$. First observe that since A_{ext} is the generator of a contraction semigroup (see Theorem 2.7) we have from the Lümer-Phillips theorem (see [2, Theorem 2.27]) that

$$\langle A_{\text{ext}}\tilde{e}, \tilde{e}\rangle \leq 0 \quad \text{for all } \tilde{e} \in D(A_{\text{ext}}) \text{ and} \tag{26}$$

$$\text{ran}\,(\lambda I - A_{\text{ext}}) = L_2((a,b);\mathbb{R}^{2n}) \quad \text{for some } \lambda > 0. \tag{27}$$

Now we can proceed to prove that A generates a contraction semigroup. To do so, we will use the same Lümer-Phillips theorem. That is, we first prove that A satisfies $\langle Ae, e\rangle \leq 0$ for any $e \in D(A)$ and next that $\text{ran}\,(\lambda I - A) = L_2((a,b);\mathbb{R}^n)$ for some $\lambda > 0$. For $e \in D(A)$, we have

$$\langle Ae, e\rangle = \langle (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)e, e\rangle = \langle \mathcal{J}, e\rangle + \langle -\mathcal{G}_R S \mathcal{G}_R^* e, e\rangle\,.$$

Define $e_p = -S\mathcal{G}_R^* e$ and observe that $e_p \in H^N((a,b);\mathbb{R}^n)$, see (25). It is now easy to see that $\left(\begin{smallmatrix} e \\ e_p \end{smallmatrix}\right) \in D(A_{\text{ext}})$, see Remark 3.3 and (19). From this and the equation above we can see that

$$\begin{aligned}
\langle Ae, e\rangle &= \langle \mathcal{J}e + \mathcal{G}_R e_p, e\rangle \\
&\leq \langle \mathcal{J}e + \mathcal{G}_R e_p, e\rangle + \langle \mathcal{G}_R^* e, S\mathcal{G}_R^* e\rangle \\
&= \langle \mathcal{J}e + \mathcal{G}_R e_p, e\rangle + \langle \mathcal{G}_R^* e, -e_p\rangle \\
&= \left\langle \begin{bmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{bmatrix} \begin{pmatrix} e \\ e_p \end{pmatrix}, \begin{pmatrix} e \\ e_p \end{pmatrix} \right\rangle \\
&= \left\langle A_{\text{ext}} \begin{pmatrix} e \\ e_p \end{pmatrix}, \begin{pmatrix} e \\ e_p \end{pmatrix} \right\rangle \leq 0,
\end{aligned}$$

where in the second step we used the fact that S is coercive, see (4), in the third step we used $e_p = -S\mathcal{G}_R^* e$, and in the last step we used (26).

Next we prove the range condition on A. That is, for a $\lambda > 0$ we have to prove that for any given $f \in L_2((a,b);\mathbb{R}^n)$ we can find an $e \in D(A)$ such that

$$f = (\lambda I - A)e.$$

In order to prove this, let

$$P = \begin{bmatrix} 0 & 0 \\ 0 & S^{-1} - \lambda I \end{bmatrix}.$$

Since S is coercive, we can find some $\lambda > 0$ such that $S^{-1} - \lambda I \geq 0$. Thus we can assume that P is a nonnegative operator. It thus follows from Lemma 3.1 that $A_{\text{ext}} - P$ generates a contraction semigroup. This in turn implies that $\text{ran}\,(\lambda I - A_{\text{ext}} + P) = L_2((a,b);\mathbb{R}^{2n})$. Thus, given any $\left(\begin{smallmatrix} f \\ 0 \end{smallmatrix}\right) \in L_2((a,b);\mathbb{R}^{2n})$ we can find $\left(\begin{smallmatrix} e \\ e_p \end{smallmatrix}\right) \in D(A_{\text{ext}})$ such that

$$\begin{aligned}
&\begin{pmatrix} f \\ 0 \end{pmatrix} = (\lambda I - A_{\text{ext}} + P)\begin{pmatrix} e \\ e_p \end{pmatrix} = \begin{bmatrix} \lambda I - \mathcal{J} & -\mathcal{G}_R \\ \mathcal{G}_R^* & S^{-1} \end{bmatrix} \begin{pmatrix} e \\ e_p \end{pmatrix} \\
\Rightarrow\quad & f = (\lambda I - \mathcal{J})e - \mathcal{G}_R e_p \quad \text{and} \\
& e_p = -S\mathcal{G}_R^* e \\
\Rightarrow\quad & f = [\lambda I - (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)]e.
\end{aligned} \tag{28}$$

Since $\left(\begin{smallmatrix} e \\ e_p \end{smallmatrix}\right) = \left(\begin{smallmatrix} e \\ -S\mathcal{G}_R^* e \end{smallmatrix}\right) \in D(A_{\text{ext}})$, it is easy to see that $e \in D(A)$. Then from (28) we can see that A satisfies the range condition. Concluding, we see that A generates a contraction semigroup. □

Following Theorem 2.7 and Theorem 3.4 we can prove the following result.

Theorem 3.5. *Let W be a $2nN \times 4nN$ matrix. If W has full rank and satisfies $W\Sigma W^T \geq 0$, where Σ is defined in (17), then the system*

$$\frac{\partial x}{\partial t}(t) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)x(t) \tag{29}$$

with input

$$u(t) = \mathcal{B}x(t) = W \begin{pmatrix} g_{f,\partial}(t) \\ g_{e,\partial}(t) \end{pmatrix} \tag{30}$$

is a boundary control system. Furthermore, the operator $A = \mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*$ with domain

$$D(A) = \left\{ e \in H^N((a,b);\mathbb{R}^n) \,\middle|\, S\mathcal{G}_R^* e \in H^N((a,b);\mathbb{R}^n), \begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix} \in \ker W \right\}. \tag{31}$$

generates a contraction semigroup.

Let $\tilde{W}$ be a full rank matrix of size $2nN \times 4nN$ with $\left(\begin{smallmatrix} W \\ \tilde{W} \end{smallmatrix}\right)$ invertible. If we define the linear mapping $\mathcal{C} : H^N((a,b),\mathbb{R}^{2n}) \to \mathbb{R}^{2nN}$ as,

$$\mathcal{C}x(t) := \tilde{W} \begin{pmatrix} g_{f,\partial}(t) \\ g_{e,\partial}(t) \end{pmatrix} \tag{32}$$

and the output as

$$y(t) = \mathcal{C}x(t), \tag{33}$$

then for $u \in C^2((0,\infty);\mathbb{R}^{2nN})$, $x(0) \in H^N((a,b),\mathbb{R}^{2n})$, and $\mathcal{B}x(0) = u(0)$ the following balance equation is satisfied:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 \leq \frac{1}{2} \begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_{W,\tilde{W}} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, \tag{34}$$

where

$$P_{W,\tilde{W}}^{-1} = \begin{bmatrix} W\Sigma W^T & W\Sigma \tilde{W}^T \\ \tilde{W}\Sigma W^T & \tilde{W}\Sigma\tilde{W}^T \end{bmatrix}. \tag{35}$$

Furthermore, we have that the matrix $\left(\begin{smallmatrix} W\Sigma W^T & W\Sigma\tilde{W}^T \\ \tilde{W}\Sigma W^T & \tilde{W}\Sigma\tilde{W}^T \end{smallmatrix}\right)$ is invertible if and only if $\left(\begin{smallmatrix} W \\ \tilde{W} \end{smallmatrix}\right)$ is invertible.

Proof. We divide the proof in three steps. In Step 1. and 2. we show that we have a boundary control system. In step 3. we prove (34) and (35), respectively. For a boundary control system we have to show that for zero inputs, the operator A generates a C_0-semigroup, and furthermore that there exists a bounded operator B mapping into the domain of $\mathcal{B}$ and such that $\mathcal{B}Bu = u$ for all $u \in \mathbb{R}^{2nN}$.

Step 1: As mentioned above, we have to show that $A = \mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*$ with domain (31) is an infinitesimal generator of a semigroup. This follows directly from Theorem 3.4.

Step 2: We have to find a bounded linear operator B such that $Bu \in D(\mathcal{B}) = H^N((a,b);\mathbb{R}^{nN})$ and $\mathcal{B}Bu = u$ for all $u \in \mathbb{R}^{2nN}$. This follows similarly as the second step in the proof of Theorem 4.5 of [3].

Step 3: By the definition of B and $D(A)$, we see that the conditions stated in the theorem are the same as $x(0) - Bu(0) \in D(A)$. Hence by Theorem 3.3.3 of [4] we have that there exists a classical solution of (29)–(30). Hence, in particular, $x(t) \in H^N((a,b), \mathbb{R}^n)$ holds pointwise in t, $x(t)$ is differentiable as a function of t, and $\dot{x}(t) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)x(t)$. Using this, we obtain

$$\begin{aligned}
\frac{d}{dt}\|x(t)\|^2 =& \frac{d}{dt}\langle x(t), x(t)\rangle \\
=& \langle \dot{x}(t), x(t)\rangle + \langle x(t), \dot{x}(t)\rangle \\
=& \langle (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)x(t), x(t)\rangle + \langle x(t), (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)x(t)\rangle
\end{aligned}$$

Define $x_p = -S\mathcal{G}_R^* x$ and observe that $x_p \in H^N((a,b); \mathbb{R}^n)$, see (31). It is now easy to see that $\left(\begin{smallmatrix} x \\ x_p \end{smallmatrix}\right) \in D(A_{\text{ext}})$, see Remark 3.3. From this and the equation above we can see that

$$\begin{aligned}
\frac{d}{dt}\|x(t)\|^2 =& \langle \mathcal{J}x(t) + \mathcal{G}_R x_p(t), x(t)\rangle + \langle x(t), \mathcal{J}x(t) + \mathcal{G}_R x_p(t)\rangle \\
\leq& \langle \mathcal{J}x(t) + \mathcal{G}_R x_p(t), x(t)\rangle + \langle x(t), \mathcal{J}x(t) + \mathcal{G}_R x_p(t)\rangle \\
&+ \langle \mathcal{G}_R^* x(t), S\mathcal{G}_R^* x(t)\rangle + \langle S\mathcal{G}_R^* x(t), \mathcal{G}_R^* x(t)\rangle \\
=& \langle \mathcal{J}x(t) + \mathcal{G}_R x_p(t), x(t)\rangle + \langle x(t), \mathcal{J}x(t) + \mathcal{G}_R x_p(t)\rangle \\
&+ \langle \mathcal{G}_R^* x(t), -x_p(t)\rangle + \langle -x_p(t), \mathcal{G}_R^* x(t)\rangle \\
=& \left\langle \begin{bmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{bmatrix} \begin{pmatrix} x(t) \\ x_p(t) \end{pmatrix}, \begin{pmatrix} x(t) \\ x_p(t) \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} x(t) \\ x_p(t) \end{pmatrix}, \begin{bmatrix} \mathcal{J} & \mathcal{G}_R \\ -\mathcal{G}_R^* & 0 \end{bmatrix} \begin{pmatrix} x(t) \\ x_p(t) \end{pmatrix} \right\rangle \\
=& \left\langle \frac{d}{dt} \begin{pmatrix} x(t) \\ x_p(t) \end{pmatrix}, \begin{pmatrix} x(t) \\ x_p(t) \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} x(t) \\ x_p(t) \end{pmatrix}, \frac{d}{dt} \begin{pmatrix} x(t) \\ x_p(t) \end{pmatrix} \right\rangle \\
=& \frac{d}{dt} \left\| \begin{pmatrix} x(t) \\ x_p(t) \end{pmatrix} \right\|^2,
\end{aligned}$$

where we used the fact that S is coercive, see (4), and $x_p = -S\mathcal{G}_R^* x$. The rest of the proof follows from Equations (22) and (23) and the fact that $\left(\begin{smallmatrix} x \\ x_p \end{smallmatrix}\right) \in D(A_{\text{ext}})$. □

Remark 3.6. Following Section 5 of [1] we can easily see that Theorem 3.5 also holds if we replace the operator $(\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)$ by $(\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*)\mathcal{L}$, where $\mathcal{L}$ is a coercive operator on $L_2((a,b); \mathbb{R}^n)$. This allows to deal with systems with different parameters or even systems with nonconstant parameters.

Example 3.7. This is a simple example of a fixed bed reactor, see [5]. The main phenomena which takes place into the reactor are the diffusion and the convection. The resulting PDE is

$$\frac{\partial C}{\partial t}(t,z) = D\frac{\partial^2 C}{\partial z^2}(t,z) - U\frac{\partial C}{\partial z}(t,z), \tag{36}$$

where $U > 0$ is the velocity of the fluid and $D > 0$ the diffusion constant. Comparing the equation above with (5a), we can easily see that in this case we have

$$\mathcal{J} = -U\frac{\partial}{\partial z}, \tag{37}$$

$$-\mathcal{G}_R S \mathcal{G}_R^* = D\frac{\partial^2}{\partial z^2}. \tag{38}$$

From this we get

$$\mathcal{G}_R = \frac{\partial}{\partial z}, \quad S = D, \quad \text{and} \quad \mathcal{G}_R^* = -\frac{\partial}{\partial z}, \tag{39}$$

and thus (see equations (6) and (11)) $N = 1$,

$$P_1 = -U,\ G_1 = 1,\ P_0 = G_0 = 0,\ \text{and}\ \widetilde{P}_1 = \begin{bmatrix} -U & 1 \\ 1 & 0 \end{bmatrix} = Q.$$

Recall that $\mathcal{G}_R^*$ is the formal adjoint of $\mathcal{G}_R$, i.e. the adjoint of $\mathcal{G}_R$ ignoring boundary variables. Then it is easy to see that equation (36) becomes

$$\frac{\partial C}{\partial t}(t,z) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*) C(t,z). \tag{40}$$

From Definition 3.2 and using (15) we obtain the boundary port variables

$$\begin{aligned} \begin{pmatrix} g_{f,\partial} \\ g_{e,\partial} \end{pmatrix} &= \frac{1}{\sqrt{2}} \begin{pmatrix} -U & 1 & U & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} C(b) \\ D\frac{\partial C}{\partial z}(b) \\ C(a) \\ D\frac{\partial C}{\partial z}(a) \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} -U(C(b) - C(a)) + D(\frac{\partial C}{\partial z}(b) - \frac{\partial C}{\partial z}(a)) \\ C(b) - C(a) \\ C(b) + C(a) \\ D(\frac{\partial C}{\partial z}(b) + \frac{\partial C}{\partial z}(a)) \end{pmatrix} \end{aligned} \tag{41}$$

Typically, the boundary conditions are chosen as a linear combination of $UC - D\frac{\partial C}{\partial z}$. Say, we want

$$D\frac{\partial C}{\partial z}(t,a) - UC(t,a) = UC_{\text{in}}(t), \quad \text{and} \quad D\frac{\partial C}{\partial z}(t,b) = 0, \tag{42}$$

where C_{in} is an input function. It is easy to see that these boundary conditions can be obtained from the port variables by premultiplying them by the following matrix

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & -U & 1 \\ 1 & U & 0 & 1 \end{pmatrix}.$$

Since this matrix satisfies $W\Sigma W^T = \left[\begin{smallmatrix} U & 0 \\ 0 & U \end{smallmatrix}\right] \geq 0$, we have that the results in this section apply to this system.

References

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