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H^∞ Control of Systems with a Single Delay via Reduction to a One-Block Problem[1]

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Abstract

In this paper the standard (4-block) H^∞ control problem for systems with a single loop delay is studied. A simple procedure of the reduction of the problem to an equivalent one-block problem having particularly simple structure is proposed. The one-block problem is then solved by the J-spectral factorization approach, resulting in the so-called dead-time compensator (DTC) form of the controller. The advantages of the proposed procedure are its simplicity, intuitively clear derivation of the DTC form of the H^∞ controller, and extensibility to the multiple delay case.

1 Introduction and problem formulation

Consider the dead-time system in Fig. 1, where $P(s)$ is a finite-dimensional generalized plant with the transfer matrix

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right],$$

e^{-sh} is the loop delay with the dead-time $h > 0$, and $K_h(s)$ is a proper part of the controller to be designed. The problem to be studied in this paper is as follows:

OP_h: Given the plant $P(s)$ and the dead time h, determine whether there exists a proper $K_h(s)$, which internally stabilizes the system in Fig. 1 and guarantees

$$\|\mathcal{F}_\ell(P, e^{-sh}K_h)\|_\infty < \gamma$$

for a given γ, and then characterize all such K_h when one exists.

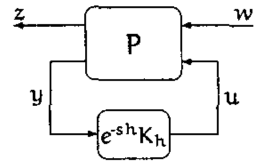

Figure 1: The "standard problem" for DT systems

Here $\mathcal{F}_\ell(G, U) \doteq G_{11} + G_{12}U(I - G_{22}U)^{-1}G_{21}$ stands for the lower linear fractional transformation of U over G, see [1].

H^∞ control of DT systems has been an active research area since mid 80's. Early frequency response methods, see [2] and the references therein, treated DT systems in the framework of the general infinite-dimensional control theory. This resulted in rather cumbersome solutions, for which implementation and analysis issues appear to be quite complicated. This fact motivated more problem-oriented approaches, exploiting the structure of DT systems [3–10], see also the review paper [11] for additional references.

In the late 90's it was shown [8, 12] that suboptimal H^∞ controllers can be presented in the so-called dead-time compensator (DTC) form, i.e., in the form of the feedback interconnection of a finite-dimensional part and an infinite-dimensional "prediction" block reminiscent of the Smith predictor. The J-spectral factorization approach used in [8, 12] produces the DTC form of the controller in an intuitively clear fashion, though the presence of several intermediate steps blurs the final formulae and the relationship with the delay-free problem.

The further simplifications were proposed in [10], where the problem is addressed by the extraction of the dead-time controllers from the known parameterization of the delay-free H^∞ controllers. This reduces the four-block problem to a Nehari problem which, in turn, is solved using the results of [13]. The original controller is then recovered in the DTC form as well. The advantage of the result of [10] lies in the transparency and "interpretability" of the resulting controller. Yet the controller recovery there is far from being intuitive. This practically prevents the extension of the approach to multiple delay problems.

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The purpose of this paper is to amalgamate the approaches of [8] and [10]. As in the latter reference the solution is based on the extraction of the dead-time controllers from the delay-free parameterization. Yet at this stage the problem is reduced not to a Nehari, but rather to a *one-block* problem, which turns out to possess some nice properties making it particularly suitable for the application of the J-spectral factorization ideas of [8]. This approach allows one to bypass the complicated math needed in the previous approaches and results in probably the simplest solution to date.

Notation: The notations used in this paper are fairly standard. Given a matrix M, M' denotes its transpose and $M^{-'}$ stands for $(M')^{-1}$ when the inverse exists. Given a transfer matrix $G(s)$, its conjugate is defined as $G^{\sim}(s) = G'(-s)$ and $\|G(s)\|_\infty$ denotes its H^∞ norm (with a slight abuse of notation, it is assumed throughout the paper that $\|G(s)\|_\infty = \infty$ whenever $G(s) \notin H^\infty$). By $\mathcal{C}_r(G,U) \doteq (G_{12} + G_{11}U)(G_{22} + G_{21}U)^{-1}$ we denote the chain-scattering (Möbius or homographic) linear fractional transformation.

For a given $G(s) = C(sI - A)^{-1}B$ the *h-completion* operator $\pi_h\{e^{-sh}G\}$ introduced in [10] is defined as

$$\pi_h\{e^{-sh}G\} = \hat{G} - e^{-sh}G \doteq \left[\begin{array}{c|c} A & B \\ \hline Ce^{-Ah} & 0 \end{array}\right] - e^{-sh}\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

It can be verified that $\pi_h\{e^{-sh}G\}$ is an entire function of s with the impulse response having support in $[0, h]$ (FIR system).

2 Reduction to one-block problem

2.1 Solution to the delay-free problem

We start with a brief review of the now classical results on the solvability of the delay-free H^∞ standard problem, i.e., OP_0. To this end, let us impose the following assumptions on the state-space realization of P:

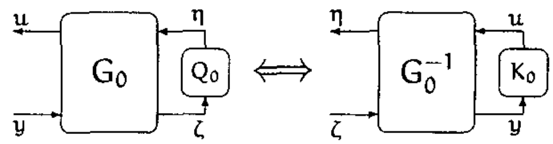
(A1): (C_2, A, B_2) is stabilizable and detectable;

(A2): $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ and $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ have full column and row rank, respectively, $\forall \omega \in \mathbb{R}$;

(A3): $D_{12}'[C_1 \;\; D_{12}] = [0 \;\; I]$ and $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}' = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

Introduce also the following two H^∞ algebraic Riccati equations:

$$XA + A'X + C_1'C_1 - XB_2B_2'X + \gamma^{-2}XB_1B_1'X = 0 \quad (1)$$



(a) K_0 as a function of Q_0 (b) Q_0 as a function of K_0

Figure 2: All admissible controllers for OP_0

and

$$AY + YA' + B_1B_1' - YC_2'C_2Y + \gamma^{-2}YC_1'C_1Y = 0. \quad (2)$$

The solutions to Riccati equations (1) and (2) are said to be stabilizing if the matrices $A_F \doteq A + \gamma^{-2}B_1B_1'X - B_2B_2'X$ and $A_L \doteq A + \gamma^{-2}YC_1'C_1 - YC_2'C_2$, respectively, are Hurwitz. Then [1] OP_0 is solvable iff

(a) there exists a stabilizing solution $X = X' \geq 0$ to ARE (1);

(b) there exists a stabilizing solution $Y = Y' \geq 0$ to ARE (2);

(c) $\rho(XY) < \gamma$.

Furthermore, if these conditions hold, then the transfer matrix

$$G_0(s) = \left[\begin{array}{c|cc} A_F & ZB_2 & ZYC_2' \\ \hline -B_2'X & I & 0 \\ C_2 & 0 & I \end{array}\right], \quad (3)$$

where $Z \doteq (I - \gamma^{-2}YX)^{-1}$, is well defined and the set of all admissible controllers is parametrized as [14]

$$K_0 = \mathcal{C}_r(G_0, Q_0), \quad (4)$$

where Q_0 must satisfy $\|Q_0\|_\infty < \gamma$ but otherwise arbitrary.

Remark 1 *Note that by construction the matrix* A_F *in* (3) *is Hurwitz. Moreover, the "A" matrix of* G_0^{-1} *is*

$$A_F + ZB_2B_2'X - ZYC_2'C_2 = ZA_LZ^{-1}, \quad (5)$$

which is Hurwitz as well. Hence, G_0 *given by* (3) *is bistable.*

2.2 From standard problem to one-block problem

The parameterization (4) of all admissible controllers can be visualized as shown in Fig. 2(a). The key property of the mapping $Q_0 \mapsto K_0$ for G_0 given by (3) is that it is an isomorphism, so that $K_0 = \mathcal{C}_r(G_0, Q_0) \iff Q_0 = \mathcal{C}_r(G_0^{-1}, K_0)$, see Fig. 2(b). It then follows that provided conditions *(a)-(c)* above hold, a controller K_0 solves OP_0 iff $\|\mathcal{C}_r(G_0^{-1}, K_0)\|_\infty < \gamma$.

On the other hand, the delay can be thought of as just an additional restriction imposed upon the controller K_0. This means that (for any $h > 0$) OP_h is solvable only if so is OP_0. Therefore, combining the parameterization of all solutions to OP_0 with the transformation in Fig. 2 the following result can formulated:

Lemma 1 *OP_h is solvable iff so is its delay-free counterpart OP_0 and, in addition, $\|\mathcal{C}_r(G_0^{-1}, e^{-sh}K_h)\|_\infty < \gamma$.*

Lemma 1 actually implies that OP_h can be converted to the following equivalent problem:

OP_{eq}:Given the bistable system $G_0(s)$ with the state-space realization (3) and the dead time h, determine whether there exists a proper $K_h(s)$, which guarantees

$$\|\mathcal{C}_r(G_0^{-1}, e^{-sh}K_h)\|_\infty < \gamma$$

for a given γ, and then characterize all such K_h when one exists.

Note that G_0^{-1} partitioned according to the signal partition in Fig. 2(b) has "square" $(1,1)$ and $(2,2)$ blocks. Hence OP_{eq} falls into the class of the so-called *one-block* problems, the solution to which is simpler than that to OP_h. In other words, Lemma 1 reduced the general (four-block) problem OP_h to a simpler one-block problem OP_{eq}. Moreover, only IO (rather than internal) stability is required for the system in Fig. 2(b), which may simplify the analysis.

Remark 2 *It is worth stressing that the reasoning above applies to any constrained version of the standard problem. Thus, any four-block problem with some constraints imposed on the controller (e.g., multiple delay problems) can be reduced to a one-block problem in a simple and intuitive way.*

3 Solution to the one-block problem

3.1 The main results

We start with the formulation of the solution to OP_{eq}. Toward this end the following symplectic matrix function is required:

$$\Sigma(t) = \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{bmatrix} \doteq \exp\left(\begin{bmatrix} A_F + ZB_2B_2'X & \frac{1}{\gamma^2}ZYC_2C_2'YZ' \\ -XB_2B_2'X & -A_F' - XB_2B_2'Z' \end{bmatrix} t\right). \quad (6)$$

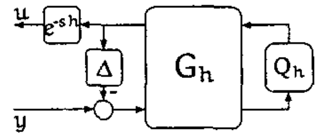


Figure 3: All admissible controllers for OP_h

For the sake of simplicity, hereafter we use Σ to mean $\Sigma(h)$. Introduce also the quantity

$$\gamma_h \doteq \left\| \left[\begin{array}{c|c} A_F + ZB_2B_2'X & ZYC_2' \\ \hline B_2'X & 0 \end{array}\right] \right\|_{L^2[0,h]}. \quad (7)$$

This is the $L^2[0,h]$-induced norm of an LTI system—a notion extensively studied in the delay and sampled-data literature, see [2,15] and the references therein. It is well known [15] that $\gamma > \gamma_h$ iff $\Sigma_{22}(t)$ is nonsingular for all $t \in [0,h]$.

We are now in the position to formulate our main result:

Theorem 1 *OP_{eq} is solvable iff $\gamma > \gamma_h$. In that case all solutions K_h to the OP_{eq} are given by*

$$K_h = \mathcal{C}_r\left(\left[\begin{smallmatrix} I & 0 \\ \Delta & I \end{smallmatrix}\right] G_h, Q_h\right) \quad (8)$$

(see Fig. 3), where

$$\Delta = \pi_h \left\{ \left[\begin{array}{cc|c} A_F + ZB_2B_2'X & \frac{1}{\gamma^2}ZYC_2C_2'YZ' & ZB_2 \\ -XB_2B_2'X & -A_F' - XB_2B_2'Z' & -XB_2 \\ \hline -C_2 & -\frac{1}{\gamma^2}C_2YZ' & 0 \end{array}\right] \right\},$$

$$G_h = \left[\begin{array}{c|cc} A_F & (Z + \Sigma_{12}\Sigma_{22}^{-1}X)B_2 & \Sigma_{22}^{-\prime}ZYC_2' \\ \hline -B_2'X & I & 0 \\ C_2(\Sigma_{22}' - \frac{1}{\gamma^2}YZ'\Sigma_{21}') & 0 & I \end{array}\right]$$

and Q_h must satisfy $\|Q_h\|_\infty < \gamma$ but otherwise arbitrary.

Having this result, the solution to OP_h can now be formulated as follows:

Corollary 1 *OP_h is solvable iff so is OP_0 and also $\gamma > \gamma_h$. In that case all solutions K_h to the OP_h are given by* (8).

Remark 3 *The formulae for $\Sigma(t)$ and Δ could be further cleaned up as shown in [10]. The reader could also find there the more conventional LFT form of parametrization* (8).

The rest of this section is devoted to the proof of Theorem 1. In §3.2 the main ideas of the proof are outlined, then, in §3.3, some technical machinery to be used in the sequel is introduced, in §3.4 we derive the necessary conditions for solvability of OP_{eq}, §3.5 is devoted to the construction of Δ and G_h, and, finally, in §3.6 we prove the validity of the formulae.

3.2 Proof outline

In the proof of Theorem 1 we use the J-spectral factorization approach. Let $J_\gamma \doteq \left[\begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix}\right]$. We are looking for a bistable W_α so that

$$\Pi_\alpha \doteq G_\alpha^\sim J_\gamma G_\alpha = W_\alpha^\sim J_\gamma W_\alpha, \quad G_\alpha \doteq G_0^{-1}\left[\begin{smallmatrix} e^{-sh} I & 0 \\ 0 & I \end{smallmatrix}\right],$$

and $G_\alpha W_\alpha^{-1}$ is J_γ-lossless, see [14, 16] for the definitions. If G_α were finite dimensional, then the existence of such a W_α would be necessary and sufficient for the solvability of $\mathbf{OP_{eq}}$ and the set of all solutions would be parameterized by $K_h = \mathcal{C}_r(W_\alpha^{-1}, Q_h)$ with $\|Q_h\|_\infty < \gamma$. Yet G_α is infinite dimensional. This complicates both the construction of W_α and the proof that the factorization above does yield the solution to $\mathbf{OP_{eq}}$. To circumvent this obstacle the approach of [8] is used. The idea is to exploit the special structure of Π_α and use it to remove the infinite-dimensional part from the factorization. To this end, note that the infinite-dimensional part of Π_α only enters the off-diagonal blocks,

$$\Pi_\alpha = \begin{bmatrix} \Pi_{11} & e^{sh}\Pi_{12} \\ e^{-sh}\Pi_{21} & \Pi_{22} \end{bmatrix}.$$

Here Π_{ij} are the subblocks of the (finite-dimensional) transfer matrix $\Pi \doteq (G_0^{-1})^\sim J_\gamma G_0^{-1}$. Also note that the J_γ-spectral factorization of Π_α can be reduced to that of

$$\Pi_\beta \doteq \left[\begin{smallmatrix} I & \Delta^\sim \\ 0 & I \end{smallmatrix}\right]\Pi_\alpha\left[\begin{smallmatrix} I & 0 \\ \Delta & I \end{smallmatrix}\right] \tag{9}$$

provided $\Delta \in H^\infty$. Indeed, one can see that W_α is a bistable J_γ-spectral factor of Π_α iff

$$W_\beta \doteq W_\alpha\left[\begin{smallmatrix} I & 0 \\ \Delta & I \end{smallmatrix}\right]$$

is a bistable J_γ-spectral factor of Π_β. The idea then is to choose Δ so as to make Π_β finite dimensional. It is easy to verify that

$$\Pi_\beta = \begin{bmatrix} \Pi_{11} - \Pi_{12}\Pi_{22}^{-1}\Pi_{21} + R^\sim\Pi_{22}R & R^\sim\Pi_{22} \\ \Pi_{22}R & \Pi_{22} \end{bmatrix}$$

for $R \doteq \Delta + e^{-sh}\Pi_{22}^{-1}\Pi_{21}$. This R is finite dimensional if we choose Δ to be the stable FIR system

$$\Delta = \pi_h\{e^{-sh}\Pi_{22}^{-1}\Pi_{21}\}$$

(incidentally, Π_{22} is invertible because of the structure of G_0). In §3.5 we show that this choice yields the Δ of Theorem 1 and that G_h defined in Theorem 1 equals $G_h = W_\beta^{-1}$ where W_β is a finite dimensional J_γ-spectral factor of Π_β.

Typically it is the existence of such $G_h = W_\beta^{-1}$ that forms the bottleneck of the proof. Here, however, we bypass this difficulty by first showing that γ must exceed γ_h if $\mathbf{OP_{eq}}$ is to have a solution, see §3.4. Therefore $\gamma > \gamma_h$ and this guarantees invertibility of Σ_{22} and, hence, existence of G_h (see Theorem 1). With G_h known to exist the rest of the proof follows fairly standard arguments. Continuity is used to show that $G_\alpha W_\alpha^{-1}$ is J_γ-lossless. Finally, as in the finite dimensional case, all solutions K_h are shown to have the form $K_h = \mathcal{C}_r(W_\alpha^{-1}, Q_h) = \mathcal{C}_r(\left[\begin{smallmatrix} I & 0 \\ \Delta & I \end{smallmatrix}\right]W_\beta^{-1}, Q_h) = \mathcal{C}_r(\left[\begin{smallmatrix} I & 0 \\ \Delta & I \end{smallmatrix}\right]G_h, Q_h)$ with $\|Q\|_\infty < \gamma$.

3.3 Preliminary: $\mathcal{S}$-transformations

Throughout this section we will extensively use the "Schur complementation" transformations $\mathcal{S}_u(O)$ and $\mathcal{S}_\ell(O)$, which are defined for a 2×2 block operator O as follows:

$$\mathcal{S}_u(O) \doteq \begin{bmatrix} O_{11}^{-1} & -O_{11}^{-1}O_{12} \\ O_{21}O_{11}^{-1} & O_{22} - O_{21}O_{11}^{-1}O_{12} \end{bmatrix},$$

$$\mathcal{S}_\ell(O) \doteq \begin{bmatrix} O_{11} - O_{12}O_{22}^{-1}O_{21} & O_{12}O_{22}^{-1} \\ -O_{22}^{-1}O_{21} & O_{22}^{-1} \end{bmatrix}.$$

In the sequel we call these transformations the upper and lower $\mathcal{S}$-transformation, respectively. It is clear that the upper (lower) $\mathcal{S}$-transformation is well-defined iff the upper left (lower right) subblock of O is nonsingular. $\mathcal{S}$-transformations can be thought of as the "swapping" of parts of the inputs and outputs, namely

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = O\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \iff \begin{cases} \begin{bmatrix} \eta_1 \\ \zeta_2 \end{bmatrix} = \mathcal{S}_u(O)\begin{bmatrix} \zeta_1 \\ \eta_2 \end{bmatrix} \\ \begin{bmatrix} \zeta_1 \\ \eta_2 \end{bmatrix} = \mathcal{S}_\ell(O)\begin{bmatrix} \eta_1 \\ \zeta_2 \end{bmatrix} \end{cases}$$

(provided the mappings are well-defined). The relations above prompt an elegant way to perform $\mathcal{S}$-transformations for systems given by their state-space realizations. Indeed, if

$$\Phi(s) = \left[\begin{array}{c|cc} A_\phi & B_{\phi1} & B_{\phi2} \\ \hline C_{\phi1} & I & 0 \\ C_{\phi2} & 0 & \frac{1}{\kappa}I \end{array}\right],$$

then the straightforward flow-tracing yields

$$\mathcal{S}_u(\Phi(s)) = \left[\begin{array}{c|cc} A_\phi - B_{\phi1}C_{\phi1} & B_{\phi1} & B_{\phi2} \\ \hline -C_{\phi1} & I & 0 \\ C_{\phi2} & 0 & \frac{1}{\kappa}I \end{array}\right], \tag{10a}$$

$$\mathcal{S}_\ell(\Phi(s)) = \left[\begin{array}{c|cc} A_\phi - \kappa B_{\phi2}C_{\phi2} & B_{\phi1} & \kappa B_{\phi2} \\ \hline C_{\phi1} & I & 0 \\ -\kappa C_{\phi2} & 0 & \kappa I \end{array}\right]. \tag{10b}$$

The signal swapping interpretation implies also the following relations when corresponding transformations exist:

$$\mathcal{S}_u(\mathcal{S}_u(O)) = \mathcal{S}_\ell(\mathcal{S}_\ell(O)) = O, \tag{11a}$$

$$\mathcal{S}_u(\mathcal{S}_\ell(O)) = \mathcal{S}_\ell(\mathcal{S}_u(O)) = O^{-1}. \tag{11b}$$

See also [14, Ch. 4], where similar transformations were introduced.

Another advantage of looking at the $\mathcal{S}$-transformation of O instead of at O itself is that

$$\mathcal{S}_\ell\left(\left[\begin{smallmatrix} I & \Delta_1^{\sim} \\ 0 & I \end{smallmatrix}\right] O \left[\begin{smallmatrix} I & 0 \\ \Delta_2 & I \end{smallmatrix}\right]\right) = \mathcal{S}_\ell(O) + \left[\begin{smallmatrix} 0 & \Delta_1^{\sim} \\ -\Delta_2 & 0 \end{smallmatrix}\right]. \tag{12}$$

This relation will be used in Subsection 3.5.

3.4 Necessary solvability conditions

We start with finding the necessary condition for the solvability of $\mathrm{OP_{eq}}$. To this end, note that given any proper K_h, the responses of $\mathcal{C}_r(G_0^{-1}, e^{-sh}K_h)$ and $\mathcal{C}_r(G_0^{-1}, 0)$ to any input coincide in the interval $[0, h]$ (the former system is actually open-loop in this interval). This, in fact, proves that $\mathrm{OP_{eq}}$ is solvable only if

$$\gamma > \|\mathcal{C}_r(G_0^{-1}, 0)\|_{L^2[0,h]}.$$

To find the state-space realization of $\mathcal{C}_r(G_0^{-1}, 0)$, note that it is equal to the $(1,2)$-subblock of $\mathcal{S}_\ell(G_0^{-1})$. Yet, according to (11), $\mathcal{S}_\ell(G_0^{-1}) = \mathcal{S}_u(G_0)$. This, together with (10a), yields that

$$\mathcal{C}_r(G_0^{-1}, 0) = \left[\begin{array}{c|c} A_F + ZB_2B_2'X & ZYC_2' \\ \hline B_2'X & 0 \end{array}\right].$$

We thus proved the following result:

Lemma 2 $\mathrm{OP_{eq}}$ *is solvable only if* $\gamma > \gamma_h$, *where* γ_h *is the* $L^2[0,h]$*-induced norm of* $\mathcal{C}_r(G_0^{-1}, 0)$ *given by (7).*

3.5 Factorization of $G_\alpha^{\sim} J_\gamma G_\alpha$

By Lemma 2 we can safely assume hereafter that $\gamma > \gamma_h$. In this subsection the assumption above is required to ensure that Σ_{22} is invertible.

Consider $\Pi = (G_0^{-1})^{\sim} J_\gamma G_0^{-1}$, which has the following state-space realization (recall (5)):

$$\Pi = \left[\begin{array}{cc|cc} ZA_LZ^{-1} & 0 & ZB_2 & ZYC_2' \\ \gamma^2C_2'C_2 - XB_2B_2'X & -Z^{-\prime}A_L'Z' & -XB_2 & -\gamma^2C_2' \\ \hline B_2'X & B_2'Z' & I & 0 \\ \gamma^2C_2 & C_2YZ' & 0 & -\gamma^2 I \end{array}\right]$$
$$\doteq \left[\begin{array}{c|cc} A_\Pi & B_{\Pi 1} & B_{\Pi 2} \\ \hline C_{\Pi 1} & I & 0 \\ C_{\Pi 2} & 0 & -\gamma^2 I \end{array}\right].$$

Note that by construction

$$\left[\begin{array}{cc} B_{\Pi 1} & B_{\Pi 2} \end{array}\right] = \hat{J}\left[\begin{array}{cc} C_{\Pi 1}' & C_{\Pi 2}' \end{array}\right], \tag{13}$$

where $\hat{J} \doteq \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$.

Now for the construction of $\Delta \doteq \pi_h\{e^{-sh}\Pi_{22}^{-1}\Pi_{21}\}$ we use the fact that $-\Pi_{22}^{-1}\Pi_{21}$ is the lower left block of $\mathcal{S}_\ell(\Pi)$. Therefore consider (using (10b))

$$\mathcal{S}_\ell(\Pi) = \left[\begin{array}{c|cc} A_\Pi + \frac{1}{\gamma^2}B_{\Pi 2}C_{\Pi 2} & B_{\Pi 1} & -\frac{1}{\gamma^2}B_{\Pi 2} \\ \hline C_{\Pi 1} & I & 0 \\ \frac{1}{\gamma^2}C_{\Pi 2} & 0 & -\frac{1}{\gamma^2}I \end{array}\right]$$

(by (5) one can see that $\Sigma(t) = e^{(A_\Pi + \gamma^{-2}B_{\Pi 2}C_{\Pi 2})t}$). Then

$$\Delta = \pi_h\left\{e^{-sh}\left[\begin{array}{c|c} A_\Pi + \gamma^{-2}B_{\Pi 2}C_{\Pi 2} & B_{\Pi 1} \\ \hline -\gamma^{-2}C_{\Pi 2} & 0 \end{array}\right]\right\}$$

and this coincides with the realization in Theorem 1. Moreover,

$$R \doteq \Delta + e^{-sh}\Pi_{22}^{-1}\Pi_{21} = \left[\begin{array}{c|c} A_\Pi + \frac{1}{\gamma^2}B_{\Pi 2}C_{\Pi 2} & \Sigma^{-1}B_{\Pi 1} \\ \hline -\frac{1}{\gamma^2}C_{\Pi 2} & 0 \end{array}\right]$$

Next, to obtain a realization of Π_β as defined in (9) we use (12) and then combine the various realizations:

$$\begin{aligned} \mathcal{S}_\ell(\Pi_\beta) &= \mathcal{S}_\ell\left(\left[\begin{smallmatrix} I & \Delta^{\sim} \\ 0 & I \end{smallmatrix}\right]\Pi_\alpha\left[\begin{smallmatrix} I & 0 \\ \Delta & I \end{smallmatrix}\right]\right) \\ &= \mathcal{S}_\ell(\Pi_\alpha) + \left[\begin{smallmatrix} 0 & \Delta^{\sim} \\ -\Delta & 0 \end{smallmatrix}\right] \\ &= \left[\begin{smallmatrix} e^{sh}I & 0 \\ 0 & I \end{smallmatrix}\right]\mathcal{S}_\ell(\Pi)\left[\begin{smallmatrix} e^{-sh}I & 0 \\ 0 & I \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & \Delta^{\sim} \\ -\Delta & 0 \end{smallmatrix}\right] \\ &= \left[\begin{array}{c|cc} A_\Pi + \frac{1}{\gamma^2}B_{\Pi 2}C_{\Pi 2} & \Sigma^{-1}B_{\Pi 1} & -\frac{1}{\gamma^2}B_{\Pi 2} \\ \hline C_{\Pi 1}\Sigma & I & 0 \\ \frac{1}{\gamma^2}C_{\Pi 2} & 0 & -\frac{1}{\gamma^2}I \end{array}\right]. \end{aligned}$$

This, together with (11a) and (10b), yields

$$\Pi_\beta = \left[\begin{array}{c|cc} A_\Pi & \Sigma^{-1}B_{\Pi 1} & B_{\Pi 2} \\ \hline C_{\Pi 1}\Sigma & I & 0 \\ C_{\Pi 2} & 0 & -\gamma^2 I \end{array}\right]$$

and the "B" and "C" matrices of Π_β satisfy

$$\left[\begin{array}{cc} \Sigma^{-1}B_{\Pi 1} & B_{\Pi 2} \end{array}\right] = \hat{J}\left[\begin{array}{cc} \Sigma'C_{\Pi 1}' & C_{\Pi 2}' \end{array}\right], \tag{14}$$

which follows from (13) and the fact that Σ is symplectic and thus $\Sigma\hat{J}\Sigma' = \hat{J}$.

To J-factorize Π_β, let $M = M'$ be any matrix satisfying the following Riccati equation

$$\left[\begin{array}{cc} -M & I \end{array}\right]\Sigma^{-1}(A_\Pi - B_{\Pi 1}C_{\Pi 1} + \tfrac{1}{\gamma^2}B_{\Pi 2}C_{\Pi 2})\Sigma\left[\begin{array}{c} I \\ M \end{array}\right] = 0$$

(we construct one such M below). Then, taking into account (14), one can verify that stable

$$W_\beta = \left[\begin{array}{c|cc} ZA_LZ^{-1} & [I\ 0]\Sigma^{-1}B_{\Pi 1} & [I\ 0]B_{\Pi 2} \\ \hline C_{\Pi 1}\Sigma\left[\begin{smallmatrix} I \\ M \end{smallmatrix}\right] & I & 0 \\ -\frac{1}{\gamma^2}C_{\Pi 2}\left[\begin{smallmatrix} I \\ M \end{smallmatrix}\right] & 0 & I \end{array}\right]$$

does satisfy $\Pi_\beta = W_\beta^{\sim} J_\gamma W_\beta$. Moreover, the "A" matrix of W_β^{-1}, say A_β, equals

$$\begin{aligned} A_\beta &= ZA_LZ^{-1} \\ &\quad - [I\ \ 0](\Sigma^{-1}B_{\Pi 1}C_{\Pi 1}\Sigma - \tfrac{1}{\gamma^2}B_{\Pi 2}C_{\Pi 2})\left[\begin{array}{c} I \\ M \end{array}\right] \\ &= [I\ \ 0](A_\Pi - \Sigma^{-1}B_{\Pi 1}C_{\Pi 1}\Sigma + \tfrac{1}{\gamma^2}B_{\Pi 2}C_{\Pi 2})\left[\begin{array}{c} I \\ M \end{array}\right] \\ &= [I\ \ 0]\Sigma^{-1}(A_\Pi - B_{\Pi 1}C_{\Pi 1} + \tfrac{1}{\gamma^2}B_{\Pi 2}C_{\Pi 2})\Sigma\left[\begin{array}{c} I \\ M \end{array}\right] \\ &= [I\ \ 0]\Sigma^{-1}\left[\begin{array}{cc} A_F & Z(\frac{1}{\gamma^2}YC_2'C_2Y - B_2B_2')Z' \\ 0 & -A_F' \end{array}\right]\Sigma\left[\begin{array}{c} I \\ M \end{array}\right] \end{aligned}$$

where equality (5) and the fact that $A_\Pi + \frac{1}{\gamma^2} B_{\Pi 2} C_{\Pi 2}$ commutes with Σ were used. Since Σ is symplectic,

$$\Sigma^{-1} = \begin{bmatrix} \Sigma'_{22} & -\Sigma'_{12} \\ -\Sigma'_{21} & \Sigma'_{11} \end{bmatrix}$$

and $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = \Sigma_{22}^{-\prime}$. Then the natural choice for M is $M = -\Sigma_{22}^{-1}\Sigma_{21}$, since this yields

$$A_\beta = \Sigma'_{22} A_F \Sigma_{22}^{-\prime},$$

which is Hurwitz, so that W_β is bistable (anothor way to find the required M is to use the J-spectral factorization of generic para-Hermitian transfer matrices [17]). Straightforward algebra yields then that $W_\beta^{-1} = G_h$ with G_h as in Theorem 1.

3.6 Necessity & sufficiency

By construction we have: $\|\mathcal{C}_r(G_0^{-1}, e^{-sh}K_h)\|_{L^\infty} < \gamma$ iff $\|Q_h\|_{L^\infty} < \gamma$ for Q_h defined as

$$Q_h \doteq \mathcal{C}_r(G_h^{-1}\left[\begin{smallmatrix} I & 0 \\ -\Delta & I \end{smallmatrix}\right], K_h).$$

Now this Q_h is proper if K_h is proper, yet the set of proper operators in L^∞ is in fact H^∞ [18] (see also [19, A6.26.c, A6.27]). So if K_h solves OP_{eq}, then necessarily $\|Q_h\|_\infty < \gamma$. This condition on Q_h is also sufficient as we shall now see. The thing to note is that

$$\Theta(h) \doteq G_0^{-1}\left[\begin{smallmatrix} e^{-sh}I & 0 \\ 0 & I \end{smallmatrix}\right]\left[\begin{smallmatrix} I & 0 \\ \Delta & I \end{smallmatrix}\right] G_h$$

is not only stable and J_γ-unitary (i.e., $\Theta(h)^\sim J_\gamma \Theta(h) = J_\gamma$) but in fact J_γ-lossless (meaning that in addition $\Theta_{22}(h)$ is bistable). Indeed, from $\Theta(h)^\sim J_\gamma \Theta(h) = J_\gamma$ it follows that $\Theta_{22}(h)^\sim \Theta_{22}(h) \geq I$, and as our $\Theta(t)$ (which by Lemma 2 exists for all $\gamma > \gamma_h$) is stable and continuous as a function of $t \in [0, h]$, and $\Theta_{22}(0) = I$ it follows that $\Theta_{22}(h)$ is bistable. It is well known that for J_γ-lossless $\Theta(h)$ we have that $Q = \mathcal{C}_r(\Theta(h), Q_h)$ is stable for any $\|Q_h\|_\infty < \gamma$, see, e.g., [8, Thm. 6.2]. Also, $K_h = \mathcal{C}_r(\left[\begin{smallmatrix} I & 0 \\ \Delta & I \end{smallmatrix}\right] G_h, Q_h)$ is proper for any stable Q_h.

4 Concluding remarks

In this note two "competing" approaches to H^∞ control for systems with a single delay have been put together and the result is probably the simplest solution to date to the problem.

Instrumental is the idea to reduce the problem to a one-block problem with a simple structure. In fact in the mean time this idea has been put to use to solve the case where there are multiple delays. This will be reported elsewhere.

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