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A Novel Lumped Spatial Model of Tire Contact

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***Abstract*—In this article, we describe a novel way to physically model the dynamics of the compliant contact between a tire and the ground. We show how the use of a spatial spring [1] results in an intuitive model and how screw bond graphs can be used to describe the various parts of the model in a power-consistent way.**

I. Introduction

In this article, we consider the modeling of the contact between a tire and the ground. This model is part of a larger project in which we model the complete dynamics of a car (Figure 1). The car model (and the contact model in particular) can be used in the design and simulation of intelligent controllers (e.g. steer-by-wire) that increase the performance, comfortability and safety of the car.

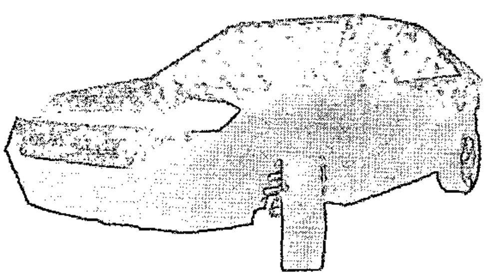

Fig. 1. 3D view in 20sim [2] of the car model.

Since both the tire and the ground are not infinitely stiff, the contact between these objects is not an ideal point contact, but instead there is an area of contact (we refer to [3] for detailed information on the dynamics of these types of tire contacts). Furthermore, the objects are slightly deformed at this area of contact, but when the contact is broken, the original shape of the objects is almost completely restored.

Even though the contact is not an ideal point contact, we start from the description of point contact, and extend it to obtain a model of the compliant contact. Ideal point contact has been described first by Montana [4], and more recently in a coordinate free way in [5]. These articles describe how to find the velocity of the contact point over the surfaces of two rigid bodies in point contact, given the relative (translational and angular) velocity of the bodies.

Work has been done before on the extension of point contact to compliant area contact. For example, Montana [6] extended his earlier ideal-contact results to include compliant contact, and Visser et al. [7] used 1D springs to model the temporary deformation of the objects.

In this article, we also consider a lumped parameter model. We use one spatial spring (instead of separate springs for each coordinate axis) to represent explicitly the potential energy, as stored during the deformation of the objects in a geometrically consistent way. We then use bond graphs to describe the connection of this spring between the tire and the ground. In this way, we ensure that the energy balance is maintained at all times. Furthermore, the model allows relatively easy extension to the case of bouncing, i.e. when the contact between tire and ground is sometimes lost and the tire is temporary airborne.

First, Section II discusses the background knowledge as required for the rest of this article. Next, Section III describes the concept of a spatial spring. Then, Section IV explains the basic concept of the spatial contact model, and Section V extends these ideas to a mathematical model. Section VI extend the presented considerations to the case of anysotropical contacts and coupling. Finally, Section VII presents the conclusions and discusses possible future work in this area.

II. Background

In this article, we deal with rigid bodies moving in the Euclidean space, which means we can describe the position and orientation of every body by an element of the matrix special Euclidean group $SE(3)$, once a reference frame has been chosen. As shown for example in [8] and [9], elements of this group can be represented by a homogeneous matrix of the form

$$H_j^i = \begin{bmatrix} R_j^i & p_j^i \\ 0 & 1 \end{bmatrix}$$

where R_j^i is a rotation matrix (element of the special orthonormal group $SO(3)$) and p_j^i is a vector in $\mathbb{R}^3$. H_j^i denotes the change of coordinates from a right-handed coordinate frame Ψ_j to another right-handed coordinate frame Ψ_i and can thus be used for example to describe the position and orientation of a body (with attached coordinate frame Ψ_j) relative to a reference (inertial) coordinate frame (Ψ_i).

The instantaneous velocity of a body i with frame Ψ_i relative to a body j with frame Ψ_j can be represented by a twist $T_i^{k,j}$, with

$$T_i^{k,j} = \begin{bmatrix} \omega_i^{k,j} \\ v_i^{k,j} \end{bmatrix}$$

where $\omega_i^{k,j}$ denotes the angular velocity of body i relative to body j expressed in coordinate frame Ψ_k, and $v_i^{k,j}$ denotes the instantaneous velocity (relative to frame Ψ_j) of the point fixed in frame Ψ_i that passes through the origin of frame Ψ_k. A twist can be regarded as the derivative of a homogeneous matrix using what is called a right translation of a Lie group [10] in the following way:

$$\tilde{T}_i^{j,j} := \begin{bmatrix} \tilde{\omega}_i^{j,j} & v_i^{j,j} \\ 0 & 0 \end{bmatrix} = \dot{H}_i^j H_j^i$$

where $\tilde{\omega}$ is the skew-symmetric matrix equivalent to $(\omega \times \cdot)$. To change the coordinate frame in which the twist is expressed, we use the adjoint representation of a Lie group which is in this case indicated with $Ad_{H_j^k}$ and such that

$$\tilde{T}_i^{k,j} = Ad_{H_j^k} T_i^{j,j} = \begin{bmatrix} R_j^k & 0 \\ \tilde{p}_j^k R_j^k & R_j^k \end{bmatrix} T_i^{j,j}.$$

More information on twists can be found e.g. in [8] and [9].

We can also define a wrench W_i^k (the dual of a twist), which describes the generalized forces acting on body i and expressed in frame Ψ_k, as

$$W_i^k = \begin{bmatrix} \tau_i^k \\ F_i^k \end{bmatrix}$$

where F_i^k denotes the linear force and τ_i^k the momentum, acting on the point in the origin of frame Ψ_k. The dual product of a twist and a wrench (when expressed in the same coordinate frame) is equal to a power flow.

The concepts of twists and wrenches can be easily visualized in terms of so-called screw bond graphs. These are generalizations of ordinary bond graphs as defined first by Paynter [11]. We do not discuss the whole screw bond-graph theory here (we refer to [9], [12], [13] for the details), but we describe the basic concepts.

In a bond graph, each bond (drawn as a half-arrow) represents the power-flow caused by two power-conjugate variables: one flow (i.e. a twist) and one effort (i.e. a wrench). The dual product of these two equals the corresponding power flowing through the bond in the direction of the arrow-head. Bonds can be interconnected by means of 1-junctions (where all connected bonds have the same flow and the algebraic sum of the efforts of all connected bonds is zero) and 0-junctions (where all connected bonds have the same effort and the algebraic sum of the flows of all connected bonds is zero) to build a complete graph of connected bonds. Furthermore, one-port elements can be connected to the graph to represent various ideal physical elements (**R** for pure dissipation, **C** for pure (temporary) storage of potential energy, etc.). Also, two-port elements can be included, like a modulated transformer (represented as **MTF**), which transforms both the flow and the effort in such a way that the power flowing into the transformer on one side is equal to the power flowing out of the transformer on the other side, hence the **MTF** is a power-continuous element.

In this article, we mainly use bond graphs to show schematically the relations between various twists. Because of the bond graph structure, we immediately get the relation between the various wrenches. This means that by looking at velocities in the model compliant contact, we get the contact forces back, which is what we want to know for simulation purposes.

III. Spatial Springs

In this article, we use a spatial spring to describe the compliant contact. Spatial springs were first introduced in [14] and later used for modeling in [15] and [1].

A spatial spring is an ideal mechanical element that stores potential energy, in a way that is completely determined by its energy function V in a coordinate free matter. The spring is connected between two frames (the hinge frames) and the energy stored in the spring depends on the relative position and orientation of these frames.

If we denote the two hinge frames by Ψ_{1c} and Ψ_{2c}, then the relative position of these two frames is described by H_{2c}^{1c} and the stored energy in the spring can be represented by

$$V : SE(3) \to \mathbb{R}; \quad H_{2c}^{1c} \mapsto V(H_{2c}^{1c})$$

For passivity reasons, the potential energy function must have a minimum. We choose the hinge frames in such a way, that this minimum is obtained when Ψ_{1c} and Ψ_{2c} coincide, i.e. when $H_{2c}^{1c} = I$.

The spatial spring can be represented in bond graph terms by a C-element: the bond represents the twist T_{2c}^{1c}, i.e. the (linear and angular) relative velocity of the hinge frames, and the wrench $W_{1c,2c}$, i.e. the forces and torques of the spring acting on the elements attached to the frames Ψ_{1c} and Ψ_{2c}.

In this work, complete spatial springs will not be necessary. A model of a contact spring is nicely described using a Lie subgroup of $SE(3)$ which is indicated with $SE(2) \times T(1)$ [14], [10], [16]. This Lie sub-group represents all the planar Euclidean motions on a plane ($SE(2)$) together with possible vertical pure translation along the normal of the plane ($T(1)$).

There are two important consideration concerning $SE(2) \times T(1)$. First of all this Lie group is NOT unique, but there are many sub-groups of $SE(3)$ with this form which are uniquely characterized once a direction is chosen. Such a direction can be seen either with an element of a Gauss sphere or as an element of RP^2, a projective two dimensional space.

Second, the Lie group $SE(2)$ and $T(1)$ are completely decoupled and the total Lie sub-group is not a semi-direct product. The first consideration will be essential because this direction can be geometrically defined based on the gauss maps of the bodies and their elastic properties. The second consideration allows, if necessary, to consider separately an elastic compression energy:

$$V_c : T(1) \to \mathbb{R}$$

and a 'shifting' energy

$$V_s : SE(2) \to \mathbb{R}$$

such that the total contact energy can be defined as:

$$V : SE(2) \times T(1) \to \mathbb{R}; (c, s) \mapsto V_c(c) + V_s(s)$$

We will see in Section VI that this can be better generalised in order to model an existing energetic coupling between the two possible motions.

IV. THE BASIC CONCEPT

In this section, we explain the basic concept of using a spatial spring to model compliant contact. In the next section, we derive the corresponding equations.

Figure 2 shows the concept of the model: two bodies (the tire and the ground) have collided, and they are now in contact over some area (shown exaggerated in the figure). Due to the collision, the objects are locally deformed near the contact area. We assume the contact to be (partially) elastic, so the deformation is only temporary, and (some of) the kinetic energy that was lost during impact has been stored in the form of potential energy. It is exactly this energy that we want to model with the spatial spring. The energy that was lost during collision is modeled with a pure dissipation.

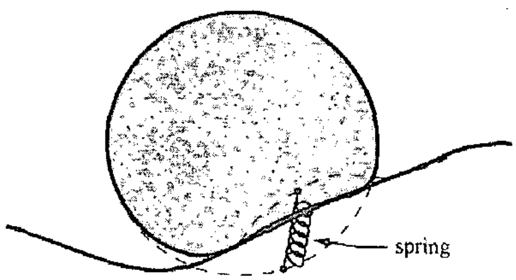

Fig. 2. Concept of the compliant contact model: we model the deformation of the two bodies as a spring between two virtual contact points, i.e. between the contact points on the undeformed bodies.

At the instant of impact, when the two bodies first touch at a single point, we attach the spring with both hinge points at the instantaneous contact point to the bodies 1 and 2. Since the hinge points coincide, the spring is initialized with zero energy (by definition), which is obviously the right thing to do, since we cannot create energy from scratch.

After the impact, the objects continue to move, and the surfaces of the objects deform. We model this deformation in the following way: the hinge-points move with the objects as if the ideal, rigid shapes are maintained and move through each other. Since in reality, the surfaces of the objects change, this means that the hinge points move away from each other, and the spring is charged.

Under the influence of external and contact forces, the objects will start to slide and roll over each other, and so the contact area will change. However, we do not want all movement of the bodies to charge the spring: pure rolling of two bodies over each other is not supposed to charge the spring, since no potential energy is stored in that case. On the other hand, when the bodies are sliding over each other (and in this way locally stretching the surfaces), we *do* want the spring to be charged, so that potential energy is stored.

The problem is that it is not so easy to distinguish between rolling and sliding if the bodies share an area of contact. In case of point contact, it would have been easy: rolling is when one body rotates relative to the other body around an axis in the tangent plane to the surfaces at the contact point, and sliding is in all other circumstances. As explained in Section II, we can completely specify the sub-group of sliding ($SE(2) \times T(1)$) by a single line, or equivalently, by the plane perpendicular to this line. For the case of point contact, this plane is exactly the tangent plane to the surfaces.

In the case of area contact, we cannot simple talk about *the* tangent plane to the surface. However, we can talk about the tangent planes at the hinge points: these are completely specified by the normal vectors to the surfaces, i.e. by the Gauss maps. But, since these tangent planes are not aligned in general, we need to take their average, which is an 'average tangent plane', somewhere in between. The exact position of this plane depends on the relative stiffnesses of the tire and the ground. We then use this average tangent plane to decompose the twist in rolling and sliding. Finally, we use the rolling component of the twist to calculate the motion of the hinge points over the surface, and the sliding component to define the spatial spring.

V. THE MATHEMATICAL MODEL

In this section, we formalize the concept model of the previous section, and take a look at the mathematical parts that constitute the full spatial compliant contact model. We use the screw bond graph in Figure 3 to illustrate the relations between the various parts, and to give a framework that can be used to implement the model, e.g. in the modeling package 20sim [2], which can handle bond graphs.

We assume to have a description of the (un-deformed) surfaces as functions $f_i : \mathbb{R}^2 \to S_i$ with $i = 1, 2$, which take coordinates $(u_i, v_i) \in \mathbb{R}^2$ and return the corresponding elements of the surfaces S_i as points expressed in the coordinate frames

Ψ_i. From these functions, we can compute $f_{iu} := \frac{\partial f_i}{\partial u_i}$ and $f_{iv} := \frac{\partial f_i}{\partial v_i}$, and we can derive the gauss maps

$$g_i = \frac{f_{iu} \times f_{iv}}{|f_{iu} \times f_{iv}|}.$$

Now consider the bond graph of the contact model. The upper part (right down to $T_2^{1,1}$) is just the computation of the relative twist, expressed in frame Ψ_1, fixed to body 1. Then, we use another MTF to express the same twist in frame Ψ_c, which is a time-varying coordinate frame with the origin at the average tangent plane (as defined in the Section IV) and its z-axis perpendicular to the plane (we explain later how we obtain this frame). Once the twist is expressed in this frame, it is easy to project onto the 4D Lie sub-algebra $se(2) \times t(1)$ corresponding to the Lie sub-group $SE(2) \times T(1)$ on which the elastic energy function will be defined: the rolling component (the part that does not charge the spring) is just the (x, y)-part of ω. This means that the projection matrix P takes the following simple form:

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

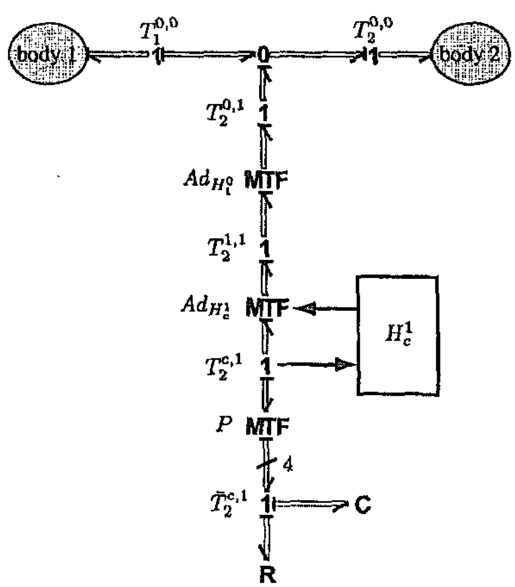

Fig. 3. Bond graph of the model of the compliant contact between two bodies.

Now we only need to calculate H_c^1, i.e. the position and orientation of the average tangent plane to the surfaces at the contact area. As said before, this plane will be some average of the two tangent planes to the (un-deformed) surfaces at the hinge points.

If we consider one of the bodies (say, body 1), we can completely specify the tangent plane, once we know the local coordinates (u_1, v_1) of the hinge point attached to that body: The orientation of the plane is given by the gauss map $g_1(u_1, v_1)$, and the location of the plane is given by the function $f_1(u_1, v_1)$.

The locations p_1 and p_2 of both tangent planes (expressed in frame Ψ_1) are

$$\begin{bmatrix} p_1 \\ 1 \end{bmatrix} = \begin{bmatrix} f_1(u_1, v_1) \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} p_2 \\ 1 \end{bmatrix} = H_2^1 \begin{bmatrix} f_2(u_2, v_2) \\ 1 \end{bmatrix}$$

and the orientations n_1 and n_2 of both tangent planes (expressed in frame Ψ_1) are

$$\begin{bmatrix} n_1 \\ 1 \end{bmatrix} = \begin{bmatrix} g_1(u_1, v_1) \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} n_2 \\ 1 \end{bmatrix} = H_2^1 \begin{bmatrix} g_2(u_2, v_2) \\ 1 \end{bmatrix}.$$

To compute the average plane, we simply take a weighted average of these two, where the weighting depends on the relative compressibility of the bodies. If a body is relatively hard to compress, then the tangent plane is relatively close to that body.

Let k_1 and k_2 denote the compressibilities of bodies 1 and 2, respectively. Then the average tangent plane is defined by the position vector

$$p_c = \frac{k_1 p_1 + k_2 p_2}{k_1 + k_2} \quad (1)$$

and the normal vector

$$n_c = \frac{k_1 n_1 - k_2 n_2}{|k_1 n_1 - k_2 n_2|} \quad (2)$$

where the minus sign is a consequence of the definition of n_2 as the normal vector pointing outward of surface S_2, so for small deformations, $-n_2$ (as opposed to $+n_2$) will be almost in the direction of n_1 and therefore the denominator of the previous equation will never be equal to zero.

We can now compute the transformation matrix H_c^1 as

$$H_c^1 = \begin{bmatrix} x_c & y_c & n_c & p_c \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3)$$

where x_c and y_c are any orthonormal vectors in the tangent plane, such that (x_c, y_c, n_c) defines a right-handed coordinate frame.

Equation 3 gives an expression for the location and orientation of the average tangent plane, once the local coordinates of the hinge points on the undeformed surfaces are known. As discussed in the previous section, the position of these hinge points is initialized at the contact point when the bodies first come into contact, and the points move over the surface if the objects are rolling over each other.

As discussed in detail in [5], the velocity of the contact points can be computed from the rolling velocity of the bodies as

$$f_{1*}\begin{bmatrix}\dot{u}_1\\ \dot{v}_1\end{bmatrix} = (g_{1*}+g_{2*})^{-1}(g_{1*}\times\omega_1^{c,2}),$$

which can be rewritten, ignoring the z-components, as

$$\begin{bmatrix}\dot{u}_1\\ \dot{v}_1\end{bmatrix} = f_{1*}^{-1}(g_{1*}+g_{2*})^{-1}\bar{g}_{1*}\begin{bmatrix}(\omega_1^{c,2})_x\\ (\omega_1^{c,2})_y\end{bmatrix}$$

where f_{1*} denotes the derivative of the mapping f (and similarly g_{1*} and g_{2*}) and $\omega_1^{c,2}$ denotes the angular rolling velocity of body 1 relative to body 2, expressed in coordinate frame Ψ_c.

Once the coordinate frame Ψ_c has been fully determined, we can calculate the 4D velocity $\bar{T}_2^{c,1} \in se(2)\times t(1)$ and also the 4D force $\bar{W}^c \in se^*(2)\times t^*(1)$ exerted by the spatial spring. This force has the following form

$$\bar{W}^c = \begin{bmatrix}m_z\\ f_x\\ f_y\\ f_z\end{bmatrix}$$

where m_z is the torque around the z-axis of Ψ_c, f_x and f_y are the tangential forces, and f_z is the compression force. The compression force can be considered independent of the other forces and torques, since $T(1)$ and $SE(2)$ are independent sub groups. So for f_z, we could simply use a 1D spring.

The other forces can be calculated (following [9] but now for $SE(2)$ instead of $SE(3)$ and assuming isotropic elasticity in the average tangent plane) as

$$\begin{aligned}
m_z &= k_z\sin(\theta)\\
f_x &= k_c\cos(\theta)+k_c\sin(\theta)-k_tx\cos(\theta)+k_ty\sin(\theta)\\
f_y &= -k_c\cos(\theta)+k_c\sin(\theta)-k_ty\cos(\theta)+k_tx\sin(\theta)
\end{aligned}$$

where k_z is the angular stiffness, k_c is a coupling stiffness between rotation and translation, k_t is the tangential stiffness, and θ is the rotation angle (the deformation around the z-axis). Note that in the previous equations, the tangential forces on the plane do also depend on the angular torsion θ which is typical of properly defined spatial springs.

VI. Coupling and Anisotropy

Due to the profile of a tire, there are clearly direction dependent stiffnesses in the contact and furthermore, considering the tangential stiffness independent from the compression is not realistic either, since a higher compression usually corresponds to a higher tangential stiffness of the contact.

In order to take these effect into account, we can associate a stiffness information to each point of the contacting surfaces and then calculate a corresponding geometrical anysotropical stiffness during contact based on them. Once this stiffness is defined, it can be used in the projection plane in order to integrate the projected twists and calculate the corresponding wrenches.

In mathematical terms we can proceed as follows. We can associate at each point of the surfaces a two covariant tensor based on $se(2)\times t(1)$ corresponding to a stiffness:

$$K_i : S_i \rightarrow (se(2)\times t(1))_2 \qquad i=1,2$$

The previous mappings are in differential geometric terms called tensor bundles.

By clearly making an approximation, we can then consider both tensors defined in the same point p_c of eq.(1).

It is then meaningful to consider

$$K(p_c) := (K_1^{-1}(p_1)+K_2^{-1}(p_2))^{-1}$$

as a representative stiffness of the contact. To understand this, it is sufficient to realise that in case one of the two contacting materials is much softer than the other, as it is the case with a tire and the ground, the resulting combined stiffness $K(p_c)$ is almost equal to the the one with the smallest stiffness.

It would then be possible to use the Lie-group structure of $SE(2)\times T(1)$ in order to use $K(p_c)$ to geometrically define an elastic energy. This procedure would be a subcase of the procedure presented in [9] for complete spatial springs.

By simply considering the K_i with coupling between compression and tangential motions we can then model a general anysotropical couple stiffness of the surfaces materials.

VII. Conclusions and Future Work

In this article, we have used a 4D spatial spring to model the compliant contact between two bodies. The energy stored in this spring represents the elastic potential energy stored physically as the two bodies deform during contact. As the objects press into each other or stretch tangentially during sliding motion, the spring is charged and applies forces and torques opposing the direction of deformation. If the objects roll over each other, the spring is not charged, but instead the hinge points are moved over the surfaces of the bodies to follow the relative movement of the bodies.

The use of the spatial spring allowed to consider the contact as one dynamical system, instead of looking at separate directions (three separate translations and one rotation) and considering separate 1D springs. The global approach also allowed for easy implementation in terms of bond graphs.

A possible direction for future research could be the modeling of stick-slip contact. This would mean that, once some threshold tangential contact force is reached, the contact is broken (the energy in the spring is lost), and the tire slides freely over the ground, i.e. it slips.

Another direction is to use this contact model as a sub-model of a complete car. So instead of one tire, we would have four connected tires. In this case, the bond graph approach will be very useful, since it allows power-consistent interconnection of subsystems.

Finally, an important addition to the model would be the detection of contact/no-contact to be able to initialize the spring at the right moment. Some interesting work has been done in this area already (e.g. [7]).

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