NEW MODEL REDUCTION TECHNIQUE FOR A CLASS OF PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT:

The aim of this paper is to provide a new model reduction (or lumping) technique for a class of parabolic type partial differential equations and to show its application. The frequency response of the temperature distribution in any multilayer solid is developed and given by a matrix expression. The distributed transfer functions (DTF) are also given between the surface variables and the ambient variables. The proposed method is based on the special high frequency properties of these DTF's. We compare the rational fraction expansion with our new lumping technique. We give numerical examples from real-time applications.

1. INTRODUCTION

A large class of parabolic partial differential equations (PDE) come from heat processes arising in many industrial areas. The general heat conduction equation in solids describing the temperature distribution has the form [2]:

$$c\rho \frac{\partial \vartheta(t,\underline{x})}{\partial t} + \mathrm{div}[-\lambda\ \mathrm{grad}(\vartheta(t,\underline{x}))] = S(t,\underline{x}) \quad (1)$$

where $\vartheta(t,\underline{x})$ is the temperature distribution in the solid, λ is the heat conductivity, c is the specific heat, ρ is the density, $S(t,\underline{x})$ denotes the internal heat source, t is time, $\underline{x}$ is the spatial variable, $\underline{x} \in \Omega$ where Ω is a closed domain of the p-dimensional Euclidean space E^p ($p \le 3$). Usually one is interested in the temperature distribution inside the solid or at the boundary of Ω (denoted by Γ). In general Eq.(1) has no analytical solution. In complicated cases only numerical techniques can be applied [6] but they do not provide a low-order process model in parametric form. However, for certain cases, as with constant heat conductivity and no inner heat source the distributed transfer functions (DTF) of the process can be developed [9] as we shall show in this paper. The technique can be used for multilayer solid structures, as walls, insulation etc. with any number of layers [10]. Based on the spacial high-frequency properties of the DTF's a new model reduction technique is described leading to a low-order and highly accurate parametric process model.

2. PROBLEM STATEMENT

For sake of simplicity we shall consider only one dimensional problems. Let us consider the multilayer structure in Fig.1. Assuming constant heat conductivity and no inner heat source (S=0) the temperature distribution in the i-th layer is described by the following second order parabolic PDE's [2,10]:

$$\frac{\partial \vartheta^i(t,x)}{\partial t} = a_i \frac{\partial^2 \vartheta^i(t,x)}{\partial x^2}\ ; \qquad i=1,2,\ldots N \quad (2)$$

where $a_i=\lambda_i/(c_i\rho_i)$ [m^2/s] and N is the number of layers. Assuming the layers to be in perfect thermal contact at their surfaces means that the the temperature and heat-flux between two layers remains the same. So the boundary conditions are:

$$\vartheta^i(t,x)\Big|_{x=L_i} = \vartheta^{i+1}(t,x)\Big|_{x=L_i}\ ; \quad i=1,2,..N-1 \quad (3)$$

$$\phi^i(t,x)\Big|_{x=L_i} = \phi^{i+1}(t,x)\Big|_{x=L_i}\ ; \quad i=1,2,..N-1 \quad (4)$$

where ϕ denotes the heat-flux [W/m^2]. At the boundary of Ω (i.e. at the left-hand side and right-hand side of the structure) we assume the general 3rd kind of BC's as given by [2]:

$$x=0 \qquad \alpha_o F_o[\vartheta_{a1} - \vartheta(t,0)] = -\lambda_1 F_o \frac{\partial \vartheta}{\partial x}\Big|_{x=0} = \phi_o \quad (5)$$

$$x=L \qquad \alpha_N F_N[\vartheta(t,L) - \vartheta_{a2}] = -\lambda_N F_N \frac{\partial \vartheta}{\partial x}\Big|_{x=L} = \phi_N \quad (6)$$

where F_i (i=0,N) denotes the surface [m^2]. Besides determining the temperature distribution one is usually interested in the interaction between the ambient variables (ϑ_{a1}, ϑ_{a2}) and the surface variables (ϑ_o, ϕ_o at x=0 for example). In the following section we shall provide a matrix method to determine the DTF's of the process.

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3. DISTRIBUTED TRANSFER MATRIX OF THE PROCESS

To determine the temperature distribution inside the multilayer structure ($x\in\Omega$) we have to solve *N coupled, partial differential equations* with N initial conditions and 2N *boundary conditions*. Taking the Laplace transform of Eq.(2) the general solution in the s-domain can be expressed as [7]:

$$\vartheta^i(s,x) = C_1^i(s)\,\sinh(q_i x) + C_2^i(s)\,\cosh(q_i x) \qquad (7)$$

where $q_i = \sqrt{s/a_i}$ [1/m] (8)

The unknown coefficients C_1^i and C_2^i can be determined from BC's. Let us assume that the temperature and heat flux *is known* at the left-hand side of the i-th layer. Then the temperature distribution $\vartheta^i(s,x)$ and heat-flux distribution $\phi^i(s,x)$ can be expressed by the following matrix equation [2,10]:

$$\begin{bmatrix} \vartheta^i(s,x) \\ \phi^i(s,x) \end{bmatrix} = \begin{bmatrix} a_{11}^i(s) & a_{12}^i(s) \\ a_{21}^i(s) & a_{22}^i(s) \end{bmatrix} \begin{bmatrix} \vartheta_{i-1}(s) \\ \phi_{i-1}(s) \end{bmatrix} \qquad (9)$$

or $$\boxed{\underline{V}(s,x) = \underline{\underline{A}}^i(s,x)\,\underline{V}_o(s)} \qquad (10)$$

where

$$\underline{\underline{A}}^i(s,x) = \begin{bmatrix} \cosh(q_i x) & -\frac{1}{\lambda_i q_i}\sinh(q_i x) \\ -\lambda_i q_i \sinh(q_i x) & \cosh(q_i x) \end{bmatrix} \qquad (11)$$

By subtituting x=d$_i$ into Eq.(9) we get a distributed transfer matrix (DTM) between the left-hand and right-hand surface variables of the i-th layer! Considering each layer as a 2x2 input-output system we can easily determine the overall DTM between the left-hand side and right-hand side surface variables (see Fig.2.):

$$\boxed{\underline{V}_N(s) = \underline{\underline{A}}(s)\,\underline{V}_o(s)} \qquad (12)$$

where

$$\underline{\underline{A}}(s) = \prod_{i=N}^{1} \begin{bmatrix} \cosh(q_i d_i) & -\frac{1}{\lambda_i q_i}\sinh(q_i d_i) \\ -\lambda_i q_i \sinh(q_i d_i) & \cosh(q_i d_i) \end{bmatrix} \qquad (13)$$

After expressing the dynamic relation between the two surfaces we have to take into account the BC's. Eq.(5) and (6) can be expressed as follows:

$$x=0 \qquad \vartheta_{a1} = \begin{bmatrix} 1, & \frac{1}{\alpha_o} \end{bmatrix} \begin{bmatrix} \vartheta_o \\ \phi_o \end{bmatrix} = \underline{b}^T \underline{V}_o \qquad (14)$$

$$x=L \qquad \vartheta_{a2} = \begin{bmatrix} 1, & \frac{1}{\alpha_N} \end{bmatrix} \begin{bmatrix} \vartheta_N \\ \phi_N \end{bmatrix} = \underline{c}^T \underline{V}_N \qquad (15)$$

where superscript T denotes matrix transpose. Substituting these expressions into Eq.(12) we get a the following matrix expression:

$$\underline{\vartheta}_a(s) = \underline{\underline{E}}(s)\,\underline{V}_o(s) \qquad (16)$$

where

$$\underline{\vartheta}_a(s) = \begin{bmatrix} \vartheta_{a1} \\ \vartheta_{a2} \end{bmatrix}; \quad \underline{\underline{E}}(s) = \begin{bmatrix} \underline{b}^T \\ \underline{c}^T\underline{\underline{A}} \end{bmatrix}; \qquad (17)$$

Finally we can express the left-hand side surface variables by the ambient variables by simple inverting matrix E:

$$\boxed{\underline{V}_o(s) = \underline{\underline{E}}^{-1}(s)\,\underline{\vartheta}_a(s) = \underline{\underline{W}}(s)\,\underline{\vartheta}_a(s)} \qquad (18)$$

or

$$\begin{bmatrix} \vartheta_o(s) \\ \phi_o(s) \end{bmatrix} = \begin{bmatrix} G_1(s,0) & H_1(s,0) \\ G_2(s,0) & H_2(s,0) \end{bmatrix} \begin{bmatrix} \vartheta_{a1}(s) \\ \vartheta_{a2}(s) \end{bmatrix} \qquad (19)$$

where $\underline{\underline{W}}(s)$ represents the 2x2 DTM between the surface- and ambient variables. The elements of $\underline{\underline{W}}(s)$ can easily be expressed by the elements of $\underline{\underline{A}}(s)$. Table 1. gives the DTF defined by Eq.(19).

REMARK 1: The elements of matrix W(s) are not independent of one another [10]:

$$\begin{aligned} G_2(s,0) &= \alpha_o\,[\,1 - G_1(s,0)\,] \\ H_2(s,0) &= -\alpha_o\,H_1(s,0) \end{aligned} \qquad (20)$$

REMARK 2: All the solutions remain valid for a linear inner heat source $S_i(t,x) = B_i\,\vartheta^i(t,x)$!

TRANSFER FUNCTION	$\omega \to 0$	$\omega \to \infty$
$G_1(s,0) = \dfrac{\alpha_o a_{22} - \alpha_o\alpha_N a_{12}}{\alpha_o a_{22} + \alpha_N a_{11} - \alpha_o\alpha_N a_{12} - a_{21}}$	$\dfrac{\alpha_o+\alpha_o\alpha_N R_w}{\alpha_o+\alpha_N+\alpha_o\alpha_N R_w}$	$\dfrac{\alpha_o}{\alpha_o+\alpha_N+q_w}$
$H_1(s,0) = \dfrac{\alpha_N}{\alpha_o a_{22} + \alpha_N a_{11} - \alpha_o\alpha_N a_{12} - a_{21}}$	$\dfrac{\alpha_N}{\alpha_o+\alpha_N+\alpha_o\alpha_N R_w}$	0
$G_2(s,0) = \dfrac{\alpha_o\alpha_N a_{11} - \alpha_o a_{21}}{\alpha_o a_{22} + \alpha_N a_{11} - \alpha_o\alpha_N a_{12} - a_{21}}$	$\dfrac{\alpha_o\alpha_N}{\alpha_o+\alpha_N+\alpha_o\alpha_N R_w}$	α_o
$H_2(s,0) = \dfrac{-\alpha_o\alpha_N}{\alpha_o a_{22} + \alpha_N a_{11} - \alpha_o\alpha_N a_{12} - a_{21}}$	$\dfrac{-\alpha_o\alpha_N}{\alpha_o+\alpha_N+\alpha_o\alpha_N R_w}$	0
$R_w = \sum_{i=1}^{N} \frac{d_i}{\lambda_i}$; $\quad q_w = \sum_{i=1}^{N} \lambda_i q_i$; $\quad q_i = \sqrt{s/a_{ti}}$;		

Table 1. Distributed transfer functions of a multilayer structure. For sake of simplicity we dropped the argument s of $a_{ij}(s)$.

REMARK 3: In a similar way we can express the DTM of the temperature- and heat-flux distribution at any location of x:

$$\underline{V}_i(s,x) = \underline{\underline{A}}^{i}(s,x) \prod_{k=i-1}^{1} \underline{\underline{A}}^{k}(s,d_k)\, \underline{\underline{E}}^{-1}(s)\, \underline{\theta}_a(s) \quad (21)$$

4. PARTIAL FRACTIONAL EXPANSION

Due to the distributed nature of the problem the DTF's have *infinite number of poles*. A classical and widely used model reduction technique is based on the partial fraction expansion of the DTF's. Then, for example, $G_1(s,x)$ can be expressed by the following fractional form:

$$G_1(s,x) = \frac{N_1(s,x)}{D_1(s)} = \sum_{k=1}^{\infty} \frac{A_k(x)}{s - p_k} \quad (22)$$

where $A_k(x)$ is the k-th reziduum and p_k is the k-th pole of the DTF [9]. The poles of the system are determined by the zeros of the denominator of the transfer functions:

$$D(s) = a_{21} + \alpha_o a_{22} + \alpha_N a_{11} - \alpha_o \alpha_N a_{12} = 0 \quad (23)$$

Note that a_{ij} (i,j=1,2) is defined by the matrix product of Eq.(13)! Based on Eq.(22) we can get a reduced order model by simply truncating the expansion and considering only the first M terms:

$$\tilde{G}_1^M(s,x) = \sum_{k=1}^{M} \frac{A_k(x)}{s - p_k} = \frac{B_{1M}(s,x)}{A_{1M}(s)} \quad (24)$$

Unfortunately we have to ancounter series problems applying this method. These are shortly mentioned as follows:

i.) we can determine the residuum only for a single layer structure [9]. It is almost impossible for a multilayer one.

ii.) we have little *a priory* information on the *convergence* of this serie (i.e. how many terms we need for a prescribed accuracy),

iii.) the resulting *reduced order model* may exhibit a *non-minimum phase* character which contradicts with the *unconditional minimum phase nature* of the physical process!

EXAMPLE 1:
To demonstrate the slow convergence rate of the partial fraction model reduction let us consider a single layer wall of an apartment building in Utrecht. The parameters are [10]: $\rho = 2500$ [kg/m^3], $c = 840$ [J/kgK], $\lambda = 1{,}9$ [W/mK], $d = 0{,}2$ [m].
Fig.3. compares the precise frequency response of $G_1(s,0)$ with different reduced order models. We can clearly see that even a 6th order model gives a quite inaccurate response in the required 4 decade frequency domain. The figure also shows the reduced order model of the author having the 4th order transfer function:

$$\tilde{G}_1(s,0) = \frac{0{,}648\ (1+180s)(1+15{,}2s)(1+1{,}54s)}{(1+500{,}5s)(1+42s)(1+4{,}6s)(1+0{,}5s)} \quad (25)$$

We have to note that in spite of its grave problems the partial fraction model reduction can be applied with great care for single layer structures in a limited frequency range [4,8].

5. NEW MODEL REDUCTION TECHNIQUE

Besides that the above mentioned method does not give very accurate result it can not be applied for multilayer structure at all. To overcome the difficulty the author propose the following new model reduction technique. The basic idea is that insteda of taking the first M terms of Eq.(22) a low order approximation of $G_1(s,0)$ can be derived by minimizing the following criteria in the frequency domain:

$$I = \min \int_{\omega_1}^{\omega_2} |G_1(s,0) - \tilde{G}_1(s)|\, d\omega \quad (26)$$

where ω_1 and ω_2 is the lower- and upper frequency limit of the desired frequency domain To determine the pole-zero location of the reduced order model $\tilde{G}_1(s)$ we can utilize the special high- frequency properties of $G_1(s,0)$:

$$\lim_{\omega \to \infty} G_1(s,0) \longrightarrow \frac{\alpha_o}{\alpha_o + \alpha_N + \sum_{i=1}^{N} \lambda_i \sqrt{s/a_i}} \quad (26)$$

In other terms this means that the slope of $G_1(s,0)$ goes to -10 [dB/dec] for high frequencies [3] as we have seen in Fig.3. [10]! Now the poles and zeros of the reduced order model can be determined to give this slope in "*average*" in the frequency domain. The first pole is determined as the first zero of Eq.(23) and the steady-state value from Table 1. All the other poles and zeros are determined by "*averaging*" the frequency response around the precise value as indicated on Fig.4. The method has several advantage:

i.) it *always* provides a *minimum phase model*,

ii.) it is easy to computerize,

iii.) it usually gives a 3rd or 4th order accurate model in a 4 decade frequency range [10].

The proposed method has extensively been used in many parctical applications at the University of Twente. The aim of these applications were to develope general building models for optimal controller and/or optimal heat delivary design [5,11]. In the building models all the internal and external walls were considered as a distributed parameter systems described by their DTF's given in Table 1. In the Netherlands most of the external walls have 4 different layers. In contrary of classical building models relying on an electric analogue network to describe the building dynamics [1] our method proved superior

in all applications [12]. In fact an optimal heat delivery system was designed based on this concept and applied in real-time in an apartment building having 120 apartments in Utrecht [11].

EXAMPLE 2: As a second example let us consider a typical 4 layer wall structure shown in Fig.5. The actual data are given in Table 2. and they belong to the external wall of a private house in Enschede [10]. Fig.6. shows how well our reduced order model approximates the precise DTF of $G_1(s,0)$ in 4 decades! In fact there is almost *no difference* between the DTF and the reduced order model! The reduced order model is given by [10]:

$$\tilde{G}_1(s,0) = 0,91 \frac{(1+112s)(1+11s)(1+1,6s)}{(1+393s)(1+25s)(1+3,8s)(1+0,6s)} \quad (27)$$

We can clearly see how the zeros of the reduced order model are imbedded between two consequtive poles leading in "*average*" to a -10 [dB/dec] slope.

i	material	ρ_i [kg/m^3]	c_i [J/kg°C]	λ_i [W/mK]	d_i [m]	R_i [m^{2}K/W]	T_{wi} [min]
1	brick	2000	840	1,00	0,10	0,10	280,0
2	insu-lation	-	-	0,059	0,05	0,85	-
3	air-gap	-	-	(0,14)	0,02	0,14	-
4	brick	2000	840	0,80	0,10	0,125	350,0
α_o = 8 [W/m^{2}K] ; α_N = 25 [W/m^{2}K] ; $T_{wi} = d_i^2/a_i$ [min]							

Table 2. Data of a 4 layer external wall in a dwelling house, Enschede.

CONCLUSIONS

We have developed the DTF's of a multilayer (or sandwitch) solid structure between the ambient- and surface variables. Whatever complex form these transfer functions have it is possible to calculate them in the frequency domain for any given number of layers. Utilizing the special high-frequency properties of the DTF between the ambient- and surface temperature we proposed a new model reduction technique. For a user specified frequency range it results in a *low order* and *minimum phase* process model. The proposed technique have been used in developing distributed parameter building models.

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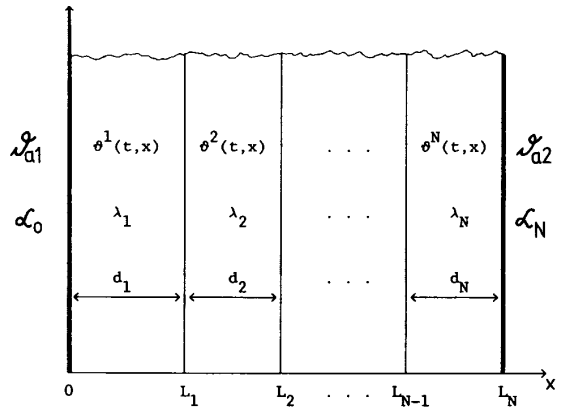

Fig.1. Model of a multilayer solid structure

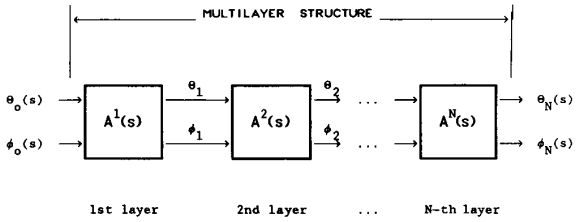

Fig.2. Interconnection of the layer's transfer matrices. Each layer is described by a 2x2 distributed transfer matrix.

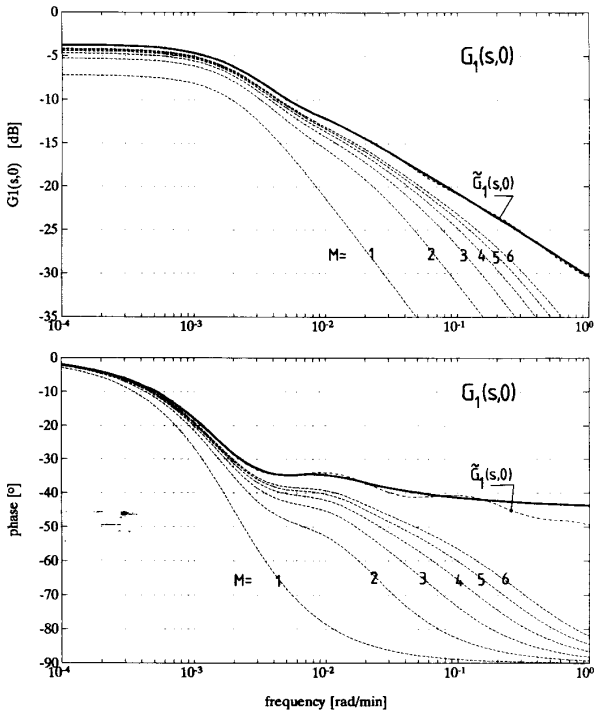

Fig.3. Frequency response of $G_1(s,0)$ and its partial fraction approximation with different order (up to 6). $\tilde{G}_1(s,0)$ denotes the author's reduced order approximation.

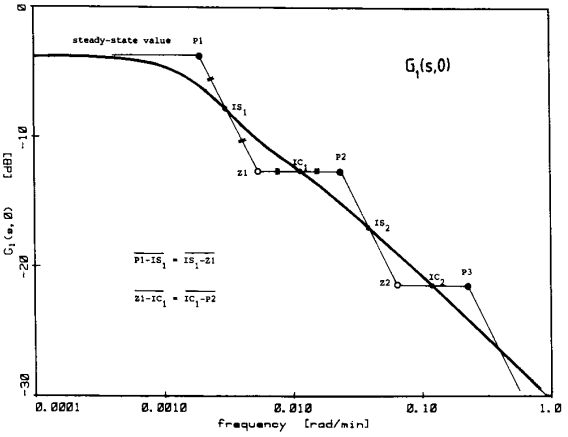

Fig.4. Pole-zero location of the new model reduction technique of $G_1(s,0)$. Note the imbedding of zeros between the consecutive poles!

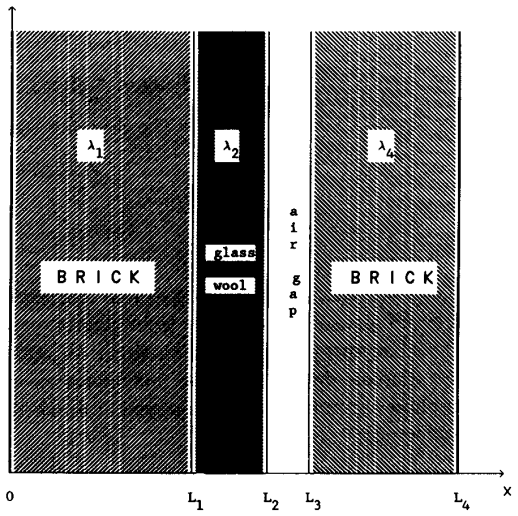

Fig.5. Internal structure of an external wall of a house in Enschede having 4 layers.

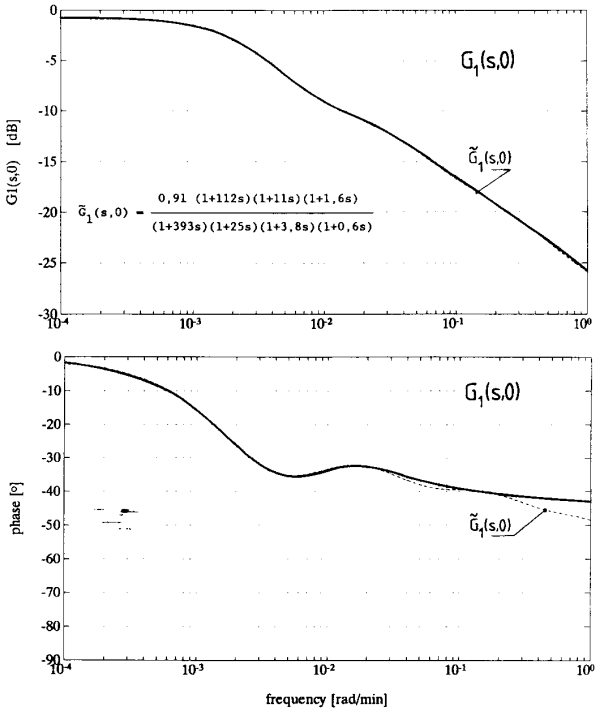

Fig.6. Frequency response of $G_1(s,0)$ in Example 2 and its low order approximation.