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Synchronization for heterogeneous time-varying networks with non-introspective, non-minimum-phase agents in the presence of external disturbances with known frequencies

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Abstract— **This paper considers regulated output synchronization for heterogeneous networks, where agents are non-introspective (i.e. agents have no access to their own states or outputs), non-minimum-phase agents in the presence of external disturbances, including process disturbances and measurement noise, with known frequencies. Moreover, the communication network is directed, weighted and time-varying. A purely decentralized linear time-invariant protocol based on a high-gain observer is designed for each agent to achieve regulated output synchronization, i.e. agents' outputs are asymptotically regulated to a given reference trajectory, even in the presence of external disturbance with known frequencies.**

I. Introduction

The problem of synchronization among agents in a multi-agent system has received substantial attention in recent years. The objective of synchronization is to secure an asymptotic agreement on a common state or output trajectory through decentralized control protocols (see [1], [9], [14], [18] and references therein). The main focus has been on synchronization for homogeneous networks (i.e., agents have identical dynamics) with introspective agents (i.e., the agent has access to part of their own states).

For the observer design, many works require the availability of an additional communication channel using the same network structure for the exchange of controller states. For example, see [10], [11], [15], [19], [4], [20] and references therein. There also exist some works that dispense with that additional communication. In [15] (homogeneous network) and [4] (heterogeneous network but introspective), this additional communication of controller states is already avoided. Recently, [2] addressed the output synchronization for heterogeneous networks with non-introspective agents without this additional communication, while agents are assumed minimum phase.

For agents that are affected by external disturbances with upper bound power, the notion of almost synchronization was brought up in [13] (introspective) and [12] (homogeneous, non-introspective), where the goal of their work is to reduce the impact of disturbances on the synchronization error to an arbitrary degree of accuracy (expressed in the $\mathcal{H}_\infty$ norm).

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These references assume the availability of an additional communication channel. [21] extended the above work to heterogeneous networks of introspective agents without exchange of controller states. When the external disturbances are stochastic noises with an upper bound rate, $\mathcal{H}_2$ norm minimization of the transfer function is utilized under a fixed network (e.g., [24]). For the disturbance with known frequencies, almost disturbance decoupling is utilized to achieve exact synchronization in a network(e.g., [22]).

The majority of the works assume the communication network is fixed, i.e., the network graph is fixed. Extensions to time-varying networks are done in the framework of switching networks. Synchronization with time-varying networks is studied utilizing concepts of dwell-time and average dwell-time (e.g., [16], [17], [8]). It is assumed that the time-varying network switches among a finite set of network graphs. Recently, an infinite set of network graphs is proposed in [23], [22] and [24]. In [25], switching laws are designed to achieve synchronization.

This paper also considers synchronization problem for time-varying multi-agent systems/networks with non-introspective agents affected by disturbances and measurement noises with known frequencies (i.e. they are generated by linear autonomous exosystems), as considered in [22]. The difference is that in this paper we allow non-minimum-phase agents, while in [22] only minimum-phase agents are considered. That means we can achieve regulated output synchronization for a larger class of networks in the presence of known-frequency disturbances. In this paper, we adopted the additional channel for the communication of controller states during the observer design.

A. Notations and definitions

Given a matrix $A \in \mathbb{C}^{m\times n}$, A' denotes its conjugate transpose, $\|A\|$ is the induced 2-norm, and $\lambda_i(A)$ denotes its $i'th$ eigenvalue when $m = n$. A square matrix A is said to be Hurwitz stable if all its eigenvalues are in the open left half complex plane. We denote by blkdiag$\{A_i\}$, a block-diagonal matrix with $A_1, \dots, A_N$ as the diagonal elements, and by col$\{x_i\}$, a column vector with $x_1, \dots, x_N$ stacked together, where the range of index i can be identified from the context. $A \otimes B$ depicts the Kronecker product between A and B. I_n denotes the n-dimensional identity matrix and 0_n denotes $n \times n$ zero matrix; sometimes we drop the subscript if the dimension is clear from the context.

A *weighted directed graph* $\mathcal{G}$ is defined by a triple $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, \dots, N\}$ is a node set, $\mathcal{E}$ is a set

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of pairs of nodes indicating connections among nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the weighting matrix, and $a_{ij} > 0$ iff $(i,j) \in \mathcal{E}$. Each pair in $\mathcal{E}$ is called an *edge*. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1,\dots,i_k\}$ such that $(i_j,i_{j+1}) \in \mathcal{E}$ for $j = 1,\dots,k-1$. A *directed tree* with *root* r is a subset of nodes of the graph $\mathcal{G}$ such that a path exists between r and every other node in this subset. A *directed spanning tree* is a directed tree containing all the nodes of the graph. For a weighted graph $\mathcal{G}$, a matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^{N} a_{ik}, i = j, \\ -a_{ij}, \quad i \neq j, \end{cases}$$

is called the *Laplacian matrix* associated with the graph $\mathcal{G}$. In the case where $\mathcal{G}$ has non-negative weights, L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $\mathbf{1}$.

Definition 1: Let $\mathcal{L}_N \subset \mathbb{R}^{N\times N}$ be the family of all possible Laplacian matrices associated to a graph with N agents. We denote by $\mathcal{G}_L$ the graph associated with a Laplacian matrix $L \in \mathcal{L}_N$. Then, a time-varying graph $\mathcal{G}_t$ with N agents has such a definition as

$$\mathcal{G}_t(t) = \mathcal{G}_{\sigma(t)},$$

where $\sigma : \mathbb{R} \to \mathcal{L}_N$ is a piecewise constant, right-continuous function with minimal dwell-time τ (see [5]), i.e. $\sigma(t)$ remains fixed for $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}$ and switches at $t = t_k$, $k = 1,2,\dots$ where $t_{k+1} - t_k \geq \tau$ for $k = 0,1,\dots$. For ease of presentation we assume $t_0 = 0$.

Definition 2: A matrix pair (A,C) is said to contain the matrix pair (S,R) if there exists a matrix Π such that $\Pi S = A\Pi$ and $C\Pi = R$.

Remark 1: Definition 2 implies that for any initial condition $\omega(0)$ of the system $\dot{\omega} = S\omega$, $y_r = R\omega$, there exists an initial condition $x(0)$ of the system $\dot{x} = Ax$, $y = Cx$, such that $y(t) = y_r(t)$ for all $t \geq 0$ ([7]).

II. Heterogeneous multi-agent systems

We consider a multi-agent system/network consisting of N non-identical non-introspective agents described by

$$\begin{cases} \dot{x}_i = A_i x_i + B_i u_i + E_i w_i, \\ y_i = C_i x_i + D_i u_i + D_{wi} w_i, \end{cases} \qquad (i = 1,\dots,N) \qquad (1)$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, and $y_i \in \mathbb{R}^p$ are the state, input and output of agent i while $w_i \in \mathbb{R}^{m_{wi}}$ is the external disturbance with known frequencies, which can be generated by the following exosystem:

$$\begin{aligned} \dot{x}_{wi} &= S_i x_{wi}, \\ w_i &= R_i x_{wi}, \end{aligned} \qquad (2)$$

where $x_{wi} \in \mathbb{R}^{n_{wi}}$. Note that no initial conditions are imposed on the exosystem, for the technique we will use in this paper will reject any disturbance with known frequencies. It is clear that $E_i w_i$ represents the process disturbance while $D_{wi} w_i$ represents the measurement noise.

We make the following assumptions on the agent dynamics.

Assumption 1: For each agent $i \in \{1,\dots,N\}$, we have

- (A_i, B_i, C_i) is right-invertible;
- (A_i, B_i) is stabilizable, and (A_i, C_i) is detectable;

It is worth noting that we do not assume agents to be minimum-phase, that is, agents can have invariant zeros in the closed-right half complex plane. In [22], we only allow minimum-phase agents.

The agents considered in this paper are non-introspective, which means that they have no access to any of their states, and that the only information that is available for the controller design comes from the network. In particular, the network provides each agent with a linear combination of its own output relative to that of other neighboring agents, that is, agent $i \in \{1,\dots,N\}$, has only access to the quantity

$$\zeta_i(t) = \sum_{j=1}^{N} a_{ij}(t)(y_i(t) - y_j(t)), \qquad (3)$$

where $a_{ij}(t) \geq 0$ and $a_{ii}(t) = 0$, is a piecewise constant and right-continuous function of time t, indicating a time-varying communication among agents. We will use a time-varying graph $\mathcal{G}_t$ to describe such a time-varying communication network among agents. At time t, the weight of graph edges is given by the coefficient $a_{ij}(t)$. The Laplacian matrix associated with $\mathcal{G}_t$ is defined as $L_t = [\ell_{ij}(t)]$. In terms of the coefficients of L_t, ζ_i can be rewritten as

$$\zeta_i(t) = \sum_{j=1}^{N} \ell_{ij}(t) y_j(t). \qquad (4)$$

III. Regulated output synchronization

In this section, we consider the regulated output synchronization problem for heterogeneous time-varying networks with non-introspective agents defined in Section II, where the outputs of agents are asymptotically regulated to a reference trajectory, even in the presence of external disturbances with known frequencies. The reference trajectory is generated by an autonomous system

$$\begin{cases} \dot{x}_0 = A_0 x_0, \qquad x_0(0) = x_{00}, \\ y_0 = C_0 x_0, \end{cases} \qquad (5)$$

where $x_0 \in \mathbb{R}^{n_0}$, $y_0 \in \mathbb{R}^p$. Moreover, we assume that (A_0, C_0) is observable, all eigenvalues of A_0 are in the closed right half complex plane.

We need an extra assumption on the agents with the given reference system.

Assumption 2: (A_i, B_i, C_i) has no invariant zeros in the closed right-half complex plane that coincide with the eigenvalues of A_0 and/or S_i.

In order to have all the agents follow the reference trajectory, it is clear that a non-empty subset of agents must have knowledge of their outputs relative to the reference trajectory y_0 generated by the reference system. Specially, let π be a subset of $\mathcal{V}$. We assume that each agent has access to the quantity

$$\psi_i = \iota_i(y_i - y_0), \qquad \iota_i = \begin{cases} 1, & i \in \pi, \\ 0, & i \notin \pi. \end{cases} \qquad (6)$$

In order to achieve regulated output synchronization for all agents, the following assumption is clearly necessary.

Assumption 3: Every node of the network graph $\mathcal{G}_t$ at time t, is a member of a directed tree which has a root contained in the set π.

In the following, we will refer to the node set π as *root set* in view of Assumption 3 (A special case is when π consists of a single element corresponding to the root of a directed spanning tree of $\mathcal{G}_t$ at time t).

Note that the reference system can be viewed as a new root node, denoted as node 0. This time-varying network with the reference system will be referred to as the augmented time-varying network, and will be described by a graph $\tilde{\mathcal{G}}_t$. Based on Assumption 3, this augmented time-varying network will contain a directed spanning tree with node 0 as its root. The associated Laplacian matrix, denoted by $\tilde{L}_t = [\tilde{\ell}_{ij}(t)]$, is defined as

$$\tilde{L}_t = \begin{pmatrix} 0 & 0 \\ -\operatorname{col}\{\iota_i\} & L_t + \operatorname{diag}\{\iota_i\} \end{pmatrix}. \tag{7}$$

In terms of the Laplacian matrix $\tilde{L}_t$, the quantity ζ_i in (4) will be updated as

$$\tilde{\zeta}_i = \sum_{j=0}^{N} \tilde{\ell}_{ij}(t) y_j, \tag{8}$$

for $i = 0, 1, \dots, N$.

Define the matrix $\bar{L}_t = [\bar{\ell}_{ij}(t)]$ as,

$$\bar{L}_t = L_t + \operatorname{diag}\{\iota_i\}.$$

Clearly, for each time t, $\bar{L}_t$ is not a Laplacian matrix associated with some graph since it does not have a zero row sum. From [3, Lemma 7], all eigenvalues of $\bar{L}_t$ at time t are in the open right-half complex plane. Moreover, these eigenvalues are the non-zero eigenvalues of the Laplacian matrix $\tilde{L}_t$ for any fixed time t.

We define next a set of network graphs for the augmented network with the external reference system as follows:

Definition 3: Given a root set π, and real values $\alpha, \beta, \varphi > 0$ and positive integer N, $\mathbb{G}^{\varphi,N}_{\alpha,\beta,\pi}$ is the set of directed graphs composed of N nodes and the node of the reference system, such that every augmented network graph $\tilde{\mathcal{G}} \in \mathbb{G}^{\varphi,N}_{\alpha,\beta,\pi}$ satisfies the properties:

- Assumption 3 holds for the root set π.
- The eigenvalues $\tilde{\lambda}_0 \leq \dots \leq \tilde{\lambda}_N$ of the Laplacian matrix $\tilde{L}$ associated with the augmented network graph $\tilde{\mathcal{G}}$ satisfy $Re\{\tilde{\lambda}_i\} > \beta$ and $|\tilde{\lambda}_i| < \alpha$ for $i \in \{1, 2, \dots, N\}$, and $\tilde{\lambda}_0 = 0$. Note that $\tilde{\lambda}_i$ ($i = 1, \dots, N$) are the eigenvalues of matrix $\bar{L}$.
- The condition number[1] of $\tilde{L}$ is bounded by φ.

Remark 2: Note that for undirected graphs the condition number of the Laplacian matrix is always bounded. Moreover, if we have a *finite* set of possible graphs each of which has a directed spanning tree then there always exists a set of the form $\mathbb{G}^{\varphi,N}_{\alpha,\beta,\pi}$ for suitable $\alpha, \beta, \varphi > 0$ and N containing these graphs. The only limitation is that we cannot find **one** protocol for a sequence of graphs converging to a graph without a spanning tree or whose Laplacian either diverges or approaches some ill-conditioned matrix.

Then, a set of time-varying network graphs for augmented time-varying communication network with the external reference system is defined as follows:

Definition 4: Given a root set π, real values $\alpha, \beta, \varphi, \tau > 0$ and a positive integer N, we define the set of time-varying graphs $\mathbb{G}^{\varphi,\tau,N}_{\alpha,\beta,\pi}$ composed of N nodes and the node of the reference system, as the set of all time-varying graphs $\tilde{\mathcal{G}}_t$ for which

$$\tilde{\mathcal{G}}_t(t) = \tilde{\mathcal{G}}_{\sigma(t)} \in \mathbb{G}^{\varphi,N}_{\alpha,\beta,\pi}$$

for all $t \in \mathbb{R}$.

Remark 3: Note that the minimal dwell-time is assumed to avoid chattering problems. However, it can be arbitrarily small.

Define $e_i := y_i - y_0$ as the regulated synchronization error for agent $i \in \{1, \dots, N\}$ and $\mathbf{e} = \operatorname{col}\{e_i\}$. Then, we formulate the problem of regulated output synchronization as follows.

Problem 1: Consider a multi-agent system (1), (3), and reference system (5). For a given root set π, a positive integer N, and real numbers $\alpha, \beta, \varphi, \tau > 0$ and defining a set of time-varying network graphs $\mathbb{G}^{\varphi,\tau,N}_{\alpha,\beta,\pi}$, the *regulated output synchronization* problem for heterogeneous networks under time-varying graphs and in the presence of external disturbances with known frequencies is to find, if possible, a linear time-invariant dynamic protocol such that, for any time-varying graph $\tilde{\mathcal{G}}_t \in \mathbb{G}^{\varphi,\tau,N}_{\alpha,\beta,\pi}$, and for all initial conditions of the agents and the reference system, the regulated output synchronization error satisfies

$$\lim_{t\to\infty} \|\mathbf{e}(t)\| = 0. \tag{9}$$

As we emphasized at the beginning, we allow information exchanging among agents by using the same communication network. Assume that the controller of agent i supplies information η_i, which will be defined exactly later on during the controller design. Then, the extra information available for controller design is

$$\hat{\zeta}_i = \sum_{j=0}^{N} \tilde{\ell}_{ij} \eta_j, \tag{10}$$

where η_0 is set zero. The main result of this paper is stated as the following theorem.

Theorem 1: Consider a multi-agent system (1) and(3), and reference system (5). Let a root set π, a positive integer N and real numbers $\alpha, \beta, \varphi, \tau > 0$ be given, and hence a set of network graphs $\mathbb{G}^{\varphi,\tau,N}_{\alpha,\beta,\pi}$ be defined.

Under Assumptions 1, 2 and 3, the regulated output synchronization problem is solvable, i.e., there exists a family of distributed dynamic protocols, parametrized in terms of

[1]In this context, we mean by condition number the minimum of $\|U\|\|U^{-1}\|$ over all possible matrices U whose columns are the (generalized) eigenvectors of the expanded Laplacian matrix $\bar{L}$.

high-gain parameter ε, of the form:

$$\begin{cases} \dot{\chi}_i = \mathcal{A}_i(\varepsilon)\chi_i + \mathcal{B}_i(\varepsilon)\begin{pmatrix}\tilde{\zeta}_i\\ \hat{\zeta}_i\end{pmatrix}, \\ \tilde{u}_i = C_i(\varepsilon)\chi_i + \mathcal{D}_i(\varepsilon)\begin{pmatrix}\tilde{\zeta}_i\\ \hat{\zeta}_i\end{pmatrix}, \end{cases} \quad i \in (1,\dots,N) \tag{11}$$

where $\chi_i \in \mathbb{R}^{q_i}$, such that for any time-varying graph $\tilde{\mathcal{G}}_t \in \mathbb{G}^{\varphi,\tau,N}_{\alpha,\beta,\pi}$, and for all initial conditions of agents and the reference system, the regulated output synchronization error satisfies (9).

In particular, there exists an $\varepsilon^* \in (0,1]$ such that for any $\varepsilon \in (0,\varepsilon^*]$, the protocol (11) solves the regulated output synchronization problem.

Proof: To regulate the outputs of agents to the reference trajectory and at the same time reject the external disturbance, we will first design a pre-compensator such that the interconnection of the pre-compensator and agents dynamics contain the reference system (5) and the exosystem (2).

Let

$$x_{ir} = \operatorname{col}\{x_0, x_{wi}\},$$

$$S_{ir} = \operatorname{blkdiag}\{A_0, S_i\}, R_{ir} = \operatorname{blkdiag}\{C_0, R_i\}.$$

Then we get the following dynamics:

$$\begin{cases} \dot{x}_{ir} = S_{ir}x_{ir}, \\ \begin{pmatrix} y_0 \\ w_i \end{pmatrix} = R_{ir}x_{ir}. \end{cases} \tag{12}$$

Let

$$\bar{R}_{0r} = \begin{pmatrix} C_0 & 0 \end{pmatrix} \text{ and } \bar{R}_{ir} = \begin{pmatrix} 0 & R_i \end{pmatrix}.$$

Then, we have

$$y_0 = \bar{R}_{0r}x_{ir} \text{ and } w_i = \bar{R}_{ir}x_{ir}.$$

When A_i and S_{ir} have common eigenvalues (modes), the common eigenvalues in S_{ir} will be removed first. For this process refer to [3]. Here we assume A_i and S_{ir} have no common eigenvalues. Then, with Assumption 2, the following regulation equations

$$\begin{aligned} A_i\Pi_i + B_i\Gamma_i + E_i\bar{R}_{ir} &= \Pi_i S_{ir}, \\ C_i\Pi_i + D_{wi}\bar{R}_{ir} + D_i\Gamma_i &= \bar{R}_{0r}, \end{aligned} \tag{13}$$

have unique solution of Π_i and Γ_i with $i = 1,\dots,N$ (see [?]). If (Γ_i, S_{ir}) is not observable, we can construct the observable subsystem $(\Gamma_{i,o}, S_{ir,o})$; otherwise simply set $\Gamma_{i,o} = \Gamma_i$ and $S_{ir,o} = S_{ir}$. Then, the pre-compensator for agent i is designed as

$$\begin{aligned} \dot{x}_{pi} &= S_{ir,o}x_{pi} + B_{pi}\tilde{u}_i, \\ u_i &= \Gamma_{i,o}x_{pi}, \end{aligned} \tag{14}$$

for $i \in \{1,\dots,N\}$, where $x_{pi} \in \mathbb{R}^{n_{pi}}$, and B_{pi} is chosen to guarantee that no invariant zeros are introduced by the pre-compensator (see [6]).

Let $\tilde{x}_i = \operatorname{col}\{x_i, x_{pi}\}$. The interconnection of agent (1) and the pre-compensator (14) can be represented as

$$\begin{aligned} \dot{\tilde{x}}_i &= \tilde{A}_i\tilde{x}_i + \tilde{B}_i\tilde{u}_i + \tilde{E}_i w_i, \\ y_i &= \tilde{C}_i\tilde{x}_i + \tilde{D}_{wi}w_i, \end{aligned} \tag{15}$$

where $\tilde{x}_i \in \mathbb{R}^{\tilde{n}_i}$ and $\tilde{n}_i = n_i + n_{pi}$. In the case that (Γ_i, S_{ir}) is observable, by simply constructing $\tilde{\Pi}_i = \operatorname{col}\{\Pi_i, I\}$ and using (13), we are able to achieve the equality

$$\begin{aligned} \tilde{A}_i\tilde{\Pi}_i + \tilde{E}_i\bar{R}_{ir} &= \tilde{\Pi}_i S_{ir}, \\ \tilde{C}_i\tilde{\Pi}_i + \tilde{D}_{wi}\bar{R}_{ir} &= \bar{R}_{0r}. \end{aligned} \tag{16}$$

When (Γ_i, S_{ir}) is not observable, the construction of $\tilde{\Pi}_i$ needs more work and is given in the Appendix.

Now define $\bar{x}_i = \tilde{x}_i - \tilde{\Pi}_i x_{ir}$. Then we find that

$$\begin{aligned} \dot{\bar{x}}_i &= \tilde{A}_i\tilde{x}_i + \tilde{B}_i\tilde{u}_i + \tilde{E}_i w_i - \tilde{\Pi}_i S_{ir}x_{ir} \\ &= \tilde{A}_i\tilde{x}_i + \tilde{B}_i\tilde{u}_i + \tilde{E}_i\bar{R}_{ir}x_{ir} - (\tilde{A}_i\tilde{\Pi}_i + \tilde{E}_i\bar{R}_{ir})x_{ir} \\ &= \tilde{A}_i\bar{x}_i + \tilde{B}_i\tilde{u}_i, \end{aligned}$$

and

$$\begin{aligned} e_i &= y_i - y_0 \\ &= \tilde{C}_i\tilde{x}_i + \tilde{D}_{wi}w_i - \bar{R}_{0r}x_{ir} \\ &= \tilde{C}_i\tilde{x}_i + \tilde{D}_{wi}\bar{R}_{ir}x_{ir} - (\tilde{C}_i\tilde{\Pi}_i + \tilde{D}_{wi}\bar{R}_{ir})x_{ir} \\ &= \tilde{C}_i\bar{x}_i. \end{aligned}$$

Design a state-feedback controller

$$\tilde{u}_i = F_i\bar{x}_i, \tag{17}$$

where F_i is chosen such that $\tilde{A}_i + \tilde{B}_iF_i$ is Hurwitz. Therefore, we achieve that $\lim_{t\to\infty}\bar{x}_i(t) = 0$ and $\lim_{t\to\infty}e_i(t) = 0$. That implies that all agents follow exactly the given reference trajectory y_0, even in the presence of external disturbances with known frequencies.

However, $\bar{x}$ is not available for the above controller design, for all agents are non-introspective. Next, we will design a high-gain observer to produce an estimation of $\bar{x}_i$, denoted by $\hat{\bar{x}}_i$ $(i = 1,\dots,N)$.

Denote $n = \max_i\{\tilde{n}_i\}$, and define a matrix

$$T_i = \begin{pmatrix} \tilde{C}_i \\ \vdots \\ \tilde{C}_i\tilde{A}_i^{n-1} \end{pmatrix}.$$

Note that T_i is not necessarily a square matrix; however, the observability of $(\tilde{A}_i, \tilde{C}_i)$ ensures that T_i is injective, which implies that $T_i'T_i$ is nonsingural. Let $\chi_i = T_i\bar{x}_i$. Then, we can write the dynamics of χ_i as follows:

$$\begin{aligned} \dot{\chi}_i &= (\mathcal{A} + \mathcal{L}_i)\chi_i + \mathcal{B}_i\tilde{u}_i, \quad \chi_i(0) = T_i\bar{x}_i(0), \\ e_i &= C\chi_i, \end{aligned} \tag{18}$$

where $i = 1,\dots,N$, and

$$\mathcal{A} = \begin{pmatrix} 0 & I_{p(n-1)} \\ 0 & 0 \end{pmatrix},\ C = \begin{pmatrix} I_p & 0 \end{pmatrix},\ \mathcal{L}_i = \begin{pmatrix} 0 \\ L_i \end{pmatrix},\ \mathcal{B}_i = T_i\tilde{B}_i$$

and where $L_i = \tilde{C}_i\tilde{A}_i^n(T_i'T_i)^{-1}T_i'$.

Let $\varepsilon \in (0,1]$ be a high-gain parameter and define $S_\varepsilon = \operatorname{blkdiag}\{I_p\varepsilon^{-1},\dots,I_p\varepsilon^{-n}\}$. The high-gain observer for the estimation of χ_i is constructed as

$$\begin{aligned} \dot{\hat{\chi}}_i &= (\mathcal{A} + \mathcal{L}_i)\hat{\chi}_i + \mathcal{B}_i\tilde{u}_i + S_\varepsilon PC'(\tilde{\zeta}_i - \hat{\zeta}_i), \\ \eta_i &= C\hat{\chi}_i. \end{aligned} \tag{19}$$

It is easy to verify that

$$\hat{\zeta}_i = \sum_{j=1}^{N} \tilde{\ell}_{ij}(t) C \hat{\chi}_i \text{ and } \hat{\bar{x}}_i = (T_i' T_i)^{-1} T_i' \hat{\chi}_i.$$

Since $(\mathcal{A}, C)$ is observable, P is the unique solution of the algebraic Riccati equation

$$\mathcal{A}P + P\mathcal{A}' - 2\beta PC'CP + I_{np} = 0. \tag{20}$$

We will first prove the following lemma.

Lemma 1: There exists an $\varepsilon^* \in (0,1]$ such that for any $\varepsilon \in (0, \varepsilon^*]$,

$$\lim_{t\to\infty} (\chi_i - \hat{\chi}_i) = 0, \tag{21}$$

for all $i \in \{1, \ldots, N\}$ and for all time-varying graphs $\tilde{\mathcal{G}}_t \in \mathbb{G}^{\varphi,\tau,N}_{\alpha,\beta,\pi}$.

Proof: For each $i \in \{1, \ldots, N\}$, let $\bar{\chi}_i = \chi_i - \hat{\chi}_i$. Then

$$\dot{\bar{\chi}}_i = (\mathcal{A} + \mathcal{L}_i)\bar{\chi}_i - S_\varepsilon PC'(\tilde{\zeta}_i - \hat{\zeta}_i). \tag{22}$$

Noting that for each $i \in \{1, \ldots, N\}$, we have $\sum_{j=0}^{N} \tilde{\ell}_{ij}(t) = 0$, and therefore

$$\begin{aligned}\tilde{\zeta}_i = \sum_{j=0}^{N} \tilde{\ell}_{ij}(t) y_j &= \sum_{j=0}^{N} \tilde{\ell}_{ij}(t)(y_j - y_0)\\ &= \sum_{j=1}^{N} \bar{\ell}_{ij}(t) e_j = \sum_{j=1}^{N} \bar{\ell}_{ij}(t) C \chi_j.\end{aligned}$$

Thus,

$$\tilde{\zeta}_i - \hat{\zeta}_i = \sum_{j=1}^{N} \bar{\ell}_{ij}(t) C \bar{\chi}_j.$$

Then, dynamics (22) can be rewritten as

$$\dot{\bar{\chi}}_i = \mathcal{A}\bar{\chi}_i + \mathcal{L}_i\bar{\chi}_i - S_\varepsilon PC'C \sum_{j=1}^{N} \bar{\ell}_{ij}(t)\bar{\chi}_i.$$

Define $\xi_i = \varepsilon^{-1} S_\varepsilon^{-1} \bar{\chi}_i$. Then, we get

$$\varepsilon\dot{\xi}_i = \mathcal{A}\xi_i + \mathcal{L}_{i\varepsilon}\xi_i - PC'C \sum_{j=1}^{N} \bar{\ell}_{ij}(t)\xi_i,$$

where

$$\mathcal{L}_{i\varepsilon} = \begin{pmatrix} 0 \\ \varepsilon^{n+1} L_i S_\varepsilon \end{pmatrix}.$$

Let $\xi = \operatorname{col}\{\xi_i\}$ and $\mathcal{L}_\varepsilon = \operatorname{blkdiag}\{\mathcal{L}_{i\varepsilon}\}$. Then, the dynamics of the complete network becomes

$$\varepsilon\dot{\xi} = [I_N \otimes \mathcal{A} + \mathcal{L}_\varepsilon - \bar{L}_t \otimes PC'C]\xi. \tag{23}$$

Define $U_t^{-1} \bar{L}_t U_t = J_t$, where J_t is the Jordan form of $\bar{L}_t$, and let $v = (U_t^{-1} \otimes I_{pn})\xi$. Then we get

$$\varepsilon\dot{v} = (I_N \otimes \mathcal{A})v + W_{\varepsilon,t} v - J_t \otimes (PC'C)v, \tag{24}$$

where $W_{\varepsilon,t} = (U_t^{-1} \otimes I_{pn})\mathcal{L}_\varepsilon(U_t \otimes I_{pn})$.

By our definition on the set of time-varying graphs, we know that J_t and J_t^{-1} are bounded. Moreover, the boundedness of the condition number guarantees that U_t and U_t^{-1} are both bounded as well. Note that when a switching of the network graph occurs, v will in most cases experiences a discontinuity (because of a sudden change in J_t and U_t). There exists a m_1 such that we will have

$$\|v(t_k^+)\| \leq m_1 \|v(t_k^-)\|$$

for any switching time t_k because of our bounds on U_t and J_t. Here

$$v(t^+) = \lim_{h\downarrow 0} v(t+h), \quad v(t^-) = \lim_{h\downarrow 0} v(t-h).$$

We find that when ε is small enough, the stability of dynamics (24) is dominant by

$$\varepsilon\dot{v} = (I_N \otimes \mathcal{A})v - J_t \otimes (PC'C)v. \tag{25}$$

It is well known that dynamics (25) is asymptotically stable if the N subsystems

$$\varepsilon\dot{\rho} = \mathcal{A}\rho - \lambda_{t,i} PC'C\rho, \qquad (i = 1, \ldots, N) \tag{26}$$

is asymptotically stable, where $\lambda_{t,i}$ $(i = 1, \ldots, N)$ are eigenvalues of $\bar{L}_t$ at time t. Let $\mathcal{A}_t = \mathcal{A} - \lambda_{t,i} PC'C$. Then, $\mathcal{A}_t$ is Hurwitz stable, since

$$\begin{aligned}&\mathcal{A}_t P + P\mathcal{A}_t'\\ &= -I_{pn} + 2\beta PC'C - 2\lambda_{t,i} PC'CP\\ &= -I_{pn} - (2\lambda_{t,i} - 2\beta)PC'CP\\ &< -I_{pn}.\end{aligned}$$

The last inequality holds because $\operatorname{Re}\{\lambda_{t,i}\} > \beta$, for any time $t > 0$ and for all $i = \{1, \ldots, N\}$.

Since dynamics (25) is asymptotically stable, there exists matrix $\tilde{P}$ and small enough $\mu > 0$, such that

$$\begin{aligned}(I_N \otimes \mathcal{A}) - J_t \otimes (PC'C)]'\tilde{P} + \tilde{P}[(I_N \otimes \mathcal{A}) - J_t \otimes (PC'C)\\ \leq -\mu\tilde{P} - I.\end{aligned}$$

Define a Lyapunov function $V = \varepsilon v' \tilde{P} v$. Then, the derivative of V is bounded by

$$\begin{aligned}\dot{V} &\leq -\mu\varepsilon^{-1} V - \|v\|^2 + 2\operatorname{Re}(v' W_{\varepsilon,t} \tilde{P} v)\\ &\leq -\mu\varepsilon^{-1} V - 1\|v\|^2 + \varepsilon r_1 \|v\|^2\\ &\leq -\mu\varepsilon^{-1} V,\end{aligned}$$

for small enough ε, where $\|W_{\varepsilon,t}\tilde{P}\| \leq \varepsilon r_1$ with a large enough r_1. By integration, we find that

$$V(t_k^-) \leq e^{-\mu\varepsilon^{-1}(t_k - t_{k-1})} V(t_{k-1}^+). \tag{27}$$

We have a potential jump at time t_{k-1} in V. However, there exists a m such that

$$V(t_{k-1}^+) \leq m V(t_{k-1}^-).$$

From the fact that $t_k - t_{k-1} > \tau$, there exists a small enough ε such that

$$V(t_k^-) \leq e^{-\mu\varepsilon^{-1}(t_k - t_{k-1})} V(t_{k-1}^-).$$

Combining these time intervals, we get

$$V(t_k^-) \leq e^{-\mu\varepsilon^{-1} t_k} V(0).$$

Assuming $t_{k+1} > t > t_k$, we have

$$V(t) \le e^{-\mu\varepsilon^{-1}(t-t_k)}V(t_k^+) \le me^{-\mu\varepsilon^{-1}(t-t_k)}V(t_k^-) \le me^{-\mu\varepsilon^{-1}t}V(0).$$

This implies that $\lim_{t\to\infty} V(t) = 0$. Given that U_t is bounded for any graph in $\mathbb{G}_{\alpha,\beta,\pi}^{\varphi,N}$ for any time t, we achieve $\lim_{t\to\infty} \bar{\chi}_i(t) = 0$, i.e., $\lim_{t\to\infty}(\chi_i(t) - \hat{\chi}_i(t)) = 0$. ∎

From the above result of Lemma 1, we have

$$\lim_{t\to\infty}(\bar{x}_i(t) - \hat{\bar{x}}_i(t)) = \lim_{t\to\infty}(T_i'T_i)^{-1}T_i'(\chi_i(t) - \hat{\chi}_i(t)) = 0,$$

for any time-varying graph $\tilde{\mathcal{G}}_t \in \mathbb{G}_{\alpha,\beta,\pi}^{\varphi,\tau,N}$ by choosing a small enough ε. ∎

Appendix

We will construct $\tilde{\Pi}_i$ for the case that (Γ_i, S_{ir}) is not observable for some $i \in \{1,\dots,N\}$. There exists a nonsingular matrix Ω_i, such that

$$\bar{S}_{ir} = \Omega_i S_{ir}\Omega_i^{-1} = \begin{pmatrix} S_{ir,no} & S_{ir,12} \\ 0 & S_{ir,o} \end{pmatrix},$$
$$\bar{\Gamma}_i = \Gamma_i\Omega_i^{-1} = \begin{pmatrix} 0 & \Gamma_{i,o} \end{pmatrix},$$

where $(\Gamma_{i,o}, S_{ir,o})$ is observable while $S_{ir,no}$ contains the unobservable modes. Choose

$$\tilde{\Pi}_i = \begin{pmatrix} \Pi_i \\ (0\ \ I)\Omega_i \end{pmatrix}$$

where $(0\ \ I)$ has the same dimension of $(0\ \ \Gamma_{i,o})$. Then, we find that

$$\tilde{A}_i\tilde{\Pi}_i + \tilde{E}_i\bar{R}_{ir} = \begin{pmatrix} A_i & B_i\Gamma_{i,o} \\ 0 & S_{ir,o} \end{pmatrix}\begin{pmatrix} \Pi_i \\ (0\ \ I)\Omega_i \end{pmatrix} + \begin{pmatrix} E_i \\ 0 \end{pmatrix}\bar{R}_{ir} = \begin{pmatrix} A_i\Pi_i + B_i\Gamma_i + E_i\bar{R}_{ir} \\ (0\ \ S_{ir,o})\Omega_i \end{pmatrix},$$

and

$$\tilde{\Pi}_i S_{ir} = \begin{pmatrix} \Pi_i \\ (0\ \ I)\Omega_i \end{pmatrix} S_{ir} = \begin{pmatrix} \Pi_i S_{ir} \\ (0\ \ I)\Omega_i S_{ir} \end{pmatrix} = \begin{pmatrix} \Pi_i S_{ir} \\ (0\ \ S_{ir,o})\Omega_i \end{pmatrix}.$$

From the regulation equations (13), we have

$$\tilde{A}_i\tilde{\Pi}_i + \tilde{E}_i\bar{R}_{ir} = \tilde{\Pi}_i S_{ir}.$$

Moreover,

$$\tilde{C}_i\tilde{\Pi}_i + \tilde{D}_{wi}\bar{R}_{ir} = \begin{pmatrix} C_i & D_i\Gamma_i \end{pmatrix}\begin{pmatrix} \Pi_i \\ (0\ \ I)\Omega_i \end{pmatrix} + D_{wi}\bar{R}_{ir} = C_i\Pi_i + D_i\Gamma_i + D_{wi}\bar{R}_{ir} = \bar{R}_{0r}.$$

References

[1] H. Bai, M. Arcak, and J. Wen. *Cooperative control design: a systematic, passivity-based approach*. Communications and Control Engineering. Springer Verlag, 2011.

[2] H.F. Grip, A. Saberi, and A.A. Stoorvogel. Synchronization in networks of minimum-phase, non-introspective agents without exchange of controller states: homogeneous, heterogeneous, and nonlinear. *Automatica*, 54:246–255, 2015.

[3] H.F. Grip, T. Yang, A. Saberi, and A.A. Stoorvogel. Output synchronization for heterogeneous networks of non-introspective agents. *Automatica*, 48(10):2444–2453, 2012.

[4] H. Kim, H. Shim, and J.H. Seo. Output consensus of heterogeneous uncertain linear multi-agent systems. *IEEE Trans. Aut. Contr.*, 56(1):200–206, 2011.

[5] D. Liberzon and A.S. Morse. Basic problem in stability and design of switched systems. *IEEE Control Systems Magazine*, 19(5):59–70, 1999.

[6] X. Liu, B.M. Chen, and Z. Lin. On the problem of general structure assignments of linear systems through sensor/actuator selection. *Automatica*, 39(2):233–241, 2003.

[7] J. Lunze. An internal-model principle for the synchronisation of autonomous agents with individual dynamics. In *Proc. Joint 50th CDC and ECC*, pages 2106–2111, Orlando, FL, 2011.

[8] Z. Meng, T. Yang, D. V. Dimarogonas, and K. H. Johansson. Coordinated output regulation of multiple heterogeneous linear systems. *Automatica*, 53(3):362–368, 2015.

[9] M. Mesbahi and M. Egerstedt. *Graph theoretic methods in multiagent networks*. Princeton University Press, Princeton, 2010.

[10] R. Olfati-Saber and R.M. Murray. Agreement problems in networks with direct graphs and switching topology. In *Proc. 42nd CDC*, pages 4126–4132, Maui, Hawaii, 2003.

[11] R. Olfati-Saber and R.M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans. Aut. Contr.*, 49(9):1520–1533, 2004.

[12] E. Peymani, H.F. Grip, and A. Saberi. Homogeneous networks of non-introspective agents under external disturbances - H_∞ almost synchronization. *Automatica*, 52:363–372, 2015.

[13] E. Peymani, H.F. Grip, A. Saberi, X. Wang, and T.I. Fossen. H_∞ almost ouput synchronization for heterogeneous networks of introspective agents under external disturbances. *Automatica*, 50(4):1026–1036, 2014.

[14] W. Ren and Y.C. Cao. *Distributed Coordination of Multi-agent Networks*. Communications and Control Engineering. Springer-Verlag, London, 2011.

[15] J.H. Seo, H. Shim, and J. Back. Consensus of high-order linear systems using dynamic output feedback compensator: Low gain approach. *Automatica*, 45(11):2659–2664, 2009.

[16] G. Shi and K.H. Johansson. Robust consensus for continuous-time multi-agent dynamics. *SIAM J. Contr. & Opt.*, 51(5):3673–3691, 2013.

[17] Y. Su and J. Huang. Stability of a class of linear switching systems with applications to two consensus problem. *IEEE Trans. Aut. Contr.*, 57(6):1420–1430, 2012.

[18] C.W. Wu. *Synchronization in complex networks of nonlinear dynamical systems*. World Scientific Publishing Company, Singapore, 2007.

[19] T. Yang, S. Roy, Y. Wan, and A. Saberi. Constructing consensus controllers for networks with identical general linear agents. *Int. J. Robust & Nonlinear Control*, 21(11):1237–1256, 2011.

[20] T. Yang, A. Saberi, A.A. Stoorvogel, and H.F. Grip. Output synchronization for heterogeneous networks of introspective right-invertible agents. *Int. J. Robust & Nonlinear Control*, 24(13):1821–1844, 2014.

[21] M. Zhang, A. Saberi, H. F. Grip, and A. A. Stoorvogel. $\mathcal{H}_\infty$ almost output synchronization for heterogeneous networks without exchange of controller states. *IEEE Trans. Control of Network Systems*, 2(4):348–357, 2015.

[22] M. Zhang, A. Saberi, and A. A. Stoorvogel. Regulated output synchronization for heterogeneous time-varying networks with non-introspective agents in presence of disturbance and measurement noise with known frequencies. In *American Control Conference*, pages 2069–2074, Chicago, IL, 2015.

[23] M. Zhang, A. Saberi, A. A. Stoorvogel, and P. Sannuti. Almost regulated output synchronization for heterogeneous time-varying networks of non-introspective agents and without exchange of controller states. In *American Control Conference*, pages 2735–2740, Chicago, IL, 2015.

[24] M. Zhang, A. A. Stoorvogel, and A. Saberi. Stochastic almost regulated output synchronization for heterogeneous time-varying networks with non-introspective agents and without exchange of controller states. In *American Control Conference*, pages 775–780, Chicago, IL, 2015.

[25] J. Zhao, D. J. Hill, and T. Liu. Synchronization of complex dynamical networks with switching topology: a switched system point of view. *Automatica*, 45(11):2502–2511, 2009.