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State synchronization in the presence of unknown, nonuniform and arbitrary large communication delays

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Abstract— **This paper studies synchronization for multi-agent systems with agents that are identical and introspective (i.e. agents have access to their own states) and coupled through a network with unknown nonuniform and arbitrary large communication delays. Exact knowledge of the network is not available but only one specific lower bound is available. When the system is running, a given constant desired trajectory is provided to one of the agents. The objective is to design a decentralized protocol such that the multi-agent system achieves state synchronization for all possible networks, for any reference trajectory consistent with the system dynamics and for any arbitrary large nonuniform communication delay.**

I. Introduction

In the past few decades, synchronization problems for multi-agent systems have received substantial attention, where the objective is to achieve asymptotic agreement on a common state (*state synchronization*) or an output trajectory (*output synchronization*) among agents of the network through decentralized control protocols. Some early results can be found in [7], [9], [13] for state synchronization problems of homogeneous networks (i.e. agents are identical), and in [1], [4], [18] for output synchronization problems for heterogeneous networks.

Recently, synchronization in a network with time delay has attracted a great deal of interest. As clarified in [3], we can identify two kinds of delay. Firstly there is *communication delay*, which results from limitations on the communication between agents. Secondly we have *input delay* which is due to computational limitations of an individual agent. Many works have focused on dealing with input delay, progressing from single- and double-integrator agent dynamics (see e.g. [8], [11], [12], [17]) to more general agent dynamics (see e.g. [10], [14], [15], [16], [19], [22]). Its objective is to derive an upper bound on the input delay such that agents can still achieve synchronization. Moreover, such an upper bound always depends on the agent dynamics and the network properties.

Communication delay is much less understood at this moment. In the case of communication delay, only for a constant synchronization trajectory do we preserve the diffusive nature of the network. This diffusive nature is an intrinsic part of the currently available design techniques and hence only this case has been studied. Some works in this area can be seen in [2], [6], [11] and [17].

The above works on communication delay only consider simple dynamics. In this paper, we deal with general agent dynamics. The network is either directed or undirected, weighted, and has unknown nonuniform communication delays, which can be also arbitrary large. We assume all agents are introspective and one agent has access to an, a priori given, constant trajectory. A distributed protocol is proposed such that state synchronization is achieved among all agents and the synchronous trajectory is the given constant trajectory.

A. Notations and definitions

Given a matrix $A \in \mathbb{C}^{m\times n}$, A' denotes its conjugate transpose while $\|A\|$ denotes the induced 2-norm. We denote by $\operatorname{diag}\{a_i\}$, a diagonal matrix with $a_1,\ldots,a_N$ as the diagonal elements, and by $\operatorname{col}\{x_i\}$, a column vector with $x_1,\ldots,x_N$ stacked together, where the range of index i can be identified from the context. $A\otimes B$ indicates the Kronecker product between A and B.

A *weighted graph* $\mathcal{G}$ is defined by a triple $(\mathcal{V},\mathcal{E},\mathcal{A})$, where $\mathcal{V}=\{1,\ldots,N\}$ is a node set, $\mathcal{E}\subseteq V\times V$ is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$ is the weighting matrix, with $a_{ij}>0$ iff $(i,j)\in\mathcal{E}$ and $a_{ii}=0$. If $a_{ij}=a_{ji}$ for all $(i,j)\in\mathcal{E}$, the graph is called *undirected*; otherwise *directed*. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1,\ldots,i_k\}$ such that $(i_j,i_{j+1})\in\mathcal{E}$ for $j=1,\ldots,k-1$. A graph is *connected* if there exists a path between every pair of nodes. A directed graph is *balanced* if $\sum_j^N a_{ij}=\sum_j^N a_{ji}$ for all $i=1,\ldots,N$. A *directed tree* with *root* r is a subset of nodes of the graph $\mathcal{G}$ such that a path exists between r and every other node in this subset. A *directed spanning tree* is a directed tree containing all the nodes of the graph. For a weighted graph $\mathcal{G}$, the matrix $L=[\ell_{ij}]$ with

$$\ell_{ij}=\begin{cases}\sum_{k=1}^N a_{ik}, & i=j,\\ -a_{ij}, & i\neq j,\end{cases}$$

is called the *Laplacian matrix* associated with the graph $\mathcal{G}$. All eigenvalues of L are located in the closed right half complex plane with at least one eigenvalue at zero which is associated with right eigenvector $\mathbf{1}$. In case the graph is strongly connected then the multiplicity of the eigenvalue at zero is 1 and all other eigenvalues are in the open right-half plane. When $\mathcal{G}$ is undirected, L is symmetric.

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II. Problem formulation for undirected graphs

The multi-agent system we will consider in this paper is composed of N identical general agents, which are denoted by Σ_i with $i \in \{1, \ldots, N\}$,

$$\Sigma_i : \quad \dot{x}_i = Ax_i + Bu_i \tag{1}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ are the state, input of agent i. It is assumed that (A, B) is controllable. Moreover, all agents are supposed to be introspective.

The network provides agent i with the following information

$$\zeta_i(t) = \sum_{j=1}^{N} a_{ij}(x_i(t) - x_j(t - \tau_{ij})), \tag{2}$$

where $\tau_{ij} \in \mathbb{R}^+$ $(i \neq j)$ represents an unknown constant communication delay from agent j to agent i. In the above $a_{ij} \geq 0$, $a_{ij} = a_{ji}$ and $a_{ii} = 0$. The above communication presented in (2) can be connected to a weighted graph $\mathcal{G}$ with each node indicating an agent in the network and the weight of an edge is given by the coefficient a_{ij}. The communication delay implies that it takes τ_{ij} seconds for agent j to transfer its state information to agent i.

Our goal is to achieve state synchronization among all agents while the synchronized dynamics should be equal to an, a priori given, constant trajectory, denoted by $x_r \in \mathbb{R}^n$. We assume that at least one agent has access to the constant trajectory information. For ease of presentation, we assume that only agent k has access to x_r. The information available to agent k is given by:

$$\bar{\zeta}_k(t) = \zeta_k(t) + (x_k - x_r). \tag{3}$$

To have consistency in the notation, we define $\bar{\zeta}_i(t) = \zeta_i(t)$ for all other agents $i \in \{1, \ldots, N\}/k$.

Note that when there is no communication delay, $\zeta_i(t)$ can be represented in terms of a Laplacian matrix, i.e.,

$$\zeta_i(t) = \sum_{j=1}^{N} \ell_{ij} x_j(t).$$

Let $\bar{\ell}_{kk} = \ell_{kk} + 1$ and $\bar{\ell}_{ij} = \ell_{ij}$ for all other $i, j \in \{1, \ldots, N\}$ in which case

$$\bar{\zeta}_i(t) = \sum_{j=1}^{N} \bar{\ell}_{ij}(x_j(t - \tau_{ij}) - x_r)$$

where $\tau_{ii} = 0$. We will refer to the matrix $\bar{L} = [\bar{\ell}_{ij}]$ as the expanded Laplacian matrix.

We would like to note that, in practice, precise information of a network communication topology is usually not available for controller design and only some rough characterization of the network can be obtained. In our case, we assume only a lower bound on the smallest eigenvalue of the expanded Laplacian is given:

Definition 1: For given real number $\beta > 0$, the set $\mathbb{G}_{\beta,N}$ consists of all strongly connected, weighted and undirected graphs composed of N nodes satisfying the following property:

- The eigenvalues of the expanded Laplacian matrix $\bar{L}$, denoted by $\lambda_1, \ldots, \lambda_N$, are real and satisfy $\lambda_i > \beta$.

Remark 1: If our undirected graph is strongly connected then all eigenvalues of $\bar{L}$ are positive (see for instance [5]). Hence each strongly connected, weighted and undirected graphs is in $\mathbb{G}_{\beta,N}$ for sufficiently small $\beta > 0$. Our protocol design will only use the β but is independent of the precise information of the network.

Note that we will not be able to track any potential constant reference signal for the state since we are constrained by the system dynamics. We define the set

$$\begin{aligned}\mathscr{X}_r &= \left\{ x \in \mathbb{R}^n \mid Ax \in \operatorname{Im} B \right\} \\ &= \left\{ x \in \mathbb{R}^n \mid \exists\, u \in \mathbb{R}^m \text{ such that } Ax + Bu = 0 \right\}.\end{aligned} \tag{4}$$

It is easily verified that only if the constant reference signal is in the set $\mathscr{X}_r$ can we possibly achieve asymptotic tracking even without complications due to the decentralized structure and the communication delays.

We formulate the problem of state synchronization for networks with unknown, nonuniform communication delays as follows.

Problem 1: Let β be a given positive real number. Consider a network with agents described by (1) and (2) associated with a graph $\mathcal{G} \in \mathbb{G}_{\beta,N}$. Let the constant reference trajectory be available to at least one agent. The *state synchronization* problem for networks with unknown, nonuniform communication delay is to find a distributed controller for each agent such that, for any graph $\mathcal{G} \in \mathbb{G}_{\beta,N}$, for any communication delay $\tau_{ij} \in \mathbb{R}^+$, and for any constant constant reference trajectory $x_r \in \mathscr{X}_r$, the state of each agent coverges to the reference trajectory i.e.,

$$\lim_{t \to \infty} (x_i(t) - x_r) = 0, \tag{5}$$

for all $i \in \{1, \ldots, N\}$.

III. State synchronization for undirected graphs

In this section, we will present a distributed controller design to achieve state synchronization for networks with unknown, nonuniform communication delays such that the state of each agent will converge to any constant trajectory $x_r \in \mathscr{X}_r$.

The main result in this section is presented in the following theorem.

Theorem 1: Let β be a given positive real number. Consider a multi-agent system with agents described by (1) and (2). Let the constant reference signal be available to agent k. Assume the above multi-agent system is associated with an undirected graph $\mathcal{G} \in \mathbb{G}_{\beta,N}$. In that case, Problem 1 is solvable. More specifically, there exists a distributed protocol of the type

$$u_i = Fx_i + H\bar{\zeta}_i$$

for each agent such that Problem 1 is solved for any undirected graph $\mathcal{G} \in \mathbb{G}_{\beta,N}$, for any constant reference trajectory x_r in the set $\mathscr{X}_r$ and for any communication delay $\tau_{ij} \in \mathbb{R}^+$.

The graph structure, as expressed in the Laplacian, combined with the communication delays is in the frequency domain connected to the following matrix

$$\bar{L}_s(\tau) = \begin{pmatrix} \bar{\ell}_{11} & \bar{\ell}_{12}e^{-\tau_{12}s} & \cdots & \bar{\ell}_{1N}e^{-\tau_{1N}s} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\ell}_{k1}e^{-\tau_{k1}s} & \bar{\ell}_{kk} & \cdots & \bar{\ell}_{kN}e^{-\tau_{kN}s} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\ell}_{N1}e^{-\tau_{N1}s} & \bar{\ell}_{N2}e^{-\tau_{N2}s} & \cdots & \bar{\ell}_{NN} \end{pmatrix}$$

where τ denotes a vector consisting of all τ_{ig} with $i, g \in \{1, \ldots, N\}$. In order to prove Theorem 1 we need the following result:

Lemma 1: Given that the lower bound on the eigenvalues of $\bar{L}$ is β, then, for all communication delays $\tau_{ig} \in \mathbb{R}^+$ $(i, g = 1, \ldots, N)$ and all $\omega \in \mathbb{R}$, the real part of all eigenvalues of $\bar{L}_{j\omega}(\tau)$ will be larger than or equal to β.

Proof: All eigenvalues of $\bar{L}_{j\omega}(\tau)$ are in the set

$$\left\{ v'\bar{L}_{j\omega}(\tau)v \mid v \in \mathbb{C}^N, \|v\| = 1 \right\}.$$

and therefore it is sufficient to establish that all elements in this set have a real part larger than or equal to β.

Since $\bar{L}$ is symmetric, we know from Definition 1, that $v'\bar{L}v$ is real and larger than or equal to β, provided $\|v\| = 1$.

Next, consider an arbitrary vector $v \in \mathbb{C}^N$. We have

$$v'\bar{L}_{j\omega}(\tau)v = \sum_{i=1}^{N} |v_i|^2 \bar{\ell}_{ii} + \sum_{i=1}^{N} \sum_{\substack{g=1 \\ g \neq i}}^{N} v_i' v_g \bar{\ell}_{ig} e^{-\tau_{ig} j\omega}.$$

Since $\bar{\ell}_{ig}$ is negative or equal to zero for $i \neq g$, we get

$$\begin{aligned} \operatorname{Re}\left(v'\bar{L}_{j\omega}(\tau)v\right) &\geq \sum_{i=1}^{N} |v_i|^2 \bar{\ell}_{ii} + \sum_{i=1}^{N} \sum_{\substack{g=1 \\ g \neq i}}^{N} |v_i' v_g| \bar{\ell}_{ig} \\ &= \begin{pmatrix} |v_1| \\ \vdots \\ |v_N| \end{pmatrix}' \bar{L} \begin{pmatrix} |v_1| \\ \vdots \\ |v_N| \end{pmatrix} \geq \beta. \end{aligned}$$

which completes the proof. ∎

Proof of Theorem 1: For each agent $i \in \{1, \ldots, N\}$, a preliminary state feedback law

$$u_i = Fx_i + v_i, \tag{6}$$

is used such that

$$\ker(A + BF) = \mathscr{X}_r. \tag{7}$$

The matrix F basically guarantees that the kernel of $A+BF$ is maximal which enables us to track the largest possible set of constant reference trajectories. For the construction of such a matrix F we note that there exists a state transformation T such that

$$TAT^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad TB = \begin{pmatrix} 0 \\ B_1 \end{pmatrix}.$$

with B_1 has full row rank. Then, we can choose $F = -B_1^r \begin{pmatrix} A_{21} & A_{22} \end{pmatrix} T$ where B_1^r denotes a right-inverse of B_1.

Combining each agent dynamics (1) and the state feedback law (6) the closed-loop system can be written as

$$\dot{x}_i = \bar{A}x_i + Bv_i, \tag{8}$$

where $\bar{A} = A + BF$. For such a closed-loop system, we develop a distributed local controller

$$v_i = -\alpha B'P\bar{\zeta}_i, \tag{9}$$

where α is a design parameter that will be chosen later and P is the positive definite solution of the algebraic Riccati equation:

$$\bar{A}'P + P\bar{A} - PBB'P + I = 0. \tag{10}$$

In the following, we will prove that the state of each agent converges to the constant trajectory x_r. Define $\bar{x}_i = x_i - x_r$ for every $i \in \{1, \ldots, N\}$. If x_r is not in the set $\mathscr{X}_r$ then it can be easily seen that even for one agent there does not exists any input u_i such that $x_i(t) \to x_r$ as $t \to \infty$. On the other hand if x_r is in the set $\mathscr{X}_r$, then we have

$$\dot{\bar{x}}_i = \bar{A}\bar{x}_i + Bv_i.$$

Moreover,

$$\begin{aligned} \bar{\zeta}_i &= \sum_{g=1}^{N} a_{ig}(x_i(t) - x_g(t - \tau_{ig})) \\ &= \sum_{g=1}^{N} a_{ig}(\bar{x}_i(t) - \bar{x}_g(t - \tau_{ig})) \end{aligned}$$

for all $i \in \{1, \ldots, N\}/k$. Similarly,

$$\begin{aligned} \bar{\zeta}_k &= \sum_{g=1}^{N} a_{kg}(x_k(t) - x_g(t - \tau_{kg})) + x_k - x_r \\ &= \sum_{g=1}^{N} a_{ig}(\bar{x}_k(t) - \bar{x}_g(t - \tau_{kg})) + \bar{x}_k. \end{aligned}$$

Let $\bar{x} = \operatorname{col}\{\bar{x}_i\}$. Then, the closed-loop system for the interconnection of agents and their distributed protocols can be written in the frequency domain as

$$s\bar{x} = (I_N \otimes \bar{A})\bar{x} - \alpha(\bar{L}_s(\tau) \otimes BB'P)\bar{x}. \tag{11}$$

To prove our result, we only need to prove (11) is asymptotically stable for any communication delay $\tau_{ig} \in \mathbb{R}^+$. The remaining proof will be done in two steps.

Step 1: In this step, we will first prove that the closed-loop system without any communication delay is asymptotically stable. This is equivalent to showing that the matrix

$$(I_N \otimes \bar{A}) - \alpha(\bar{L} \otimes BB'P) \tag{12}$$

is Hurwitz stable. As in Definition 1, λ_i $(i = 1, \ldots, N)$ denote the eigenvalues of $\bar{L}$. Then, from [20], the stability of (12) is equivalent to the Hurwitz stability of

$$\bar{A} - \alpha\lambda_i BB'P.$$

for all $i = 1, \ldots, N$. From Lemma 2 in the appendix, we conclude these matrices are Hurwitz stable provided

$$\alpha > \frac{1}{2\lambda_i} \tag{13}$$

for all i. The eigenvalues λ_i are not available for the controller design but since $\mathcal{G} \in \mathbb{G}_{\beta,N}$, it is sufficient to choose $\alpha > \frac{1}{2\beta}$ to guarantee (13) for all i.

Step 2: In this step, we need to prove the closed-loop system is asymptotically stable in the presence of communication delays. Since the system without delays is asymptotically stable, according to Lemma 3 in the appendix, the closed-loop system is asymptotically stable for any communication delay $\tau_{ig} \in \mathbb{R}^+$, if

$$\det[j\omega I - (I_N \otimes \bar{A}) + \alpha \bar{L}_{j\omega}(\tau) \otimes BB'P] \neq 0 \tag{14}$$

for all $\omega \in \mathbb{R}$ and any communication delay $\tau_{ig} \in \mathbb{R}^+$.

For Condition (14) it is clearly sufficient to show that

$$(I_N \otimes \bar{A}) - \alpha \bar{L}_{j\omega}(\tau) \otimes BB'P \tag{15}$$

does not have eigenvalues on the imaginary axis. Lemma 1 implies that all eigenvalues of $\bar{L}_{j\omega}(\tau)$ have real part larger than or equal to β. This implies that for $\alpha > \frac{1}{2\beta}$, we obtain that all eigenvalues of

$$\alpha \bar{L}_{j\omega}(\tau)$$

have a real part larger than $\frac{1}{2}$. According to Lemma 2 in the appendix, this implies that (15) is Hurwitz stable for any communication delay $\tau_{ig} \in \mathbb{R}^+$. As noted before, this implies condition (14) is satisfied. Hence, the closed-loop system is asymptotically stable for any communication delay $\tau_{ig} \in \mathbb{R}^+$. ∎

IV. State synchronization for directed graphs

In this section, we will investigate Problem 1 for a multi-agent system with a given directed graph $\mathcal{G}$. The associated Laplacian matrix L is then in general nonsymmetric, and the expanded Laplacian matrix $\bar{L}$ is defined exactly the same as in Section III. However, we should note that we now have to require that agent k is a root agent in the sense there exists a spanning tree for the graph with k as the root node.

We note that if our directed graph is balanced then the results of Theorem 1 basically still hold if we define β as the smallest eigenvalue of $\bar{L} + \bar{L}'$. However, in general the derivation presented before might not be valid. In that case, we design a protocol for one individual graph (instead of for a set):

Theorem 2: Consider a multi-agent system described by (1) and (2) associated with a given directed graph $\mathcal{G}$ which has a directed spanning tree. Let the constant reference signal be available to agent k which is a root agent. Then, Problem 1 for a multi-agent system with a given directed graph $\mathcal{G}$ is solvable. More specifically, there exists a distributed protocol of the form $u_i = Fx_i + H\bar{\zeta}_i$ for each agent such that Problem 1 is solvable for the given directed graph $\mathcal{G}$, for any constant reference trajectory x_r in the set $\mathscr{X}_r$ and for any communication delay $\tau_{ij} \in \mathbb{R}^+$.

Remark 2: In Theorem 1 we only used limited information about the network to design our distributed protocol. In the proof of the above theorem we make explicit use of knowledge of the network to design our protocol (more specifically, to find a lower bound for our design parameter α. If we have a finite set of possible graphs then we can still find a protocol that works for every graph in this finite set (use as a lower bound for α, the maximum of the lower bounds for each individual graph in the set).

The protocol requires the design of a parameter α large enough. For the undirected case we can connect the smallest eigenvalue of the expanded Laplacian to a lower bound for α. This connection could not be established in the directed case.

Proof: For each $i = 1, \ldots, N$, the distributed protocol is designed as

$$u_i = Fx_i - \alpha B'P\bar{\zeta}_i, \tag{16}$$

where F and P are chosen exactly as in Section III, while α is a design parameter we will choose differently in this section. Following the proof of Theorem 1, we need to design parameter α such that the eigenvalues of

$$\alpha \bar{L}_{j\omega}(\tau) \tag{17}$$

have a real part larger than $\frac{1}{2}$ for any communication delay $\tau_{ig} \in \mathbb{R}^+$, $(i, j = 1, \ldots, N)$.

Since k corresponds to a root of a spanning tree we know the expanded Laplacian matrix $\bar{L}$ is invertible and has its eigenvalues in the open right half plane. Since $\bar{L}$ is nonsymmetric we need a slightly different approach. Given that $\bar{L}$ is an invertible M-matrix, there exists a diagonal positive matrix $D = \operatorname{diag}\{d_i\}$ such that

$$D\bar{L} + \bar{L}'D > 0 \tag{18}$$

(in the case of undirected graphs we can simply choose $D = I$). Since this matrix (18) is positive definite we find that:

$$\operatorname{Re}(v'D\bar{L}v) = \frac{1}{2}v'(D\bar{L} + \bar{L}'D)v$$

is larger or equal to some positive constant β for all v with $\|v\| = 1$. Following the proof in Lemma 1, we obtain

$$\operatorname{Re}(v'D\bar{L}_{j\omega}(\tau)v) \geq \beta. \tag{19}$$

Let λ be an eigenvalue of $\bar{L}_{j\omega}(\tau)$ with eigenvector v. In other words, $\bar{L}_{j\omega}(\tau)v = \lambda v$. Combining with inequality (19), we get

$$\operatorname{Re}(\lambda(\max d_i)v'v) \geq \operatorname{Re}(\lambda v'Dv) = \operatorname{Re} v'D\bar{L}_{j\omega}(\tau)v \geq \beta$$
$$\Rightarrow \quad \operatorname{Re}(\lambda) \geq \frac{\beta}{\max d_i}.$$

When choosing

$$\alpha > \frac{\max d_i}{2\beta},$$

condition (17) is satisfied for any communication delay $\tau_{ig} \in \mathbb{R}^+$, $(i, g = 1, \ldots, N)$. Hence, our result is proved. ∎

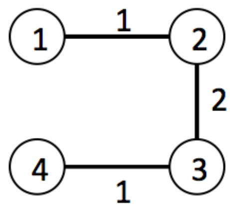

Fig. 1. The undirected weighted network with 4 agents

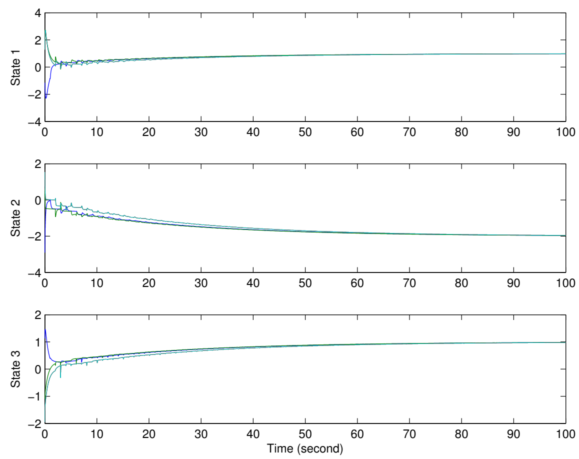

Fig. 2. The state trajectories of 4 agents

V. Example

In this section, we will give an example of state synchronization for an undirected weighted network where agents are identical and introspective.

Consider a network with $N = 4$ agents, illustrated in Figure 1. The network graph is undirected and weighted, and belongs to a set of graphs $\mathcal{G}_{\beta,N}$ with $\beta = 0.1$. We allow any nonuniform arbitrarily large communication delays in the network communication. In this example, we choose $\tau_{12} = 1$, $\tau_{21} = 2$, $\tau_{23} = 3$, $\tau_{32} = 2$, $\tau_{34} = 3$, and $\tau_{43} = 4$. The linear agent model is described by the matrices

$$A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

We choose

$$F = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

which yields:

$$\mathscr{X}_r = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} \right\}.$$

By choosing $\alpha = 10$, which is larger than $\frac{1}{2\beta}$, the distributed local controller for each agent i ($i = 1,\ldots,4$) is designed as

$$v_i = \begin{pmatrix} -5.0208 & 2.1156 & -12.2472 \\ -2.9057 & -10.9348 & -7.2258 \end{pmatrix} \bar{\zeta}_i.$$

Our target is to have all agents track to constant reference signal given by $x_r = [1; -2; 1]$ which is available to agent 2. It is easily verified that $x_r \in \mathscr{X}_r$.

Figure 2 shows that the states of all agents converge to the reference trajectory x_r asymptotically.

VI. Conclusion

In this paper, we developed a theory enabling us to achieve state synchronization and regulation for introspective, identical agents. We can handle more complex dynamics compared to the existing literature which studied single- and double-integrators. Obviously, the first extension that we will consider is output synchronization. However, the future goal is to be able to address communication delays for nonintrospective, nonindentical agents with a network which is time-varying. The latter can, for instance, address the case when certain communication links fail.

Appendix

Synchronization is connected to a robust stabilization problem as presented in the following lemma which can be found in [20].

Lemma 2: Consider a linear uncertain system

$$\dot{x} = Ax + \lambda Bu,$$

where (A, B) is stabilizable with $\lambda \in \mathbb{C}$ unknown. Consider, the state feedback $u = \alpha F x$ where $F = -B'P$, and P is the unique positive definite solution of the algebraic Riccati equation:

$$A'P + PA - PB'BP + I = 0.$$

Then, we have that $A - \alpha\lambda BB'P$ is Hurwitz stable for any

$$\lambda \in \left\{ s \in \mathbb{C} \mid \text{Re}(s) \geq \frac{1}{2\alpha} \right\}.$$

The following lemma is a useful tool to check the stability of a delay system and can be found in [21].

Lemma 3: Consider a linear time-delay system

$$\dot{x} = Ax + \sum_{i=1}^{N} A_{d,i} x(t - \tau_i). \tag{20}$$

Assume

$$A_d + \sum_{i=1}^{N} A_{d,i}$$

is Hurwitz stable. In that case, the delay system (20) is globally asymptotically stable for any $\tau_1, \ldots, \tau_N \in [0, \bar{\tau}]$ if

$$\det\left[j\omega I - A - \sum_{i=1}^{N} e^{-j\omega\tau_i} A_{d,i} \right] \neq 0,$$

for all $\omega \in \mathbb{R}$ and $\tau_1, \ldots, \tau_N \in [0, \bar{\tau}]$.

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