SAMPLED-DATA L^∞ SMOOTHING: FIXED-SIZE *ARE* SOLUTION WITH FREE HOLD FUNCTION*

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Abstract. The problem of estimating an analog signal from its noisy sampled measurements is studied in the L^∞ (induced L^2-norm) framework. The main emphasis is placed on relaxing causality requirements. Namely, it is assumed that l future measurements are available to the estimator, which corresponds to the fixed-lag smoothing formulation. A closed-form solution to the problem is derived. The solution has the complexity of $\mathcal{O}(l)$ and is based on two discrete algebraic Riccati equations, whose size does not depend on the smoothing lag l.

Key words. Sampled-data systems, fixed-lag smoothing, L^∞ optimization, generalized hold functions, signal reconstruction.

1. Introduction. This paper studies the problem of estimating an analog signal v from sampled measurements of a related signal y. We assume that v and y are generated by an analog LTI system $\mathcal{G}$, driven by a common exogenous signal w_v as shown in Fig. 1.1. The

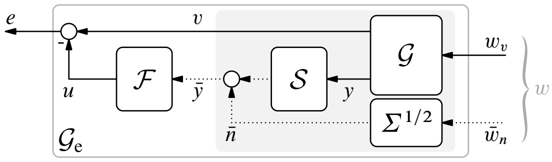

FIG. 1.1. *Sampled-data estimation setup*

measured discrete signal $\bar{y}$ is the sampled version of y ($\mathcal{S}$ denotes the ideal sampler) with a constant sampling period $h > 0$, corrupted by a discrete measurement noise $\bar{n}$. The latter may reflect roundoff errors and its intensity is modeled by the matrix $\Sigma = \Sigma' \geq 0$. The D/A converter $\mathcal{F}$ (estimator), which generates an estimate u of v, is our design parameter. We quantify the estimation performance in terms of the L^∞ norm of the *error system*

$$\mathcal{G}_{\mathrm{e}} := \begin{bmatrix} \mathcal{G}_v & 0 \end{bmatrix} - \mathcal{F} \begin{bmatrix} \mathcal{S}\mathcal{G}_y & \Sigma^{1/2} \end{bmatrix}, \tag{1.1}$$

which maps the aggregate exogenous signal $w := \left[\begin{smallmatrix} w_v \\ \bar{w}_n \end{smallmatrix}\right]$ (see Fig. 1.1) to the estimation error $e := v - u$ (here $\mathcal{G}_v$ and $\mathcal{G}_y$ are the rows of $\mathcal{G}$ corresponding to v and y, respectively). This L^∞ norm is the induced operator norm $L^2(\mathbb{R}) \times \ell^2(\mathbb{Z}) \to L^2(\mathbb{R})$.

The main theme of this study is the relaxation of causality constraints imposed upon $\mathcal{F}$. We say that $\mathcal{F}$ is *l-causal* if its output $u(t)$, at a time instance $t \in \mathbb{R}$, depends only on $\bar{y}[k]$ for all $k \leq t/h + l$. In other words, an l-causal estimator has access to l "future" measurements of $\bar{y}$ (l steps preview). Estimation problems in which the estimator is constrained to be l-causal for some $l \in \mathbb{N}$ are referred to as *fixed-lag smoothing* and l is called the *smoothing lag*, see [1, 17] and the references therein. The smoothing problem may also be interpreted as the estimation of the lh-delayed version of v by a causal estimator, so the problem is frequently referred to as the H^∞ fixed-lag smoothing, which reflects the causality of $\mathcal{G}_{\mathrm{e}}$ in this formulation.

The incentive for relaxing causality constraints on $\mathcal{F}$ is the potential for an improved estimation performance [1]. This comes at the price of a more complex $\mathcal{F}$ and, especially,

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a more knotty analysis compared with corresponding filtering ($l = 0$) and fixed-interval smoothing ($l = \infty$) results. Even in pure continuous- and discrete-time settings unrestricted solutions to the L^∞ (H^∞) fixed-lag smoothing problems were derived only in '00s [13, 21], more than a decade after the corresponding filtering and fixed-interval smoothing results [18, 19]. Sampled-data counterparts of these results are yet more challenging. To the best of our knowledge, there is no L^∞ fixed-lag smoothing solution for the setup in Fig. 1.1 in the literature. The filtering problem in this setting was solved in [20] in the case of $\Sigma = I$ and then in [15] for a general, possibly singular, Σ. The design of non-causal D/A converters (fixed-interval smoothing) is addressed in [9]. In the special case of $l = 1$ (and $\Sigma = 0$), [14] derives the solvability conditions, but not formulae for $\mathcal{F}$.

We address the sampled-data L^∞ fixed-lag smoothing problem via the lifting technique [4], which converts it to an equivalent pure discrete problem, some parameters of which are operators over infinite-dimensional spaces. We then start with a formal solution in terms of these operators and then rewrite such a solution in terms of the original parameters of $\mathcal{G}$. The latter procedure, called peeling-off, is rather nontrivial and its successful completion is the main technical contribution of this paper. Technical challenges of the peeling-off step in the smoothing case go far beyond those in the filtering case [15], owing chiefly to a more elaborate solution to the discrete smoothing problems.

It is well known [2, Sec. 7.3] that discrete fixed-lag smoothing can in principle be cast as a filtering ($l = 0$) problem by incorporating the delay z^{-l} in the "v" channel into the signal generator. This approach, however, increases the problem dimension and might blur properties of the resulting solution. In the H^2 (Kalman smoothing) case, the structure of the filtering formulae can be exploited to derive a solution that is based on fixed-size (independent of l) Riccati equation and whose computational burden is $\mathcal{O}(l)$, see [2, Sec. 7.3]. A similar approach, however, does not work so smoothly in the H^∞ optimization because the corresponding Riccati equation in this case is more involved, see [3, 7, 23] for solutions derived via this method and [21, §III-B] for a discussion about their limitations. Moreover, the application of this approach to the sampled-data problem is complicated by the fact that the "v" channel is intrinsically infinite dimensional in the lifted domain. To the best of our knowledge, the only complete solution to the discrete H^∞ fixed-lag smoothing problem available in the literature is the result of [21]. It provides necessary and sufficient solvability conditions and does not introduce restrictive assumptions about the signal generator. This solution, however, involves several intermediate steps, which impedes its use as a starting point for the peeling-of procedure. This motivates us to derive alternative discrete state-space formulae in [12] following the steps of [16].

The solution of [12] also involves several intermediate calculations. These calculations, however, appear to be more suitable for the use in sampled-data applications. As a result, in the current paper we succeed in deriving a numerically tractable and transparent solution to the L^∞ sampled-data problem. Our solution is based on two discrete algebraic Riccati equations, which are independent of the smoothing lag l and one of which does not depend on the achievable performance level γ either. Similarly to other sampled-data H^∞ solutions [4], our solvability conditions involve the verification of the non-singularity of a matrix function built upon blocks of a matrix exponential over the whole interval $(0, h]$. This part is the most involved numerically part of the solvability conditions. The others are just plain conditions based on the corresponding H^∞ Riccati equation. The suboptimal solution is then the cascade of a discrete filter and a zero-order generalized hold. The latter actually coincides with the D/A part of the optimal L^2 solution of [10].

Notation. For any set $\mathbb{A}$, its indicator function $\mathbb{1}_{\mathbb{A}}(t)$ is 1 if $t \in \mathbb{A}$ and is zero elsewhere. The space $L^2(\mathbb{R})$ is the set of functions $f : \mathbb{R} \to \mathbb{C}^{n_f}$ that have finite norm $\|f\|_2 :=$

$(\int_{t\in\mathbb{R}}\|f(t)\|^2\,\mathrm{d}t)^{1/2}$, where $\|\cdot\|$ denotes the standard Euclidean norm. $\ell^2(\mathbb{Z})$ is the set of $\bar{f}:\mathbb{Z}\to\mathbb{C}^{n_f}$ with finite norm $\|\bar{f}\|_2 := (\sum_{k\in\mathbb{Z}}\|\bar{f}[k]\|^2)^{1/2}$.

2. Problem Formulation. Consider the system in Fig. 1.1. Throughout the paper we assume that $\mathcal{G}$ is a causal finite-dimensional LTI system given in terms of its *minimal* state-space realization

$$G(s) = \begin{bmatrix} G_v(s) \\ G_y(s) \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C_v & D_v \\ C_y & 0 \end{array}\right] \tag{2.1}$$

and the estimator $\mathcal{F}:\bar{y}\mapsto u$ is shift invariant and l-causal, i.e., is in the form

$$u(t) = \sum_{i=-\infty}^{\lfloor t/h\rfloor + l} \phi(t-ih)\bar{y}[i], \quad t\in\mathbb{R} \tag{2.2}$$

for some *hold function* (interpolation kernel) $\phi(t)$ and sampling period $h>0$. We say that $\mathcal{F}$ is stable if it is bounded as an operator $\ell^2(\mathbb{Z})\to L^2(\mathbb{R})$ and stabilizing if the error system $\mathcal{G}_\mathrm{e}$ in (1.1) is bounded as an operator $L^2(\mathbb{R})\times\ell^2(\mathbb{Z})\to L^2(\mathbb{R})$. The induced norm of the error system is referred to as the L^∞ norm (see [8]) and denoted as $\|\mathcal{G}_\mathrm{e}\|_\infty$. We also assume that the realization in (2.1) satisfies

$\boldsymbol{\mathcal{A}_1}$: (C_y, e^{Ah}) is detectable,

$\boldsymbol{\mathcal{A}_2}$: $\begin{bmatrix} C_y & \Sigma \end{bmatrix}$ has full row rank.

Assumption $\boldsymbol{\mathcal{A}_1}$ is necessary and sufficient for the existence of a stabilizing $\mathcal{F}$. $\boldsymbol{\mathcal{A}_2}$ says that the measurements are not redundant and hence can be made without loss of generality. In addition, we effectively assume that $G_y(s)$ is strictly proper, which guarantees the boundedness of the ideal sampling operation.

The problem studied in this paper is formulated as follows:

RP$_{\boldsymbol{\gamma,l}}$: Let signal generators $\mathcal{G}$ and $\Sigma\geq 0$, satisfying $\boldsymbol{\mathcal{A}_{1,2}}$, and a constant $l\in\mathbb{N}$ be given and let $\mathcal{S}$ be the ideal sampler. Find whether there is a stable and stabilizing l-causal estimator $\mathcal{F}$ of form (2.2) such that

$$\|\mathcal{G}_\mathrm{e}\|_\infty < \gamma$$

for a given $\gamma > 0$.

RP$_{\boldsymbol{\gamma,0}}$ corresponds to the filtering problem solved in [15, 20], whereas **RP**$_{\boldsymbol{\gamma,\infty}}$—to the fixed-interval smoothing problem solved in [9, Sec. III].

3. Main Result. To solve the smoothing problem for this system we need two (symplectic) matrix exponentials:

$$\Lambda(t) = \begin{bmatrix} \Lambda_{11}(t) & \Lambda_{12}(t) \\ 0 & \Lambda_{22}(t) \end{bmatrix} := \exp\left(\begin{bmatrix} A & BB' \\ 0 & -A' \end{bmatrix} t\right)$$

and

$$\begin{bmatrix} \Gamma_{11}(t) & \Gamma_{12}(t) \\ \Gamma_{21}(t) & \Gamma_{22}(t) \end{bmatrix} := \exp(H_\gamma t),$$

where

$$H_\gamma := \begin{bmatrix} A & BB' \\ 0 & -A' \end{bmatrix} + \begin{bmatrix} BD_v' \\ -C_v' \end{bmatrix}(\gamma^2 I - D_v D_v')^{-1}\begin{bmatrix} C_v & D_v B' \end{bmatrix}.$$

To shorten the notation, we omit the argument when $t = h$, so that Λ_{ij} and Γ_{ij} stand for $\Lambda_{ij}(h)$ and $\Gamma_{ij}(h)$, respectively.

In the solution we need two discrete algebraic Riccati equations (DAREs). The first one is the DARE associated with the Kalman filter solution:

$$Y = \Lambda_{11}\bigl(Y - YC_y'(\Sigma + C_yYC_y')^{-1}C_yY\bigr)\Lambda_{11}' + \Lambda_{12}\Lambda_{11}'. \tag{3.1}$$

It is known [10] that if $\boldsymbol{\mathcal{A}}_{\mathbf{1,2}}$ hold, (3.1) admits a stabilizing solution $Y = Y' > 0$ for which

$$\bar{A}_1 := \Lambda_{11}\bigl(I - YC_y'(\Sigma + C_yYC_y')^{-1}C_y\bigr) \tag{3.2}$$

is Schur. The discrete Lyapunov equation

$$X = \bar{A}_1'X\bar{A}_1 + C_y'(\Sigma + C_yYC_y')^{-1}C_y \tag{3.3}$$

is then always solvable by an $X = X' \geq 0$. Denote then $P := I - YC_y'(\Sigma + C_yYC_y')^{-1}C_y$ and define the matrix

$$\begin{bmatrix} S_{\gamma,11} & S_{\gamma,12} \\ S_{\gamma,12}' & S_{\gamma,22} \end{bmatrix} := -\begin{bmatrix} 0 & 0 \\ 0 & \bar{A}_1'X\bar{A}_1 \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & P' \end{bmatrix}\begin{bmatrix} -X & \Gamma_{22} + \Gamma_{21}PY \\ I - YX & \Gamma_{12} + \Gamma_{11}PY \end{bmatrix}^{-1}\begin{bmatrix} I & \Gamma_{21}P \\ Y & \Gamma_{11}P \end{bmatrix} \tag{3.4}$$

($S_{\gamma,11} = S_{\gamma,11}' \leq 0$ and $S_{\gamma,22} = S_{\gamma,22}' \leq 0$). The second DARE,

$$Y_\gamma = S_{\gamma,12}(I + Y_\gamma S_{\gamma,22})^{-1}Y_\gamma S_{\gamma,12}' - S_{\gamma,11}, \tag{3.5}$$

is γ-dependent and its solution, which exists if γ is sufficiently large, is said to be stabilizing if $\det(I + Y_\gamma S_{\gamma,22}) \neq 0$ and the matrix $S_{\gamma,12}(I + Y_\gamma S_{\gamma,22})^{-1}$ is Schur.

The main result of this paper is then formulated as follows:

THEOREM 3.1. *Let the signal generator $\mathcal{G}$ be given by* (2.1) *and assumptions $\boldsymbol{\mathcal{A}}_{\mathbf{1,2}}$ hold. Then* $\mathbf{RP}_{\boldsymbol{\gamma},\boldsymbol{l}}$ *is solvable iff γ satisfies the following conditions:*

1. $\gamma > \|D_v\|$,
2. $\Gamma_{12}(t) + \Gamma_{11}(t)PY$ *is nonsingular* $\forall t \in (0, h]$,
3. $\rho\bigl((I - Y(\Gamma_{22} + \Gamma_{21}PY)(\Gamma_{12} + \Gamma_{11}PY)^{-1})(I - YX)\bigr) < 1$,
4. *there is a stabilizing solution* $Y_\gamma = Y_\gamma' \geq 0$ *to the DARE* (3.5) *and* $\rho(Y_\gamma S_{\gamma,22}) < 1$,
5. $\rho(Y_\gamma \bar{A}_l' X \bar{A}_l) < 1$, *where* $\bar{A}_i := \bar{A}_1^i$

(the first two conditions guarantee the well-posedness of (3.4)*). If these conditions hold, then*

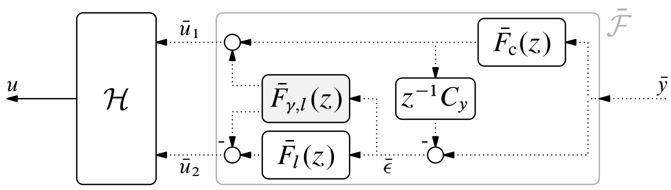


FIG. 3.1. *γ-suboptimal solution*

the estimator depicted in Fig. 3.1 solves the problem. It is the cascade of a discrete estimator, $\bar{\mathcal{F}}$, and a generalized zero-order hold, $\mathcal{H}$, with the hold function

$$\phi_h(t) := \begin{bmatrix} C_v & 0 \end{bmatrix}\Lambda(t - h)\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}\mathbb{1}_{[0,h)}(t). \tag{3.6}$$

The components of the discrete filter are

$$\bar{F}_c(z) = z \left[\begin{array}{c|c} \bar{A}_1 & \Lambda_{11} Y C_y' (\Sigma + C_y Y C_y')^{-1} \\ \hline I & 0 \end{array} \right], \tag{3.7a}$$

$$\bar{F}_{\gamma,l}(z) = z^{l+1} \left[\begin{array}{c|c} \bar{A}_1 \Delta_{l+1}^{-1} \Delta_l & \bar{A}_1 \Delta_{l+1}^{-1} Y_\gamma \bar{A}_l' C_y' (\Sigma + C_y Y C_y')^{-1} \\ \hline I & 0 \\ X - \bar{A}_l' X \bar{A}_l' & 0 \end{array} \right], \tag{3.7b}$$

$$\bar{F}_l(z) = \sum_{i=0}^{l-1} \bar{A}_{l-1-i}' C_y' (\Sigma + C_y Y C_y')^{-1} z^{l-i}, \tag{3.7c}$$

where $\Delta_i := I - Y_\gamma \bar{A}_i' X \bar{A}_i$.

Proof. Omitted because of space limitations. ☐

Some remarks are in order:

Remark 3.1 (solvability conditions). The first four conditions of Theorem 3.1 do not depend on the smoothing lag l. These are the necessary and sufficient conditions for the solvability of the L^∞ *fixed-interval* smoothing problem ($l \to \infty$). The fifth solvability condition of Theorem 3.1 reflects then constraints imposed by a finite preview. Because $\bar{A}_1$ is Schur, $\rho(Y_\gamma \bar{A}_l' X \bar{A}_l)$, as a function of l, is upperbounded by an exponentially decreasing function. Hence, whenever Y_γ is bounded, there exist a finite l for which the causality constraint becomes inactive. ▽

Remark 3.2 (solvability for $l = \infty$). It can be shown [9] that γ satisfies the first four conditions of Theorem 3.1 iff

$$\gamma > \gamma_h := \left\| \left[\begin{array}{cc} \mathcal{G}_v & 0 \end{array} \right] - \mathcal{G}_v (\mathcal{S}\mathcal{G}_y)^* (\Sigma + \mathcal{S}\mathcal{G}_y (\mathcal{S}\mathcal{G}_y)^*)^{-1} \left[\begin{array}{cc} \mathcal{S}\mathcal{G}_y & \Sigma^{1/2} \end{array} \right] \right\|_{L^2 \times \ell^2 \to L^2}.$$

In the case when $\Sigma > 0$, this γ_h can be characterized via the self-adjoint operator $\breve{O}_\gamma(\mathrm{e}^{\mathrm{j}\theta})$, described by the following two-point boundary condition system [5]:

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} A & BB' \\ 0 & -A' \end{bmatrix} x(t) + \begin{bmatrix} BD_v' \\ -C_v' \end{bmatrix} u(t), \quad \mathrm{e}^{\mathrm{j}\theta} x(0) = \begin{bmatrix} I & 0 \\ C_y' \Sigma^{-1} C_y & I \end{bmatrix} x(h) \\ y(t) = \begin{bmatrix} C_v & D_v B' \end{bmatrix} x(t) + D_v D_v' u(t) \end{cases}$$

Namely, $\gamma > \gamma_h$ iff $\breve{O}_\gamma(\mathrm{e}^{\mathrm{j}\theta}) < \gamma^2 I$ for all $-\pi < \theta \le \pi$. Thus, γ_h is the largest γ for which the symplectic matrix

$$M_\gamma := \begin{bmatrix} I & 0 \\ C_y' \Sigma^{-1} C_y & I \end{bmatrix} \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}$$

has unit circle eigenvalues. The matrix M_γ is actually similar to the symplectic matrix associated with the sampled-data H^∞ filtering Riccati equation in [15, 20] and it becomes the symplectic matrix associated with (3.1) as $\gamma \to \infty$. ▽

Remark 3.3 (recovering the L^2 solution). The only difference between the the L^∞ estimator of Theorem 3.1 and the L^2 solution of [10] is the presence of $\bar{F}_{\gamma,l}$, the gray block in Fig. 3.1, in the former. This block vanishes in two limiting cases. First, because $\bar{A}_1$ is Schur, $\lim_{l \to \infty} \bar{A}_l = 0$ and the fixed-interval solution is independent of γ (provided it satisfies the first four conditions of Theorem 3.1, of course) and approaches the L^2-optimal solution. Second, it follows from the proof of Theorem 3.1 that

$$\lim_{\gamma \to \infty} \begin{bmatrix} S_{\gamma,11} & S_{\gamma,12} \\ S_{\gamma,12}' & S_{\gamma,22} \end{bmatrix} = \begin{bmatrix} 0 & \bar{A}_1 \\ \bar{A}_1' & 0 \end{bmatrix}.$$

In this case (3.5) reads $Y_\gamma = \bar{A}_1 Y_\gamma \bar{A}_1'$ and its stabilizing solution is $Y_\gamma = 0$. Hence, the gray block vanishes for $\gamma \to \infty$ too (in this case the conditions of Theorem 3.1 hold $\forall h > 0$). ▽

4. Example: causal L^∞ cubic splines. Consider the problem with

$$G_v(s) = G_y(s) = \frac{1}{s^2} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \end{array}\right] \quad \text{and} \quad \Sigma = 0,$$

which does satisfy assumptions $\boldsymbol{\mathcal{A}}_{\mathbf{1,2}}$. Without loss of generality we may assume that $h = 1$. In the non-causal case ($l = \infty$) this setting reproduces the *cardinal cubic B-splines* [22], which are perhaps the most thoroughly studied polynomial splines. It is worth emphasizing that in that case the L^2 and L^∞ criteria result in identical estimators, which is a known property of non-causal solutions [6, §10.4.2]. Then, in [10], we studied the L^2 version of the problem under causality constraints, i.e., in the fixed-lag smoothing setup. The impulse response of the resulting estimators could then be regarded as causal cubic splines. If causality constraints are present, L^2 (mean-square) solutions are no longer identical to L^∞ (minmax) solutions. It is therefore of interest to see how cardinal cubic splines evolve under causality constraints in the L^∞ setting. This is the main goal of this section.

Although $\mathbf{RP}_{\gamma,l}$ studies a suboptimal solution ($\|\mathcal{G}_{\mathrm{e}}\|_\infty < \gamma$), in this section we consider the optimal case corresponding to $\|\mathcal{G}_{\mathrm{e}}\|_\infty \le \gamma$. This is done by addressing the limiting γ, in which case the DARE (3.5) no longer has a stabilizing solution, but still has a real positive definite one. In general, this might be a delicate procedure [16], but it works for this specific example painlessly, with the last condition of Theorem 3.1 replaced with $\rho(Y_\gamma \bar{A}_l' X \bar{A}_l) \le 1$.

First, let us calculate the matrices associated with the L^2 solution. They are

$$\Lambda(t) = \left[\begin{array}{cc|cc} 1 & t & -t^3/6 & t^2/2 \\ 0 & 1 & -t^2/2 & t \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & -t & 1 \end{array}\right], \quad Y = \frac{1}{6}\begin{bmatrix} 2+\sqrt{3} & 3+\sqrt{3} \\ 3+\sqrt{3} & 6+\sqrt{3} \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} \sqrt{3}-3 & 1 \\ \sqrt{3}-3 & 1 \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & 0 \\ \sqrt{3}-3 & 1 \end{bmatrix}, \quad \text{and} \quad X = \begin{bmatrix} 6\sqrt{3}-6 & 3-3\sqrt{3} \\ 3-3\sqrt{3} & \sqrt{3} \end{bmatrix}.$$

Then the hold function defined by (3.6) is

$$\phi_h(t) = \begin{bmatrix} 1 & -1+t & t(-t^2+3t+\sqrt{3})/6 & t(3t+\sqrt{3})/6 \end{bmatrix} \mathbb{1}_{[0,1)}(t).$$

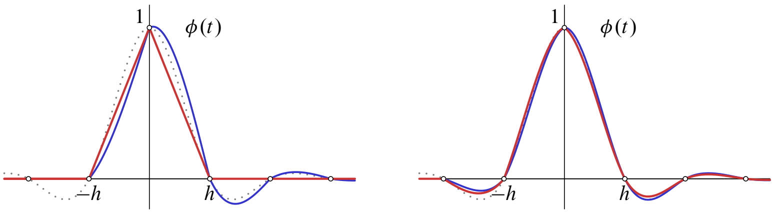
Using the arguments of Remark 3.2, it can be shown that the minimal achievable $\|\mathcal{G}_{\mathrm{e}}\|_\infty$ in the non-causal case is $\gamma = 1/\pi^2 \approx 0.1013$. For this γ the first three conditions of Theorem 3.1 hold, the matrix

$$\Gamma = \left[\begin{array}{cc|cc} (\sinh\frac{\pi}{2})^2 & \frac{1}{2\pi}\sinh\pi & -\frac{1}{3\pi^3}\sinh\pi & \frac{1}{2\pi^2}(1+\cosh\pi) \\ \frac{\pi}{2}\sinh\pi & (\sinh\pi)^2 & -\frac{1}{2\pi^2}(1+\cosh\pi) & \frac{1}{2\pi}\sinh\pi \\ \hline -\frac{\pi^3}{2}\sinh\pi & -\frac{\pi^2}{2}(1+\cosh\pi) & (\sinh\frac{\pi}{2})^2 & -\frac{\pi}{2}\sinh\pi \\ \frac{\pi^2}{2}(1+\cosh\pi) & \frac{\pi}{2}\sinh\pi & -\frac{1}{2\pi}\sinh\pi & (\sinh\frac{\pi}{2})^2 \end{array}\right]$$

and then

$$S_\gamma = \left[\begin{array}{cc|cc} -15.410 & -17.939 & -4.271 & 3.369 \\ -17.939 & -20.935 & -4.624 & 3.647 \\ \hline -4.271 & -4.624 & -1.045 & 0.825 \\ 3.369 & 3.647 & 0.825 & -0.650 \end{array}\right] \quad \text{and} \quad Y_\gamma = \begin{bmatrix} 25.900 & 29.296 \\ 29.296 & 33.229 \end{bmatrix}.$$

The latter is positive definite and such that the eigenvalues of $S_{\gamma,12}(I + Y_\gamma S_{\gamma,22})^{-1}$ are $\{-1, 0\}$ ($S_{\gamma,12}$ is singular and Y_γ is a semi-stabilizing solution of (3.5) now).



(a) $l = 1$ (b) $l = 2$

FIG. 4.1. *Hold functions $\phi(t)$ (red lines: L^∞, blue lines: L^2, dotted gray lines: $l = \infty$)*

Now,

$$\rho(Y_\gamma \bar{A}_1' X \bar{A}_1) = \rho\left(\begin{bmatrix} \pi^2(3+\sqrt{3})/6 & -\pi^2(2+\sqrt{3})/6 \\ \pi^2 - 3 + \sqrt{3} & 1 - \pi^2(3+\sqrt{3})/6 \end{bmatrix}\right) = 1,$$

which implies that the fixed-interval performance $\gamma = 1/\pi^2$ is achievable for every $l \in \mathbb{N}$.

We consider two cases: $l = 1$ and $l = 2$. The discrete filters $\bar{\mathcal{F}}$ in Fig. 3.1 for these smoothing lags have transfer functions

$$\bar{F}(z) = \begin{bmatrix} z \\ z-1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \bar{F}(z) = \begin{bmatrix} -z^3 + (4-\sqrt{3})z^2 + 2(2+\sqrt{3})z - 2 - \sqrt{3} \\ -(z-1)(z^2 - (3-\sqrt{3})z - 3 - \sqrt{3}) \\ -6(z-1)^3 \\ 6z(z-1)^2 \end{bmatrix} \frac{1}{4z+1},$$

respectively. Note that the dynamics of the causal part of $\bar{\mathcal{F}}$ depend on l. In fact, as l increases, their pole approaches $\alpha := \sqrt{3} - 2$ via the sequence $\{-\frac{1}{4}, -\frac{4}{15}, -\frac{15}{56}, -\frac{56}{209}, -\frac{209}{780}, \ldots\}$. This is in contrast to the L^2 case, where the causal pole is located at α at every l.

The resulting hold functions are presented in Fig. 4.1 by red lines. For the sake of comparison, blue lines there show the corresponding L^2 solutions of [10] and dotted lines show the fixed-interval solution (cardinal cubic B-spline). It is seen from Fig. 4.1(a) that in the case of $l = 1$ we end up with the *predictive first-order hold* (linear interpolator), whose hold function

$$\phi(t) = (1 - |t|)\mathbb{1}_{[-1,1]}(t)$$

is linear in t. This is surprising because this function is both L^2 and L^∞ optimal also in the case when $G_v(s) = G_y(s) = 1/s$ for every $l \in \mathbb{N}$ [11, Sec. III]. For $l > 1$ the optimal holds of Theorem 3.1 are cubic in t and are qualitatively closer to the corresponding L^2 solutions.

It is worth emphasizing that the L^∞ hold functions shown in Fig. 4.1, as well as every L^∞ hold for $l > 2$, attain the very same $\|\mathcal{G}_\mathrm{e}\|_\infty = 1/\pi^2$. Yet as l increases, the L^2 performance improves, see [13, Sec. 4]. For example, if $l = 1$, the L^∞ estimator attains $\|\mathcal{G}_\mathrm{e}\|_2 \approx 0.1054$, which amounts to some 120% of the optimal L^2 performance level for $l = 1$. If $l = 2$, we have $\|\mathcal{G}_\mathrm{e}\|_2 \approx 0.0773$, which amounts to about 101.3% of the corresponding optimal value.

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