

Consensus in the network with nonuniform constant input delay

Anton A. Stoorvogel¹, Ali Saberi²

Abstract. This paper studies consensus among identical agents that are at most critically unstable and coupled through networks with nonuniform constant input delay. An upper bound for delay tolerance is obtained which explicitly depends on agent dynamics. For any delay satisfying this upper bound, a controller design methodology without exact knowledge of the network topology is proposed so that multi-agent consensus in a set of unknown networks can be achieved.

I. INTRODUCTION

The consensus problem in a network has received substantial attention in recent years, partly due to the wide applications in areas such as sensor networks and autonomous vehicle control. A relatively complete coverage of earlier work can be found in the survey paper of [8], the recent books by [14], [10] and references therein.

Consensus in a network with time delay has been extensively studied in the literature. Most results consider the agent model as described by single-integrator dynamics [1], [11], [9], or double-integrator dynamics [12], [4], [2]. Specifically, it is shown by [9] that a network of single-integrator agents subject to uniform constant input delay can achieve consensus with a particular linear local control protocol if and only if the delay is bounded by a maximum that is inversely proportional to the largest eigenvalue of the graph Laplacian associated with the network. Sufficient conditions for consensus among agents with first order dynamics were also obtained in [11]. The results in [9] were extended in [4], [2] to double integrator dynamics. An upper bound on the maximum network delay tolerance for second-order consensus of multi-agent systems with any given linear control protocol was obtained. In the paper [13] we established for homogeneous networks an explicit design of a protocol which achieves consensus for the network given a constant, uniform but unknown delay provided the delay satisfies an explicit upper bound.

The result for single-integrator networks was later on generalized in [1] to non-uniform constant or time-varying delays. Also [6], [7] have recently presented interesting results on robust consensus of linear multi-agent systems (MAS) subject to non-uniform feedback delays. These works are more general and realistic because of the nonuniformity

of the delays. However, the latter papers only give a protocol design methodology for single integrator networks while, for a general network, they only present methods to verify robustness to input delays.

The objective of this paper is to extend [13] to the case of nonuniform delays. We study the multi-agent consensus problem with nonuniform input delays. The agents are assumed to be multi-input and multi-output and at most critically unstable, i.e. each agent has all its eigenvalues in the closed left half plane. In other words, we allow the agents to have eigenvalues on the imaginary axis. We find a sufficient condition on the tolerable input delay for agents with high-order dynamics, which has an explicit dependence on the agent dynamics and network topology. Moreover, in a special case where the agents only have zero eigenvalues, such as single- and double-integrator dynamics, arbitrarily large but bounded delay can be tolerated. Another layer of contribution is that for delays satisfying the proposed upper bound, we present a controller design methodology without precise knowledge of network topology so that the multi-agent consensus in a set of unknown networks can be achieved.

II. PROBLEM FORMULATION

Consider a network of N identical agents

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + Bu_i(t - \tau_i), & i = 1, \dots, N, \\ z_i(t) = \sum_{j=1}^N \ell_{ij} x_j(t). \end{cases} \quad (1)$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $z_i \in \mathbb{R}^n$, τ_1, \dots, τ_N are unknown constants satisfying $\tau_i \in [0, \bar{\tau}]$ for $i = 1, \dots, n$. The coefficients ℓ_{ij} are such that $\ell_{ij} \leq 0$ for $i \neq j$ and $\ell_{ii} = -\sum_{j \neq i}^N \ell_{ij}$. In (1), each agent collects a delayed measurement z_i of the state of neighboring agents through the network, which we refer to as *full-state coupling*.

It is also common that z_i consists of the outputs of neighboring agents instead of the complete states which can be formulated as follows:

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + Bu_i(t - \tau_i), \\ y_i(t) = Cx_i(t), \\ z_i(t) = \sum_{j=1}^N \ell_{ij} y_j(t), \end{cases} \quad i = 1, \dots, N, \quad (2)$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $y_i, z_i \in \mathbb{R}^p$. We refer to the agents in this case as having *partial-state coupling*.

The goal is to make the agents asymptotically converge to a reference trajectory. In the full-state coupling case, the reference trajectory in this paper is generated by an autonomous exosystem of the form:

$$\dot{x}_r = Ax_r, \quad x_r(0) = x_{r0}, \quad (3)$$

^{*}Department of Electrical Engineering, Mathematics and Computer Science, University of Twente, P.O. Box 217, Enschede, The Netherlands.
E-mail: A.A.Stoorvogel@utwente.nl.

[†]School of Electrical Engineering and Computer Science, Washington State University, Pullman, WA 99164-2752, U.S.A.
E-mail: saberi@eecs.wsu.edu.

where $x_r \in \mathbb{R}^n$.

Definition II.1 Consider the network described by (1). The agents in the network achieve **regulated state consensus** if

$$\lim_{t \rightarrow \infty} (x_i(t) - x_r(t)) = 0, \quad \forall i = 1, \dots, N.$$

Remark. Note that in the case of state-coupling if the network graph has a directed spanning tree then the root agent can serve as exosystem. The latter implies that the root agent sets its input to zero and serves as the exosystem for the other agents. This clearly requires some obvious modifications to our definitions of L and ψ_i .

In the partial-state coupling case, the reference trajectory in this paper is generated by an autonomous exosystem of the form:

$$\begin{aligned} \dot{x}_r &= Sx_r, & x_r(0) &= x_{r0}, \\ y_r &= Rx_r \end{aligned} \quad (4)$$

where $x_r \in \mathbb{R}^r$ with (R, S) observable while S has only eigenvalues in the closed left half plane.

Definition II.2 Consider the network described by (2). The agents in the network achieve **regulated partial-state consensus** if

$$\lim_{t \rightarrow \infty} (y_i(t) - y_r(t)) = 0, \quad \forall i = 1, \dots, N.$$

In order to achieve our goal, it is clear that a non-empty subset of agents must have knowledge of their output relative to the reference trajectory y_r generated by the reference system. Specially, each agent has access to the quantity

$$\psi_i = \iota_i(x_i - x_r), \quad \iota_i = \begin{cases} 1, & i \in \nu, \\ 0, & i \in \nu, \end{cases} \quad (5)$$

in the full-state case or

$$\psi_i = \iota_i(y_i - y_r), \quad \iota_i = \begin{cases} 1, & i \in \nu, \\ 0, & i \in \nu, \end{cases} \quad (6)$$

in the case of partial-state coupling. In the above, ν is a subset of $\{1, \dots, N\}$ which we will refer to as the root set.

The Laplacian matrix $L = \{\ell_{ij}\} \in \mathbb{R}^{N \times N}$ defines a *communication topology* that can be captured by a weighted graph $G = (\mathcal{N}, \mathcal{E})$ where $(j, i) \in \mathcal{E} \iff \ell_{ij} < 0$. The graph G is, in general, directed. However, in the special case where G is undirected, we obtain a symmetric matrix L . Based on the Laplacian matrix L of our network graph G we define the expanded Laplacian

$$\bar{L} = L + \text{diag}\{\iota_i\} = [\bar{\ell}_{ij}].$$

Note that \bar{L} clearly is not a Laplacian matrix associated to some graph since it does not have a zero row sum.

It should be noted that, in practice, perfect information of the communication topology is usually not available for controller design and that only some rough characterization of the network can be obtained. Using the non-zero eigenvalues of L as a ‘‘measure’’ for the graph, we can introduce

the following definition to characterize a set of unknown communication topologies.

Definition II.3 For given root set ν , $\beta, \gamma > 0$ and N , the set $\mathbb{C}_{\beta, \gamma}^{N, \nu}$ is the set of undirected graphs composed of N nodes satisfying the following property:

The eigenvalues of the associated expanded Laplacian \bar{L} , denoted by $\lambda_1, \dots, \lambda_N$, satisfy $\beta < \lambda_i < \gamma$ for $i = 1, \dots, N$.

Remark. The fact that we deal with an undirected graph implies that the expanded Laplacian \bar{L} is symmetric and hence its eigenvalues are all real. The fact that all eigenvalues are nonzero is equivalent (see [3, Lemma 7]) to the condition that the associated network graph is such that each agent is part of a directed spanning tree with a root agent which is in the root set ν of agents with direct access to information about the exosystem.

Assumption II.4 The following assumptions are made throughout the paper:

- (i) The agents are at most critically unstable, that is, A has all its eigenvalues in the closed left half plane;
- (ii) (A, B) is stabilizable and (A, C) is detectable;

Two consensus problems for agents with full-state coupling (1) and partial-state coupling (2) can be formulated for this set of networks respectively as follows.

Problem II.5 Consider a network of agents (1) with full state coupling. The consensus problem, given a set of possible communication topologies $\mathbb{C}_{\beta, \gamma}^{N, \nu}$ and a delay upper bound $\bar{\tau}$, is to design linear static controllers $u_i = Fz_i$ for $i = 1, \dots, N$ such that the agents (1) with $u_i = Fz_i$ achieve consensus with any communication topology belonging to $\mathbb{C}_{\beta, \gamma}^{N, \nu}$ and for $\tau_1, \dots, \tau_N \leq \bar{\tau}$.

Problem II.6 Consider a network of agents (2) with partial state coupling. The consensus problem with a set of possible communication topologies $\mathbb{C}_{\beta, \gamma}^{N, \nu}$ and a delay upper bound $\bar{\tau}$ is to design linear dynamic control protocols of the form:

$$\begin{cases} \dot{\chi}_i = A_c \chi_i + B_c z_i \\ u_i = C_c \chi_i, \end{cases} \quad (7)$$

for $i = 1, \dots, N$ such that the agents (2) with controller (7) achieve consensus with any communication topology belonging to $\mathbb{C}_{\beta, \gamma}^{N, \nu}$ and for $\tau_1, \dots, \tau_N \leq \bar{\tau}$.

III. CONSENSUS WITH FULL-STATE COUPLING

In this section, we consider the almost regulated output synchronization problem for homogeneous multi-agents systems defined in (1), where the goal is to make the agents asymptotically converge to a reference trajectory in the presence of external disturbances. The reference trajectory in this paper is generated by an autonomous system (3).

For a given set of networks $\mathbb{G}_{\beta,\gamma}^{N,v}$, we design a decentralized local consensus controller for any network in $\mathbb{G}_{\beta,\gamma}^{N,v}$ as follows:

$$u_i = -\alpha B' P_\varepsilon (z_i + \psi_i). \quad (8)$$

Here P_ε is the positive definite solution of the algebraic Riccati equation:

$$A' P_\varepsilon + P_\varepsilon A - P_\varepsilon B B' P_\varepsilon + \varepsilon I = 0. \quad (9)$$

and ε , as well as α , are design parameters which will be chosen according to β and γ so that the multi-agent consensus can be achieved with any communication topology belonging to $\mathbb{G}_{\beta,\gamma}^{N,v}$. Let

$$\omega_{\max} = \begin{cases} 0, & A \text{ is Hurwitz.} \\ \max\{\omega \in \mathbb{R} \mid \det(j\omega I - A) = 0\}, & \text{otherwise.} \end{cases}$$

Theorem III.1 For a given set $\mathbb{G}_{\beta,\gamma}^{N,v}$ with $\beta > 0$ and $\bar{\tau} > 0$, consider the agents (1) and any coupling network belonging to the set $\mathbb{G}_{\beta,\gamma}^{N,v}$. In that case Problem II.5 is solvable if,

$$\bar{\tau} \omega_{\max} < \frac{\pi}{2}. \quad (10)$$

Moreover, it can be solved by the consensus controller (8) if (10) holds. Specifically, for given $\mathbb{G}_{\beta,\gamma}^{N,v}$ and given $\bar{\tau}$ satisfying (10), there exist $\alpha > 0$ and $\varepsilon^* > 0$ such that for this α and any $\varepsilon \in (0, \varepsilon^*]$, the agents (1) with controller (8) achieve consensus for any communication topologies in $\mathbb{G}_{\beta,\gamma}^{N,v}$ and $\tau_1, \dots, \tau_N \in [0, \bar{\tau}]$.

Proof : Define $\tilde{x}_i := x_i - x_r$ as the regulated output synchronization error for agent $i \in \mathcal{V} := \{1, \dots, N\}$ and $\tilde{x} = \text{col}\{\tilde{x}_i\}$. We can write the full closed-loop system as:

$$\dot{\tilde{x}} = \tilde{A} \tilde{x} - \alpha \tilde{B} (D \bar{L} \otimes B' P_\varepsilon) \tilde{x}. \quad (11)$$

where

$$\tilde{A} = I_N \otimes A, \quad \tilde{B} = I_N \otimes B, \quad \tilde{P}_\varepsilon = I_N \otimes P_\varepsilon,$$

while D represents the delays:

$$D = \text{diag}\{D_i\} \quad \text{where} \quad (D_i z_i)(t) = z_i(t - \tau_i). \quad (12)$$

Based on the above delays, we define:

$$\tilde{D}(\omega) = \text{diag}\{e^{-j\omega\tau_i}\}. \quad (13)$$

Choose α such that

$$2\alpha\beta \cos(\bar{\tau}\omega_{\max}) > 1. \quad (14)$$

which is possible since $\bar{\tau}\omega_{\max} < \frac{\pi}{2}$. This implies in particular that $2\alpha\beta > 1$ and hence:

$$2\alpha\bar{L} \geq I \quad (15)$$

for any extended Laplacian \bar{L} associated to a network graph in $\mathbb{G}_{\beta,\gamma}^{N,v}$ since the smallest eigenvalue of the symmetric matrix \bar{L} is larger than β . We obtain that:

$$\begin{aligned} & \tilde{P}_\varepsilon (\tilde{A} - \alpha \tilde{B} (\bar{L} \otimes B' P_\varepsilon)) + (\tilde{A} - \alpha \tilde{B} (\bar{L} \otimes B' P_\varepsilon))' \tilde{P}_\varepsilon \\ & = -\varepsilon I + (I - 2\alpha\bar{L}) \otimes P_\varepsilon B B' P_\varepsilon \leq -\varepsilon I \end{aligned}$$

and hence $\tilde{A} - \alpha \tilde{B} (\bar{L} \otimes B' P_\varepsilon)$ is Hurwitz. It follows from [16] that system (11) is asymptotically stable if

$$\det \left[j\omega I - \tilde{A} + \alpha \tilde{B} (\tilde{D}(\omega) \bar{L} \otimes B' P_\varepsilon) \right] \neq 0, \quad (16)$$

for all $\omega \in \mathbb{R}$, for all $\tau_1, \dots, \tau_N \in [0, \bar{\tau}]$ and all possible \bar{L} associated to a network graph in $\mathbb{G}_{\beta,\gamma}^{N,v}$.

We note that given (14), there exists a $\delta > 0$ such that

$$2\alpha\beta \cos(\bar{\tau}(\omega_{\max} + \delta)) > 1. \quad (17)$$

Next we will split the proof of (16) in two cases where $|\omega| < \omega_{\max} + \delta$ and $|\omega| \geq \omega_{\max} + \delta$ respectively.

If $|\omega| \geq \omega_{\max} + \delta$, we have $\det(j\omega I - \tilde{A}) \neq 0$, which yields $\underline{\sigma}(j\omega I - \tilde{A}) > 0$. Hence, there exists $\mu > 0$ such that

$$\underline{\sigma}(j\omega I - \tilde{A}) > \mu, \quad \forall \omega, \text{ s.t. } |\omega| \geq \omega_{\max} + \delta. \quad (18)$$

To see this, note that for ω satisfying $|\omega| > \bar{\omega} := \max\{\|\tilde{A}\| + 1, \omega_{\max} + \delta\}$, we have $\underline{\sigma}(j\omega I - \tilde{A}) > |\omega| - \|\tilde{A}\| > 1$. But for ω with $|\omega| \in [\omega_{\max} + \delta, \bar{\omega}]$, there exists $\mu \in (0, 1]$ such that $\underline{\sigma}(j\omega I - \tilde{A}) \geq \mu$, which is due to the fact that $\underline{\sigma}(j\omega I - \tilde{A})$ depends continuously on ω .

Given α , there exists $\varepsilon^* > 0$ such that

$$\|\alpha \tilde{B} (\bar{L} \otimes B' P_\varepsilon)\| \leq \mu/2 \quad (19)$$

for $\varepsilon < \varepsilon^*$. Note that μ and ε^* can be chosen independent of \bar{L} but only relying on the upper bound γ for the largest eigenvalue of \bar{L} . Combining (18) and (19) we obtain:

$$\underline{\sigma}(j\omega I - \tilde{A} - \alpha \tilde{B} (\bar{L} \otimes B' P_\varepsilon)) \geq \frac{\mu}{2}$$

Therefore, (16) holds for $|\omega| \geq \omega_{\max} + \delta$.

It remains to verify (16) with $|\omega| < \omega_{\max} + \delta$. We will prove through a Lyapunov argument that

$$\tilde{A} - \alpha \tilde{B} (\tilde{D}(\omega) \bar{L} \otimes B' P_\varepsilon) \quad (20)$$

is Hurwitz for any fixed ω satisfying $|\omega| < \omega_{\max} + \delta$. This will clearly imply (16). We define:

$$\tilde{Q}_\varepsilon = \bar{L} \otimes P_\varepsilon$$

We obtain:

$$\begin{aligned} & \tilde{Q}_\varepsilon (\tilde{A} - \alpha \tilde{B} (\tilde{D}(\omega) \bar{L} \otimes B' P_\varepsilon)) \\ & \quad + (\tilde{A} - \alpha \tilde{B} (\tilde{D}(\omega) \bar{L} \otimes B' P_\varepsilon))' \tilde{Q}_\varepsilon \\ & = \bar{L} \otimes (A' P_\varepsilon + P_\varepsilon A) \\ & \quad - \alpha (\bar{L} \otimes P_\varepsilon B) ([\tilde{D}(\omega) + \tilde{D}^*(\omega)] \otimes I) (\bar{L} \otimes B' P_\varepsilon) \\ & = -\varepsilon (\bar{L} \otimes I) \\ & + (I \otimes P_\varepsilon B) \left(\left\{ \bar{L} - \alpha \bar{L} [\tilde{D}(\omega) + \tilde{D}^*(\omega)] \bar{L} \right\} \otimes I \right) (I \otimes B' P_\varepsilon) \\ & \leq -\varepsilon (\bar{L} \otimes I) \end{aligned}$$

In the last step we use that:

$$\bar{L} - \alpha \bar{L} [\tilde{D}(\omega) + \tilde{D}^*(\omega)] \bar{L} < 0 \quad (21)$$

To establish this, we note that:

$$\tilde{D}(\omega) + \tilde{D}^*(\omega) = \text{diag}(2 \cos(\omega\tau_i)) > \text{diag}(2 \cos(\omega\bar{\tau}))$$

The above implies that:

$$\alpha \left(\tilde{D}(\omega) + \tilde{D}^*(\omega) \right) > \bar{L}^{-1}$$

given (17) and the fact that the smallest eigenvalue of \bar{L} is larger than β . This implies

$$\alpha \bar{L} [\tilde{D}(\omega) + \tilde{D}^*(\omega)] \bar{L} > \bar{L} \bar{L}^{-1} \bar{L} = \bar{L}$$

and we obtain (21).

This establishes that

$$\begin{aligned} \tilde{Q}_\varepsilon \left(\tilde{A} - \alpha \tilde{B} (\tilde{D}(\omega) \bar{L} \otimes B' P_\varepsilon) \right) \\ + \left(\tilde{A} - \alpha \tilde{B} (\tilde{D}(\omega) \bar{L} \otimes B' P_\varepsilon) \right)^* \tilde{Q}_\varepsilon < 0 \end{aligned}$$

and hence (20) is Hurwitz for all ω satisfying $|\omega| < \omega_{\max} + \delta$ which implies (16). This completes the proof. ■

Remark. The consensus controller design depends only on the agent model and parameters $\bar{\tau}, \beta$ and γ and is independent of specific network topology.

IV. CONSENSUS WITH PARTIAL-STATE COUPLING

Next, we consider the case of partial-state coupling and design a controller of the form (7) which solves Problem II.6. We first define a modified version of ω_{\max} :

$$\begin{aligned} \tilde{\omega}_{\max} = \max\{\omega \in \mathbb{R} \mid \det(j\omega I - A) = 0 \\ \text{or } \det(j\omega I - S) = 0\}. \end{aligned}$$

Note that in partial-state coupling, we present an exosystem (4) which generates signals that the network needs to be tracked. It is kind of intuitive that if we increase the frequency of these reference signals then this will reduce in a reduced capability to withstand delays. In the case of state-coupling the exosystem had the same dynamics as the agents and hence the exosystem does not impose additional constraints on the delays.

We first design a precompensator. The objective is to find a precompensator of the form:

$$\begin{aligned} \dot{p}_i &= A_1 p_i + B_1 \tilde{u}_i, \\ u_i &= C_1 p_i, \end{aligned} \quad (22)$$

such that the interconnection of (2) and (22) is of the form:

$$\begin{cases} \dot{x}_{e,i}(t) = A_e x_{e,i}(t) + B_e \tilde{u}_i(t - \tau_i), \\ y_i(t) = C_e x_{e,i}(t), \\ z_i(t) = \sum_{j=1}^N \ell_{ij} y_j(t), \end{cases} \quad i = 1, \dots, N, \quad (23)$$

and there exists $\tilde{\Pi}$ such that:

$$\tilde{\Pi} S = A_e \tilde{\Pi}, \quad C_e \tilde{\Pi} = R \quad (24)$$

We start with Π and Γ which are uniquely determined by the so-called regulator equations:

$$\begin{aligned} A\Pi + B\Gamma &= \Pi S, \\ C\Pi &= R. \end{aligned} \quad (25)$$

The design of this precompensator is quite straightforward if, a priori, the agent has no dynamics in common with

the exosystem and (Π, S) is detectable. In that case, the precompensator is given by: $A_1 = S, C_1 = \Gamma$ while B_1 is chosen according to the technique presented in [5] to guarantee that the interconnection of (2) and (22) is stabilizable and observable. However, if A and S have common eigenvalues the design is a bit more involved. For details we refer to [15].

To design a dynamic low-gain consensus controller we start with the following algebraic Riccati equation:

$$A'_e P_{e,\varepsilon} + P_{e,\varepsilon} A_e - P_{e,\varepsilon} B_e B'_e P_{e,\varepsilon} + \varepsilon I = 0. \quad (26)$$

which has a unique solution $P_{e,\varepsilon} > 0$ for any $\varepsilon > 0$. Our controller is then constructed as

$$\begin{cases} \dot{\chi}_i = (A_e + K C_e) \chi_i - K z_i \\ \tilde{u}_i = -\alpha B'_e P_{e,\varepsilon} \chi_i, \end{cases} \quad (27)$$

where K is such that $A_e + K C_e$ is Hurwitz stable. α and ε are design parameters to be chosen later.

Our consensus controller is then the interconnection of (22) and (27) which together is a special form of (7). We will show that this consensus controller solves Problem II.6:

Theorem IV.1 For a given set $\mathbb{G}_{\beta,\gamma}^{N,v}$ with $\beta > 0$ and $\bar{\tau} > 0$, consider the agents (2) with any communication topology belonging to $\mathbb{G}_{\beta,\gamma}^{N,v}$. In that case, Problem II.6 is solvable if,

$$\bar{\tau} \tilde{\omega}_{\max} < \frac{\pi}{2}. \quad (28)$$

Moreover, it can be solved by the consensus controller consisting of (22) and (27) if (28) holds. Specifically, for given β, γ and $\bar{\tau}$ satisfying (28), there exist $\alpha > 0$ and ε^* such that for any $\varepsilon \in (0, \varepsilon^*)$, the agents (2) with controller (22) and (27) achieve consensus for any communication topology in $\mathbb{G}_{\beta,\gamma}^{N,v}$ and $\tau_1, \dots, \tau_N \in [0, \bar{\tau}]$.

Proof : The precompensator design ensures that there exist $\tilde{\Pi}$ satisfying (24). We define $\tilde{x}_{e,i} := x_{e,i} - \tilde{\Pi} x_r$ as the regulated output synchronization error for agent $i \in \{1, \dots, N\}$. Using the definition $\chi = \text{col}\{\tilde{x}_{e,1}, \dots, \tilde{x}_{e,n}, \chi_1, \dots, \chi_N\}$ we find:

$$\dot{\chi} = [\mathcal{A} + \mathcal{B}(D\bar{L} \otimes I)\mathcal{F}_\varepsilon] \chi \quad (29)$$

where

$$\begin{aligned} \mathcal{A} &= \begin{bmatrix} \tilde{A}_e & 0 \\ -\tilde{K}\tilde{C}_e & \tilde{A}_e + \tilde{K}\tilde{C}_e \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} \tilde{B}_e \\ 0 \end{bmatrix}, \\ \mathcal{F}_\varepsilon &= \begin{bmatrix} 0 & -\alpha \tilde{B}'_e \tilde{P}_{e,\varepsilon} \end{bmatrix}. \end{aligned}$$

while

$$\begin{aligned} \tilde{A}_e &= I_N \otimes A_e, \quad \tilde{B}_e = I_N \otimes B_e, \quad \tilde{C}_e = I_N \otimes C_e, \\ \tilde{K} &= I_N \otimes K, \quad \tilde{P}_{e,\varepsilon} = I_N \otimes P_{e,\varepsilon}, \end{aligned}$$

with the operator D defined in (12). Note that in obtaining the above we have used the fact that a delay D_i commutes with a linear time-invariant system, i.e. applying a signal to a time-invariant system and delaying the resulting output has the same effect as delaying the signal before applying

it to a time-invariant system providing you match the initial conditions of the system appropriately.

The system (29) can also be expressed as:

$$\dot{\tilde{\chi}} = \left[\tilde{\mathcal{A}} + \tilde{\mathcal{B}}(D\bar{L} \otimes I)\tilde{\mathcal{F}}_\varepsilon \right] \tilde{\chi} \quad (30)$$

using

$$\tilde{\chi} = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \chi$$

where

$$\tilde{\mathcal{A}} = \begin{bmatrix} \tilde{A}_e & 0 \\ 0 & \tilde{A}_e + \tilde{K}\tilde{C}_e \end{bmatrix}, \quad \tilde{\mathcal{B}} = \begin{bmatrix} \tilde{B}_e \\ -\tilde{B}_e \end{bmatrix},$$

$$\tilde{\mathcal{F}}_\varepsilon = -\alpha \begin{bmatrix} \tilde{B}'_e \tilde{P}_{e,\varepsilon} & \tilde{B}'_e \tilde{P}_{e,\varepsilon} \end{bmatrix}.$$

Choose α such that

$$\alpha\beta \cos(\bar{\tau}\tilde{\omega}_{\max}) > 1. \quad (31)$$

which is possible since $\bar{\tau}\tilde{\omega}_{\max} < \frac{\pi}{2}$. As noted before this implies:

$$\alpha\bar{L} \geq I \quad (32)$$

Consider the nominal system without delay. We will first establish that this system is asymptotically stable. Consider:

$$V(\tilde{\chi}) = \tilde{\chi}' \begin{pmatrix} \tilde{P}_{e,\varepsilon} & 0 \\ 0 & \mu Q \end{pmatrix} \tilde{\chi}$$

where Q is such that

$$\begin{aligned} & \left[\tilde{A}_e + \tilde{K}\tilde{C}_e + \alpha\tilde{B}'_e(\bar{L} \otimes B'_e P_{e,\varepsilon}) \right]' Q \\ & + Q \left[\tilde{A}_e + \tilde{K}\tilde{C}_e + \alpha\tilde{B}'_e(\bar{L} \otimes B'_e P_{e,\varepsilon}) \right] \leq -I \end{aligned} \quad (33)$$

Note that such a Q exists independent of \bar{L} and ε provided the eigenvalues of \bar{L} are smaller than γ and $\varepsilon < \varepsilon^*$. The existence of Q is obvious since $\tilde{A}_e + \tilde{K}\tilde{C}_e$ is Hurwitz stable while, for small ε^* , we can guarantee that $P_{e,\varepsilon}$ is arbitrarily small with \bar{L} is uniformly bounded. We get:

$$\begin{aligned} \dot{V} = & \tilde{\chi}'_1 \left[\tilde{A}'_e \tilde{P}_{e,\varepsilon} + \tilde{P}_{e,\varepsilon} \tilde{A}_e - 2\alpha \tilde{P}_{e,\varepsilon} \tilde{B}_e (\bar{L} \otimes I) \tilde{B}'_e \tilde{P}_{e,\varepsilon} \right] \tilde{\chi}_1 \\ & + 2\mu\alpha \tilde{\chi}'_1 \tilde{P}_{e,\varepsilon} \tilde{B}_e (\bar{L} \otimes I) \tilde{B}'_e Q \tilde{\chi}_2 \\ & - 2\alpha \tilde{\chi}'_1 \tilde{P}_{e,\varepsilon} \tilde{B}_e (\bar{L} \otimes I) \tilde{B}'_e \tilde{P}_{e,\varepsilon} \tilde{\chi}_2 \\ & + \mu \tilde{\chi}'_2 \left[Q(\tilde{A}_e + \tilde{K}\tilde{C}_e) + (\tilde{A}_e + \tilde{K}\tilde{C}_e)' Q \right. \\ & \left. + \alpha Q \tilde{B}_e (\bar{L} \otimes I) \tilde{B}'_e \tilde{P}_{e,\varepsilon} + \alpha \tilde{P}_{e,\varepsilon} \tilde{B}_e (\bar{L} \otimes I) \tilde{B}'_e Q \right] \tilde{\chi}_2 \end{aligned}$$

Using (26) and (33), we obtain:

$$\begin{aligned} \dot{V} \leq & -\varepsilon \tilde{\chi}'_1 \tilde{\chi}_1 - \tilde{v}' \tilde{v} - \mu \tilde{\chi}'_2 \tilde{\chi}_2 + 2\tilde{v} [(I - \alpha\bar{L}) \otimes I] \tilde{v} \\ & + 2\mu\alpha \tilde{v}' (\bar{L} \otimes I) \tilde{B}'_e Q \tilde{\chi}_2 - 2\alpha \tilde{v}' (\bar{L} \otimes B'_e P_{e,\varepsilon}) \tilde{\chi}_2 \end{aligned}$$

where $\tilde{v} = \tilde{P}_{e,\varepsilon} \tilde{\chi}_1$. Using (32), we get:

$$\begin{aligned} \dot{V} \leq & -\varepsilon \tilde{\chi}'_1 \tilde{\chi}_1 - \tilde{v}' \tilde{v} + 2\mu\alpha \tilde{v}' (\bar{L} \otimes I) \tilde{B}'_e Q \tilde{\chi}_2 \\ & - 2\alpha \tilde{v}' (\bar{L} \otimes B'_e P_{e,\varepsilon}) \tilde{\chi}_2 - \mu \tilde{\chi}'_2 \tilde{\chi}_2 \end{aligned}$$

Clearly, $\|\alpha(\bar{L} \otimes I) \tilde{B}'_e Q\| \leq M$ for some M and we choose μ such that $8\mu M^2 = 1$. Next, choose ε small enough such that

$$\|\alpha(\bar{L} \otimes B'_e P_{e,\varepsilon})\| \leq \frac{1}{8M}$$

Note that we can choose ε independent of the extended Laplacian matrix \bar{L} since we know this matrix is bounded given the upper bound γ for its largest eigenvalue. We get:

$$\begin{aligned} & 2\mu\alpha \tilde{v}' (\bar{L} \otimes I) \tilde{B}'_e Q \tilde{\chi}_2 - 2\alpha \tilde{v}' (\bar{L} \otimes B'_e P_{e,\varepsilon}) \tilde{\chi}_2 \\ & \leq \frac{1}{2M} \|\tilde{v}\| \|\tilde{\chi}_2\| \leq \tilde{v}' \tilde{v} + \frac{1}{16M^2} \tilde{\chi}'_2 \tilde{\chi}_2 \end{aligned}$$

Using this we obtain:

$$\dot{V} \leq -\varepsilon \tilde{\chi}'_1 \tilde{\chi}_1 - \frac{1}{16M^2} \tilde{\chi}'_2 \tilde{\chi}_2$$

which is obviously negative. This shows the system is stable without delays.

Next, we consider the case with delays. Based on [16], we only need to check whether:

$$\det \left[j\omega I - \tilde{\mathcal{A}} - \tilde{\mathcal{B}}(\tilde{D}(\omega)\bar{L} \otimes I)\tilde{\mathcal{F}} \right] \neq 0 \quad (34)$$

where $\tilde{D}(\omega)$ is defined by (13), is satisfied for all $\omega \in \mathbb{R}$, for all $\tau_1, \dots, \tau_N \in [0, \bar{\tau}]$ and all possible \bar{L} associated to a network graph in $\mathbb{G}_{\beta,\gamma}^{N,v}$.

As in the full-state case, we note that given (31), there exists a $\delta > 0$ such that

$$2\alpha\beta \cos(\bar{\tau}(\tilde{\omega}_{\max} + \delta)) > 1. \quad (35)$$

Next we will split the proof of (34) in two cases where $|\omega| < \tilde{\omega}_{\max} + \delta$ and $|\omega| \geq \tilde{\omega}_{\max} + \delta$ respectively.

If $|\omega| \geq \tilde{\omega}_{\max} + \delta$, we have $\det(j\omega I - \tilde{\mathcal{A}}) \neq 0$, which yields $\underline{\sigma}(j\omega I - \tilde{\mathcal{A}}) > 0$. Hence, there exists $\mu > 0$ such that

$$\underline{\sigma}(j\omega I - \tilde{\mathcal{A}}) > \mu, \quad \forall \omega, \text{ s.t. } |\omega| \geq \tilde{\omega}_{\max} + \delta. \quad (36)$$

Given α , there exists $\varepsilon^* > 0$ such that

$$\|\tilde{\mathcal{B}}(\tilde{D}(\omega)\bar{L} \otimes I)\tilde{\mathcal{F}}\| \leq \mu/2 \quad (37)$$

for $\varepsilon < \varepsilon^*$. Combining (36) and (37) we obtain:

$$\underline{\sigma} \left[j\omega I - \tilde{\mathcal{A}} - \tilde{\mathcal{B}}(\tilde{D}(\omega)\bar{L} \otimes I)\tilde{\mathcal{F}} \right] \geq \frac{\mu}{2}$$

Therefore, (34) holds for $|\omega| \geq \tilde{\omega}_{\max} + \delta$.

It remains to verify (34) with $|\omega| < \tilde{\omega}_{\max} + \delta$. We will prove through a Lyapunov argument that

$$\tilde{\mathcal{A}} - \tilde{\mathcal{B}}(\tilde{D}(\omega)\bar{L} \otimes I)\tilde{\mathcal{F}} \quad (38)$$

is Hurwitz for any fixed ω satisfying $|\omega| < \tilde{\omega}_{\max} + \delta$. This will clearly imply (34). Consider:

$$V(\tilde{\chi}) = \tilde{\chi}' \begin{pmatrix} \bar{L} \otimes P_{e,\varepsilon} & 0 \\ 0 & \mu \tilde{Q} \end{pmatrix} \tilde{\chi}$$

where \tilde{Q} is such that

$$\begin{aligned} & \left[\tilde{A}_e + \tilde{K}\tilde{C}_e + \alpha\tilde{B}'_e(\tilde{D}(\omega)\bar{L} \otimes B'_e P_{e,\varepsilon}) \right]^* \tilde{Q} \\ & + \tilde{Q} \left[\tilde{A}_e + \tilde{K}\tilde{C}_e + \alpha\tilde{B}'_e(\tilde{D}(\omega)\bar{L} \otimes B'_e P_{e,\varepsilon}) \right] \leq -I \end{aligned} \quad (39)$$

Note that such a \tilde{Q} exists independent of \bar{L} , ε , ω and the delays τ_1, \dots, τ_N provided the eigenvalues of \bar{L} are less than γ , $|\omega| < \tilde{\omega}_{\max} + \delta$, $\tau_1, \dots, \tau_N < \bar{\tau}$ and $\varepsilon < \varepsilon^*$. The existence of \tilde{Q} follows from the fact that $\tilde{A}_e + \tilde{K}\tilde{C}_e$ is Hurwitz stable while for small ε^* we can guarantee that $P_{e,\varepsilon}$ is arbitrarily small while \bar{L} and $\tilde{D}(\omega)$ are uniformly bounded. We get:

$$\begin{aligned} \dot{V} = & \tilde{\chi}'_1 \left[\bar{L} \otimes (A'_e P_{e,\varepsilon} + P_{e,\varepsilon} A_e) \right. \\ & \left. - \alpha (\bar{L} \otimes P_{e,\varepsilon} B_e) ([\tilde{D}(\omega) + \tilde{D}^*(\omega)] \otimes I) (\bar{L} \otimes B'_e P_{e,\varepsilon}) \right] \tilde{\chi}_1 \\ & + 2\mu\alpha \tilde{\chi}'_1 \tilde{P}_{e,\varepsilon} \tilde{B}_e (\tilde{D}(\omega) \bar{L} \otimes I) \tilde{B}'_e Q \tilde{\chi}_2 \\ & - 2\alpha \tilde{\chi}'_1 \tilde{P}_{e,\varepsilon} \tilde{B}_e (\tilde{D}(\omega) \bar{L} \otimes I) \tilde{B}'_e \tilde{P}_{e,\varepsilon} \tilde{\chi}_2 \\ & + \mu \tilde{\chi}'_2 \left[Q (\tilde{A}_e + \tilde{K}\tilde{C}_e) + (\tilde{A}_e + \tilde{K}\tilde{C}_e)' Q \right. \\ & \left. - \alpha Q \tilde{B}_e (\tilde{D}(\omega) \bar{L} \otimes I) \tilde{B}'_e \tilde{P}_{e,\varepsilon} \right. \\ & \left. - \alpha \tilde{P}_{e,\varepsilon} \tilde{B}_e (\bar{L} \tilde{D}^*(\omega) \otimes I) \tilde{B}'_e Q \right] \tilde{\chi}_2 \end{aligned}$$

Using (26) and (39), we obtain:

$$\begin{aligned} \dot{V} \leq & -\varepsilon \tilde{\chi}'_1 \tilde{\chi}_1 - \tilde{v}' \tilde{v} \\ & + \tilde{v}' \left[(\bar{L} - \alpha \bar{L} [\tilde{D}(\omega) + \tilde{D}^*(\omega)] \bar{L}) \otimes I \right] \tilde{v} \\ & + 2\mu\alpha \tilde{v}' (\tilde{D}(\omega) \bar{L} \otimes I) \tilde{B}'_e Q \tilde{\chi}_2 \\ & - 2\alpha \tilde{v}' (\tilde{D}(\omega) \bar{L} \otimes B'_e P_{e,\varepsilon}) \tilde{\chi}_2 - \mu \tilde{\chi}'_2 \tilde{\chi}_2 \end{aligned}$$

Using (21), we get:

$$\begin{aligned} \dot{V} \leq & -\varepsilon \tilde{\chi}'_1 \tilde{\chi}_1 - \tilde{v}' \tilde{v} + 2\mu\alpha \tilde{v}' (\tilde{D}(\omega) \bar{L} \otimes I) \tilde{B}'_e Q \tilde{\chi}_2 \\ & - 2\alpha \tilde{v}' (\tilde{D}(\omega) \bar{L} \otimes B'_e P_{e,\varepsilon}) \tilde{\chi}_2 - \mu \tilde{\chi}'_2 \tilde{\chi}_2 \end{aligned}$$

Clearly,

$$\|\alpha (\tilde{D}(\omega) \bar{L} \otimes I) \tilde{B}'_e Q\| \leq \tilde{M}$$

for some \tilde{M} and we choose μ such that $8\mu\tilde{M}^2 = 1$. Next, choose ε small enough such that

$$\|\alpha (\tilde{D}(\omega) \bar{L} \otimes B'_e P_{e,\varepsilon})\| \leq \frac{1}{8\tilde{M}}$$

We can choose ε independent of ω and \bar{L} since $\tilde{D}(\omega)$ and \bar{L} are uniformly bounded given our upper bound γ for the largest eigenvalue of \bar{L} . We get:

$$\begin{aligned} 2\mu\alpha \tilde{v}' (\tilde{D}(\omega) \bar{L} \otimes I) \tilde{B}'_e Q \tilde{\chi}_2 - 2\alpha \tilde{v}' (\tilde{D}(\omega) \bar{L} \otimes B'_e P_{e,\varepsilon}) \tilde{\chi}_2 \\ \leq \frac{1}{2\tilde{M}} \|\tilde{v}\| \|\tilde{\chi}_2\| \leq \tilde{v}' \tilde{v} + \frac{1}{16\tilde{M}^2} \tilde{\chi}'_2 \tilde{\chi}_2 \end{aligned}$$

Using this we obtain:

$$\dot{V} \leq -\varepsilon \tilde{\chi}'_1 \tilde{\chi}_1 - \frac{1}{16\tilde{M}^2} \tilde{\chi}'_2 \tilde{\chi}_2$$

which is obviously negative. This establishes that (38) is Hurwitz for all ω satisfying $|\omega| < \tilde{\omega}_{\max} + \delta$ which implies (34). This completes the proof. ■

Remark. The low-gain compensator (27) is constructed based on the agent model, the upper bound $\bar{\tau}$ for the delays and the network characteristics β and γ . Explicit knowledge of the network or delays are not needed.

Corollary IV.2 For a given set $\mathbb{G}_{\beta,\gamma}^{N,v}$ with $\beta > 0$ and $\bar{\tau} > 0$, consider the agents (2) with any communication topology

belonging to $\mathbb{G}_{\beta,\gamma,v}$. Suppose the eigenvalues of A are either zero or in the open left half plane. In that case, Problem II.6 is solvable by the consensus controller (27). Specifically, for given β , γ and $\bar{\tau} > 0$, there exist $\alpha > 0$ and $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*]$, the agents (2) with controller (27) achieve consensus for any communication topology in $\mathbb{G}_{\beta,\gamma}^{N,v}$ and $\tau \in [0, \bar{\tau}]$.

V. CONCLUDING REMARKS

In this paper, we study the multi-agent consensus with nonuniform constant input delay for agents with high-order dynamics. A sufficient bound on the delay is derived under which the multi-agent consensus is attainable. Whenever this condition is satisfied, a controller without the exact knowledge of network topology can be constructed such that consensus can be achieved in a set of networks. The next objective is to see whether a similar result can be obtained for directed networks.

REFERENCES

- [1] P. BLIMAN AND G. FERRARI-TRECCATE, "Average consensus problems in networks of agents with delayed communications", *Automatica*, 44 (2008), pp. 1985–1995.
- [2] B.M. CHEN, "On properties of the special coordinate basis of linear systems", *Int. J. Contr.*, 71 (1998), pp. 981–1003.
- [3] H.F. GRIP, T. YANG, A. SABERI, AND A.A. STORVOGEL, "Output synchronization for heterogeneous networks of non-introspective agents", *Automatica*, 48 (2012), pp. 2444–2453.
- [4] P. LIN, Y. JIA, J. DU, AND S. YUAN, "Distributed Consensus Control for Second-Order Agents with Fixed Topology and Time-Delay", in *Chinese Control Conference*, Hunan, China, 2007, pp. 577–581. in Chinese.
- [5] X. LIU, B.M. CHEN, AND Z. LIN, "On the problem of general structure assignments of linear systems through sensor/actuator selection", *Automatica*, 39 (2003), pp. 233–241.
- [6] U. MÜNZ, A. PAPACHRISTODOULOU, AND F. ALLGÖWER, "Delay robustness in consensus problems", *Automatica*, 46 (2010), pp. 1252–1265.
- [7] ———, "Delay robustness in non-identical multi-agent systems", *IEEE Trans. Aut. Contr.*, 57 (2012), pp. 1597–1603.
- [8] R. OLFATI-SABER, J.A. FAX, AND R.M. MURRAY, "Consensus and cooperation in networked multi-agent systems", *Proc. of the IEEE*, 95 (2007), pp. 215–233.
- [9] R. OLFATI-SABER AND R.M. MURRAY, "Consensus problems in networks of agents with switching topology and time-delays", *IEEE Trans. Aut. Contr.*, 49 (2004), pp. 1520–1533.
- [10] W. REN AND Y.C. CAO, *Distributed Coordination of Multi-agent Networks*, Communications and Control Engineering, Springer-Verlag, London, 2011.
- [11] Y.-P. TIAN AND C.-L. LIU, "Consensus of multi-agent systems with diverse input and communication delays", *IEEE Trans. Aut. Contr.*, 53 (2008), pp. 2122–2128.
- [12] ———, "Robust consensus of multi-agent systems with diverse input delays and asymmetric interconnection perturbations", *Automatica*, 45 (2009), pp. 1347–1353.
- [13] X. WANG, A. SABERI, A.A. STORVOGEL, H.F. GRIP, AND T. YANG, "Consensus in the network with uniform constant communication delay", *Automatica*, 49 (2013), pp. 2461–2467.
- [14] C.W. WU, *Synchronization in complex networks of nonlinear dynamical systems*, World Scientific Publishing Company, Singapore, 2007.
- [15] T. YANG, H.F. GRIP, A. SABERI, M. ZHANG, AND A.A. STORVOGEL, "Synchronization in time-varying networks of non-introspective agents without exchange of controller states", Submitted for publication, 2014.
- [16] J.R. ZHANG, C.R. KNOSPE, AND P. TSIOTRAS, "Stability of linear time-delay systems: a delay-dependent criterion with a tight conservatism bound", in *American Control Conference*, Chicago, IL, 2000, pp. 1458–1462.