Analysis of the Three Dimensional Heat Conduction in Nano- or Microscale

Hanif Heidari[†,*] Hans Zwart[‡] Alaeddin Malek

***Abstract*— The Dual-Phase-Lagging (DPL) equation is formulated as an abstract differential equation. In the absence of a heat source term the DPL equation with homogeneous boundary conditions generates a contraction semigroup. The exact expression of the semigroup is achieved. It is proved that the associated eigenfunctions form a Riesz basis. The stability of semigroup is proved. Moreover, it is also shown that the spectrum of DPL equation contains an interval. This implies that the infinitesimal generator associated to the DPL equation is not a Riesz spectral operator. Therefore, the known test for approximate controllability cannot be used. Several controllability properties are investigated.**

Keywords: Thin film, DPL equation, Abstract formulation, Stability, Exact controllability

I. Introduction

The demand on high switching speed of electronic devices has pushed the reduction of the device size to micro-scale. The side effect of this device size reduction is the increase of heat generation, leading to a higher thermal load on the micro device. Hence studying the thermal behavior of thin films is essential for predicting the performance of a microelectronic device and for designing a desired micro structure. The properties of heat conduction at the micro-scale level are different from classical heat conduction [4]. Qui and Tien derived a partial differential equation model for the heat transfer at the micro-scale level [7]. This Dual-Phase-Lagging (DPL) equation is based on the hypothesis that the input energy is absorbed by the electrons and the lattice in the substance. In this presentation we derive a closed analytical solution of the dual-phase-lagging differential equation (DPL) by using semigroup theory. Furthermore, we investigate system theoretic properties of this equation.

This presentation is organized as follows. In section II the heat conduction equation in microgeometries is described by the DPL equation. The semigroup formulation and closed analytical solution of this equation are achieved in section III. In section IV, the stability of the heat conduction in microscopic regions is investigated and it is also shown that the spectrum of the DPL equation has a continuous part. So, the infinitesimal generator associated to the DPL is not a Riesz spectral operator in the sense of Curtain and Zwart [1]. Since the infinitesimal generator is not a Riesz spectral operator, the test given in [1] is not applicable for proving approximate controllability. In section V, it is proved that DPL equation is not exact controllable.

II. DPL Equation Description

We consider the physical domain to be a thin film, which its thickness at nano or micro scale, i.e.,

$$\Omega = \{(x, y, z) \mid 0 \le x \le l, 0 \le y \le h, 0 \le z \le \epsilon\}$$

and ϵ is of the order to 0.01nm or 0.01μm. If all the thermo-physical material properties are assumed to be constant, the dual-phase-lagging heat conduction equation given by, [2]:

$$\frac{1}{\alpha}(\frac{\partial u}{\partial t}+\tau_q\frac{\partial^2 u}{\partial t^2}) = \nabla^2 u + \tau_q(\frac{\partial^3 u}{\partial tx^2} + \frac{\partial^3 u}{\partial ty^2}) + \tau_u\frac{\partial^3 u}{\partial tz^2} + s, \qquad (1)$$

where α is thermal diffusivity of the material, $u(x, y, z, t)$ is temperature at position (x, y, z) and time t, τ_q and τ_u are the time lags of the heat flux and temperature gradient, respectively, and s represents the internal heat sources. The parameters α, τ_q and τ_u are positive constants, [5]. The initial conditions are given by

$$u(x, y, x, 0) = f_1(x, y, z) \qquad (2)$$
$$\frac{\partial u}{\partial t}(x, y, z, 0) = f_2(x, y, z) \qquad (3)$$

with f_1 and f_2 real-valued functions. The boundary conditions are given by

$$\frac{\partial u}{\partial t}(0, y, z, t) = 0 \qquad \frac{\partial u}{\partial t}(l, y, z, t) = 0 \qquad (4)$$
$$\frac{\partial u}{\partial t}(x, 0, z, t) = 0 \qquad \frac{\partial u}{\partial t}(x, h, z, t) = 0 \qquad (5)$$
$$\frac{\partial u}{\partial t}(x, y, 0, t) = 0 \qquad \frac{\partial u}{\partial t}(x, y, \epsilon, t) = 0 \qquad (6)$$

for $t > 0$.

III. Analytical Solution

The system of equations (1)–(6) can be transformed to an abstract differential equation. As state space we choose the energy space $\mathcal{H}$, which is the Hilbert space $H_0^1(\Omega) \times L^2(\Omega)$ with the inner product

$$\left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle_e = \frac{1}{2}\int_\Omega \frac{\alpha}{\tau_q}\nabla u_1 \cdot \nabla w_1 + u_2 w_2 \; dX, \quad (7)$$

where $dX = dxdydz$.

[†]Supported by Department of Scholarship and Student Affairs in Abroad, Ministry of Science, Research and Technology, Islamic Republic of Iran

[*]Department of Applied Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, Tehran, Iran, h.heidari@modares.ac.ir

[‡]University of Twente, P.O. Box 217, 7500 AE, Enschede, The Netherlands, h.j.zwart@math.utwente.nl

Department of Applied Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, Tehran, Iran, mala@modares.ac.ir

On this state space we write (1)–(6) as

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} u \\ u_t \end{pmatrix} = A\begin{pmatrix} u \\ u_t \end{pmatrix} + Bs \\ \begin{pmatrix} u \\ u_t \end{pmatrix}|_{t=0} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \end{cases} \tag{8}$$

with $u_t = \frac{\partial u}{\partial t}$, $B = \begin{pmatrix} 0 \\ \frac{\alpha}{\tau_q} I \end{pmatrix}$ and A is given by

$$A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ \alpha \mathrm{div}(u_3) - \frac{1}{\tau_q} u_2 \end{pmatrix}, \tag{9}$$

where $u_3 = \frac{1}{\tau_q}\nabla u_1 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\tau_u}{\tau_q} \end{pmatrix} \nabla u_2$, and

$$D(A) = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H_0^1(\Omega) \oplus H_0^1(\Omega) \mid u_3 \in D(\mathrm{div}) \right\} \tag{10}$$

Lemma 3.1: Let A and it's domain given by (9) and (10), respectively. The adjoint of A is given by

$$A^* \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_2 \\ \alpha \mathrm{div}(v_3) - \frac{1}{\tau_q} v_2 \end{pmatrix} \tag{11}$$

with the following domain

$$D(A^*) = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in H_0^1(\Omega) \oplus H_0^1(\Omega) \mid v_3 \in D(\mathrm{div}) \right\} \tag{12}$$

where $v_3 = -\frac{1}{\tau_q}\nabla v_1 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\tau_u}{\tau_q} \end{pmatrix} \nabla v_2$.

Proof: For $\left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right) \in D(A)$, we have that

$$\left\langle A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle_e = \left\langle \begin{pmatrix} u_2 \\ \alpha \mathrm{div}(u_3) - \frac{1}{\tau_q} u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle_e$$
$$= \frac{1}{2}\int_\Omega \frac{\alpha}{\tau_q}\nabla u_2 \cdot \nabla v_1 + (\alpha \mathrm{div}(u_3) - \frac{1}{\tau_q} u_2) v_2 dX. \tag{13}$$

We know that $\left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right) \in D(A^*)$ if and only if for all $\left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right) \in D(A)$ we can write (13) as

$$\left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle_e = \frac{1}{2}\int_\Omega \frac{\alpha}{\tau_q}\nabla u_1 \cdot \nabla w_1 + u_2 w_2 dX \tag{14}$$

for some $(w_1, w_2) \in H_0^1(\Omega) \times L^2(\Omega)$.

It is easy to see that $\left(\begin{smallmatrix} u_1 \\ 0 \end{smallmatrix}\right) \in D(A)$ if and only if $u_1 \in H^2(\Omega) \cap H_0^1(\Omega)$. For this element in $D(A)$, equation (13) becomes $\int_\Omega \frac{\alpha}{\tau_q}\mathrm{div}(\nabla u_1) v_2 dX$. This can be written as (14) if and only if $v_2 \in H_0^1(\Omega)$. Hence, if $\left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right) \in D(A^*)$, then $v_2 \in H_0^1(\Omega)$. Using this we can write (13) for a general $\left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right) \in D(A)$ as

$$\left\langle A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle_e =$$
$$\frac{1}{2}\int_\Omega \frac{\alpha}{\tau_q}\nabla u_2 \cdot \nabla v_1 - \alpha u_3 \cdot \nabla v_2 - \frac{1}{\tau_q} u_2 v_2 dX =$$
$$\frac{1}{2}\int_\Omega \frac{\alpha}{\tau_q}\nabla u_2 \cdot \nabla v_1 - \left[\frac{\alpha}{\tau_q}\nabla u_1 \cdot \nabla v_2 + \right.$$
$$\left. \alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\tau_u}{\tau_q} \end{pmatrix} \nabla u_2 \cdot \nabla v_2 \right] - \frac{1}{\tau_q} u_2 v_2 dX. \tag{15}$$

We define $v_3 \in L^2(\Omega)$ as $v_3 = -\frac{1}{\tau_q}\nabla v_1 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\tau_u}{\tau_q} \end{pmatrix} \nabla v_2$ and write (15) as

$$\frac{1}{2}\int_\Omega -\frac{\alpha}{\tau_q}\nabla u_1 \cdot \nabla v_2 - \nabla u_2 \cdot (\alpha v_3) - \frac{1}{\tau_q} u_2 v_2 dX. \tag{16}$$

Equation (16) can be written in the form (14) if and only if $v_3 \in D(\mathrm{div})$. Hence, the domain of A^* is given by (12), and if $\left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right) \in D(A^*)$, then

$$\left\langle A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle_e = \left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} -v_2 \\ \alpha \mathrm{div}(v_3) - \frac{1}{\tau_q} v_2 \end{pmatrix} \right\rangle_e \tag{17}$$

Thus we have proved the assertion. ■

Using this lemma, it is not hard to show that A generates a contraction semigroup on $\mathcal{H}$.

Theorem 3.2: The operator A as defined in (9) and (10) is the infinitesimal generator of a strongly continuous contraction semigroup on $\mathcal{H}$.

Proof: We check that both A and A^* are dissipative on $\mathcal{H}$. Then the result follows from Lumer-Phillips Theorem [6].

$$\left\langle A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\rangle_e$$
$$= \left\langle \begin{pmatrix} u_2 \\ \alpha \mathrm{div}(u_3) - \frac{1}{\tau_q} u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\rangle_e$$
$$= \frac{1}{2}\int_\Omega \frac{\alpha}{\tau_q}\nabla u_2 \cdot \nabla u_1 + (\alpha \mathrm{div}x(u_3) - \frac{1}{\tau_q} u_2) u_2 dX$$
$$= \frac{1}{2}\int_\Omega \frac{\alpha}{\tau_q}\nabla u_1 \cdot \nabla u_2 - \alpha u_3 \cdot \nabla u_2 - \frac{1}{\tau_q} u_2^2 dX$$
$$= \frac{1}{2}\int_\Omega -\alpha \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\tau_u}{\tau_q} \end{pmatrix} \nabla u_2 \right) \cdot \nabla u_2 - \frac{1}{\tau_q} u_2^2 dX, \tag{18}$$

where we used integration by parts and the fact that u_1 and u_2 are zero at the boundary. Since the right hand side of (18) is less than or equal to zero, we see that A is dissipative on $\mathcal{H}$.

The proof that A^* is dissipative on $\mathcal{H}$ is done in a similar way. ■

Now, we find the solution of the abstract differential equation (8). We start by obtaining the solution for the homogeneous case, i.e., for $s = 0$. We obtain the solution by showing that the normalized eigenfunctions of A form a Riesz basis in $\mathcal{H}$, and thus the solution can be written with respect to this basis.

We begin by calculating the eigenvalues and eigenfunctions of A. From (9) we have that

$$A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Longleftrightarrow \begin{cases} u_2 = \lambda u_1 \\ \alpha \mathrm{div}(u_3) - \frac{1}{\tau_q} u_2 = \lambda u_2. \end{cases} \tag{19}$$

Therefore, $u_2 = \lambda u_1$ and

$$\alpha \mathrm{div}\left(\frac{1}{\tau_q}\nabla u_1 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\tau_u}{\tau_q} \end{pmatrix} \nabla u_2 \right) = (\lambda + \frac{1}{\tau_q}) u_2 \Longleftrightarrow$$
$$\alpha \mathrm{div}\left(\frac{1}{\tau_q}\nabla u_1 + \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\tau_u}{\tau_q} \end{pmatrix} \nabla u_1 \right) = \lambda(\lambda + \frac{1}{\tau_q}) u_1 \tag{20}$$

which is equivalent to

$$\begin{cases} u_1 \in H_0^1(\Omega)\cap H^2(\Omega), \\ (\frac{\alpha}{\tau_q}+\lambda\alpha)(\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2})+(\frac{\alpha}{\tau_q}+\lambda\frac{\alpha\tau_u}{\tau_q})\frac{\partial^2 u_1}{\partial z^2} = \\ \quad (\lambda\frac{1}{\tau_q}+\lambda^2)u_1. \end{cases} \tag{21}$$

We want to find all solutions of (21). Therefore, we first obtain a set of solutions. It is easily seen that $\varphi_{nmk}(x,y,z) = \sin(\frac{n\pi x}{l})\sin(\frac{m\pi y}{h})\sin(\frac{k\pi z}{\epsilon})$ lies in $H_0^1(\Omega)$. Furthermore, it satisfies (21) if and only if λ_{nmk} satisfies

$$\lambda_{nmk}^2+(\alpha[(\frac{n\pi}{l})^2+(\frac{m\pi}{h})^2+\frac{\tau_u}{\tau_q}(\frac{k\pi}{\epsilon})^2]+\frac{1}{\tau_q})\lambda_{nmk} \tag{22}$$
$$+\frac{\alpha}{\tau_q}\left[(\frac{n\pi}{l})^2+(\frac{m\pi}{h})^2+(\frac{k\pi}{\epsilon})^2\right]=0.$$

The solution of above equation is denoted as follows:

$$\lambda_{+nmk} = \frac{1}{2}(-b+\sqrt{\Delta}) \quad n\in\mathbb{N}, m\in\mathbb{N}, k\in\mathbb{N} \tag{23}$$
$$\lambda_{-nmk} = \frac{1}{2}(-b-\sqrt{\Delta}) \quad n\in\mathbb{N}, m\in\mathbb{N}, k\in\mathbb{N}, \tag{24}$$

where $b = \alpha[(\frac{n\pi}{l})^2+(\frac{m\pi}{h})^2+\frac{\tau_u}{\tau_q}(\frac{k\pi}{\epsilon})^2]+\frac{1}{\tau_q}$ and $\Delta = b^2-\frac{4\alpha}{\tau_q}[(\frac{n\pi}{l})^2+(\frac{m\pi}{h})^2+(\frac{k\pi}{\epsilon})^2]$.

For $\lambda_{\pm nmk}$ defined by (23) and (24), it is easy to see that

$$\varphi_{\pm nmk}(x,y,z) = \begin{pmatrix} \sin(\frac{n\pi x}{l})\sin(\frac{m\pi y}{h})\sin(\frac{k\pi z}{\epsilon}) \\ \lambda_{\pm nmk}\sin(\frac{n\pi x}{l})\sin(\frac{m\pi y}{h})\sin(\frac{k\pi z}{\epsilon}) \end{pmatrix} \tag{25}$$

lies in the domain of A, and satisfies $A\varphi_{\pm nmk} = \lambda_{\pm nmk}\varphi_{\pm nmk}$. Hence, $\varphi_{\pm nmk}$ is an eigenfunction of A. If $n\neq\tilde{n}$, or $m\neq\tilde{m}$, or $k\neq\tilde{k}$, then

$$\langle\varphi_{\pm nmk},\varphi_{\pm\tilde{n}\tilde{m}\tilde{k}}\rangle_e = 0. \tag{26}$$

Furthermore,

$$\langle\varphi_{+nmk},\varphi_{+nmk}\rangle_e = \frac{lh\epsilon}{16}\left(\frac{\alpha}{\tau_q}\left((\frac{n\pi}{l})^2+(\frac{m\pi}{h})^2+(\frac{k\pi}{\epsilon})^2\right)+\lambda_{+nmk}^2\right) \tag{27}$$
$$\langle\varphi_{-nmk},\varphi_{-nmk}\rangle_e = \frac{lh\epsilon}{16}\left(\frac{\alpha}{\tau_q}\left((\frac{n\pi}{l})^2+(\frac{m\pi}{h})^2+(\frac{k\pi}{\epsilon})^2\right)+\lambda_{-nmk}^2\right) \tag{28}$$

If the Δ is unequal to zero, then we have a Riesz basis of eigenfunctions. This condition is very weak and will hold generally, see Section IV,

Lemma 3.3: If for all $n,m,k \in \mathbb{N}$ we have that $\lambda_{+nmk}\neq\lambda_{-nmk}$, then the normalized set of of eigenvectors $\{\frac{\varphi_{+nmk}}{\|\varphi_{+nmk}\|}, \frac{\varphi_{-nmk}}{\|\varphi_{-nmk}\|}, n,m,k\in\mathbb{N}\}$ forms a Riesz basis of $\mathcal{H} = H_0^1(\Omega)\times L^2(\Omega)$.

Proof: It is well-known that $\{\sqrt{\frac{8}{lh\epsilon}}\sin(\frac{n\pi x}{l})\sin(\frac{m\pi y}{h})\sin(\frac{k\pi z}{\epsilon}), n,m,k\in\mathbb{N}\}$ forms an orthonormal basis of $L^2(\Omega)$. Similarly, we have that the vectors $\{\frac{1}{\sqrt{\mu_{nmk}}}\sin(\frac{n\pi x}{l})\sin(\frac{m\pi y}{h})\sin(\frac{k\pi z}{\epsilon}), n,m,k\in\mathbb{N}\}$, with

$$\mu_{nmk} = \frac{lh\epsilon}{8}\left((\frac{n\pi}{l})^2+(\frac{m\pi}{h})^2+(\frac{k\pi}{\epsilon})^2\right) \tag{29}$$

forms an orthonormal basis of $H_0^1(\Omega)$.

Let $w = \binom{w_1}{w_2} \in \mathcal{H}$. By the above, there exist $\{c_{1,nmk}\}_{n,m,k\in\mathbb{N}}$ and $\{c_{2,nmk}\}_{n,m,k\in\mathbb{N}}$ in ℓ^2 such that

$$w_1(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} c_{1,nmk}\frac{1}{\sqrt{\mu_{nmk}}}\sin(\frac{n\pi x}{l})\sin(\frac{m\pi y}{h})\sin(\frac{k\pi z}{\epsilon}) \tag{30}$$
$$w_2(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} c_{2,nmk}\sqrt{\frac{8}{lh\epsilon}}\sin(\frac{n\pi x}{l})\sin(\frac{m\pi y}{h})\sin(\frac{k\pi z}{\epsilon}). \tag{31}$$

Using the normalized eigenfunctions, we see that we can write (30), (31) as

$$w = \sum_{n,m,k=1}^{\infty} d_{+nmk}\frac{\varphi_{+nmk}}{\|\varphi_{+nmk}\|}+d_{-nmk}\frac{\varphi_{-nmk}}{\|\varphi_{-nmk}\|}. \tag{32}$$

with

$$\begin{cases} \frac{d_{+nmk}}{\|\varphi_{+nmk}\|}+\frac{d_{-nmk}}{\|\varphi_{-nmk}\|} = \frac{c_{1,nmk}}{\sqrt{\mu_{nmk}}} \\ \lambda_{+nmk}\frac{d_{+nmk}}{\|\varphi_{+nmk}\|}+\lambda_{-nmk}\frac{d_{-nmk}}{\|\varphi_{-nmk}\|} = c_{2,nmk}\sqrt{\frac{8}{lh\epsilon}} \end{cases} \tag{33}$$

This we write in a matrix notation

$$\begin{pmatrix} c_{1,nmk} \\ \sqrt{\frac{8}{lh\epsilon}}c_{2,nmk} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{\mu_{nmk}}}{\|\varphi_{+nmk}\|} & \frac{\sqrt{\mu_{nmk}}}{\|\varphi_{-nmk}\|} \\ \frac{\lambda_{+nmk}}{\|\varphi_{+nmk}\|} & \frac{\lambda_{-nmk}}{\|\varphi_{-nmk}\|} \end{pmatrix}\begin{pmatrix} d_{+nmk} \\ d_{-nmk} \end{pmatrix} \tag{34}$$

The set $\{\frac{\varphi_{+nmk}}{\|\varphi_{+nmk}\|}, \frac{\varphi_{-nmk}}{\|\varphi_{-nmk}\|}, n,m,k\in\mathbb{N}\}$ forms a Riesz basis of $\mathcal{H} = H_0^1(\Omega)\times L^2(\Omega)$ if and only if $\{d_{\pm nmk}\}_{nmk}\in\ell^2$ whenever $\{c_{\pm nmk}\}_{nmk}\in\ell^2$. This holds if and only if the matrix in (34) is (uniformly) bounded and (uniformly) boundedly invertible. Using (29), (27), and (28), we see that

$$\mu_{nmk}\leq\frac{\alpha}{2\tau_q}\|\varphi_{+nmk}\|^2, \quad \mu_{nmk}\leq\frac{\alpha}{2\tau_q}\|\varphi_{-nmk}\|^2 \quad \text{and}$$
$$\lambda_{+nmk}^2\leq\frac{16}{lh\epsilon}\|\varphi_{+nmk}\|^2, \quad \lambda_{-nmk}^2\leq\frac{16}{lh\epsilon}\|\varphi_{-nmk}\|^2.$$

So the coefficients of the matrix in (34) are (uniformly) bounded, which implies that the same holds for the matrix.

Since $\lambda_{+nmk}\neq\lambda_{-nmk}$, we have that for all $n,m,$ and k the matrix is invertible. Now we investigate its limit behaviour. We have that, see (23),

$$-2\lambda_{+nmk} = b-\sqrt{\Delta} = \frac{b^2-\Delta}{b+\sqrt{\Delta}} = \frac{\frac{32\alpha}{\tau_q lh\epsilon}\mu_{nmk}}{b+\sqrt{\Delta}} \leq \frac{32\alpha}{\tau_q lh\epsilon}\frac{\mu_{nmk}}{b}.$$

From this it is easily seen that λ_{+nmk} is bounded. Since this is bounded, $\frac{\lambda_{+nmk}}{\|\varphi_{+nmk}\|}$ converges to zero for $n,m,k\to\infty$. Furthermore, we obtain that, see (27),

$$\inf_{n,m,k}\frac{\mu_{nmk}}{\|\varphi_{+nmk}\|^2} > 0.$$

From (28) we see that

$$\frac{\lambda_{-nmk}}{\|\varphi_{-nmk}\|} = \frac{\lambda_{-nmk}}{\sqrt{\frac{\alpha}{2\tau_q}\mu_{nmk}+\frac{lh\epsilon}{16}\lambda_{-nmk}^2}} = \frac{-1}{\sqrt{\frac{\alpha}{2\tau_q}\frac{\mu_{nmk}}{\lambda_{-nmk}^2}+\frac{lh\epsilon}{16}}}.$$

Since $\lambda_{-nmk}^2 \geq \frac{b^2}{4}$, we find that

$$\sup_{n,m,k} \frac{\lambda_{-nmk}}{\|\varphi_{-nmk}\|} \leq \sup_{n,m,k} \frac{-1}{\sqrt{\frac{2\alpha}{\tau_q}\frac{\mu_{nmk}}{b^2} + \frac{lh\epsilon}{16}}} < 0.$$

So we see that the diagonal of the matrix in (34) is bounded away from zero, whereas the lower triangular element converges to zero. Together with the boundedness of all the elements, we conclude that this matrix is (uniformly) boundedly invertible.

Hence we conclude that $\{\frac{\varphi_{+nmk}}{\|\varphi_{+nmk}\|}, \ \frac{\varphi_{-nmk}}{\|\varphi_{-nmk}\|}, n, m, k \in \mathbb{N}\}$ forms a Riesz basis of $\mathcal{H}$. ■

Since the normalized eigenfunctions $\{\frac{\varphi_{+nmk}}{\|\varphi_{+nmk}\|}, \ \frac{\varphi_{-nmk}}{\|\varphi_{-nmk}\|}, n, m, k \in \mathbb{N}\}$ form a Riesz basis of $\mathcal{H}$, we have that they are all the eigenfunctions. We summarize this in a corollary.

Corollary 3.4: The eigenvalues and eigenfunctions of the operator A as defined in (9) and (10) are given by (23),(24), and (25), respectively.

Knowing that the eigenfunctions form a Riesz basis, it is easy to derive the formula for the C_0-semigroup, and thus for the solution of (8).

IV. Stability and the spectrum of A

Corollary 3.4 gives the eigenvalues of A. Since the eigenvectors form a Riesz basis, we have that the semigroup will be exponentially stable if and only if the eigenvalues are in the left-half plane, and if they are bounded away from the imaginary axis, see [1, Theorem 2.3.5].

Lemma 4.1: The semigroup generated by A as defined in (9) and (10) is exponentially stable.

Proof: Since the coefficients of the quadratic polynomial (22) are positive, all solutions of (22) have negative real part. So to prove the exponential stability, we have to exclude the possibility that the (real part of the) eigenvalues converges to zero. If these eigenvalues would be non-real, then this is not possible, since for a non-real zero of (22) the real part equals b. This is not converging to zero, and so we can only approach zero over the real axis.

Let λ be a real, (very) small solution of (22), then

$$\lambda = \frac{\lambda^2 - \frac{\alpha}{\tau_q}\left[(\frac{n\pi}{l})^2 + (\frac{m\pi}{h})^2 + (\frac{k\pi}{\epsilon})^2\right]}{\alpha[(\frac{n\pi}{l})^2 + (\frac{m\pi}{h})^2 + \frac{\tau_u}{\tau_q}(\frac{k\pi}{\epsilon})^2] + \frac{1}{\tau_q}}.$$

Since λ is small, the right-hand side is approximately equal to

$$\frac{-\frac{1}{\tau_q}\left[(\frac{n\pi}{l})^2 + (\frac{m\pi}{h})^2 + (\frac{k\pi}{\epsilon})^2\right]}{[(\frac{n\pi}{l})^2 + (\frac{m\pi}{h})^2 + \frac{\tau_u}{\tau_q}(\frac{k\pi}{\epsilon})^2] + \frac{1}{\tau_q}}.$$

This is bounded away from zero for all $n, m, k \in \mathbb{N}$, and so λ cannot be very small. Concluding, we have that all the zeros of (22) are negative and are bounded away from zero. Thus the semigroup generated by A is exponentially stable. ■

In Lemma 3.3 we have the assumption that $\lambda_{+nmk} \neq \lambda_{-nmk}$. From (23) and (24), we see that this holds if and only if $\Delta \neq 0$. We can write Δ as

$$\begin{aligned}\Delta =& b^2 - \frac{4\alpha}{\tau_q}\left[(\frac{n\pi}{l})^2 + (\frac{m\pi}{h})^2 + (\frac{k\pi}{\epsilon})^2\right] \\ =& \left(\alpha\left[(\frac{n\pi}{l})^2 + (\frac{m\pi}{h})^2 + \frac{\tau_u}{\tau_q}(\frac{k\pi}{\epsilon})^2\right] - \frac{1}{\tau_q}\right)^2 + \\ & \frac{4\alpha}{\tau_q}(\frac{k\pi}{\epsilon})^2(\frac{\tau_u}{\tau_q} - 1). \qquad (35)\end{aligned}$$

If $\tau_u \geq \tau_q$, then (35) implies that $\Delta > 0$ for all $n, m, k \in \mathbb{N}$. If $\tau_u \ll \tau_q$, then (35) may be zero or negative for $n = m = k = 1$. However, since the first term grows like k^4, whereas the last grows like as k^2, there can only the finitely many triple (n, m, k) for which (35) is zero or negative. Concluding, we see that in general Δ will be positive.

We assume that $\Delta > 0$ for all $n, m, k \in \mathbb{N}$, and so we assume that all eigenvalues are real and simple. Even without this assumption, the following result follows immediately.

Lemma 4.2: The operator A as defined in (9) and (10) is the infinitesimal generator of an analytic semigroup on $\mathcal{H}$.

Proof: This follows directly from the fact that A has a Riesz basis of eigenfunctions, and that all, but finitely many eigenvalues lie on the negative real axis. ■

Next we concentrate some more on the spectrum of A. For this we need the following lemma.

Lemma 4.3: Let β, δ and γ be positive constants, and let $\mathbb{P}$ be defined as

$$\begin{aligned}\mathbb{P} := \{p \in [0, \infty) \mid & \text{ there exists a sequence } (n, m, k) \in \mathbb{N}^3 \\ & \text{converging to infinity such that} \\ & \frac{\gamma k^2}{\beta n^2 + \delta m^2} \to p\}. \qquad (36)\end{aligned}$$

Then $\mathbb{P}$ equals $[0, \infty)$.

Proof: Consider the function $f(x) = \frac{\gamma x^2}{\beta + \delta}$. Then f is continuous and maps $[0, \infty)$ onto $[0, \infty)$. Thus for $p \in [0, \infty)$ there exists a x_p such that $f(x_p) = p$. By the continuity, we have that for any $\varepsilon > 0$, there exists a rational x_r such that $|f(x_r) - p| < \varepsilon$. We write x_r as $\frac{k}{m}$, and we choose $n = m$, then

$$\frac{\gamma k^2}{\beta n^2 + \delta m^2} = \frac{\gamma k^2}{\beta m^2 + \delta m^2} = \frac{\gamma(\frac{k}{m})^2}{\beta + \delta} = f(x_r).$$

This lies within a distance ε from p, and since p is arbitrary, we see that every $p \in [0, \infty)$ can be approximated by $f(x_r)$ with x_r rational. Since $x_r = \frac{k}{m} = \frac{kN}{mN}$, we have that without loss of generality, we may assume that there is a sequence (n, m, k) converging to infinity such that $\frac{\gamma k^2}{\beta n^2 + \delta m^2} \to p$. ■

Lemma 4.4: If $\tau_q \neq \tau_u$, then the interval between $-2\tau_u^{-1}$ and $-2\tau_q^{-1}$ lies in the spectrum of A. This implies that A is not a Riesz spectral operator in the sense of Curtain and Zwart, [1, Section 2.3].

Proof: To prove this result, we consider the limit behavior of λ_{+nmk}. For this we introduce $(\frac{n\pi}{l})^2 + (\frac{m\pi}{h})^2 =$

F and $(\frac{k\pi}{\epsilon})^2 = G$. From (23), we see that

$$\begin{aligned}\lambda_{+nmk} =& -\alpha(F + \frac{\tau_u}{\tau_q}G) - \frac{1}{\tau_q} + \\ & \sqrt{(\alpha(F + \frac{\tau_u}{\tau_q}G) + \frac{1}{\tau_q})^2 - \frac{4\alpha}{\tau_q}(F+G)} \\ =& \left(-\frac{4\alpha}{\tau_q}(F+G)\right) \cdot \\ & \left(\sqrt{([\alpha(F + \frac{\tau_u}{\tau_q}G)] + \frac{1}{\tau_q})^2 - \frac{4\alpha}{\tau_q}(F+G)}\right. \\ & \left. +(\frac{1}{\tau_q} + \alpha(F + \frac{\tau_u}{\tau_q}G))\right)^{-1}. \end{aligned} \tag{37}$$

Next we introduce, the set $\mathbb{P} := \{p \in [0, \infty) \mid$ there exists a sequence $(n, m, k) \in \mathbb{N}^3$ converging to infinity such that $\frac{G}{F} \to p\}$. By Lemma 4.3 we know that $\mathbb{P} = [0, \infty)$. Furthermore, we see that

$$\lim_{F,G\to\infty, \frac{G}{F}\to p} \lambda_{+nmk} = \frac{-\frac{4\alpha}{\tau_q}(1+p)}{2\alpha(1 + \frac{\tau_u}{\tau_q}p)} = \frac{-2(1+p)}{\tau_q + p\tau_u}. \tag{38}$$

The spectrum A is a closed set in $\mathbb{C}$. Thus for all $p \in \mathbb{P}$ we have that

$$\frac{-2(1+p)}{\tau_q + p\tau_u} \in \sigma(A), \tag{39}$$

where $\sigma(A)$ denotes the spectrum of A. The set $\mathbb{P}$ equals $[0, \infty)$, thus the interval between $\frac{-2}{\tau_q}$ and $\frac{-2}{\tau_u}$ lies in spectrum of A. ∎

As stated in the above lemma, the operator A is not a Riesz spectral operator although it has a Riesz basis of eigenvectors. This implies that extra care should be taken when applying results on Riesz spectral operators from [1]. We remark that the results on semigroups, growth bound, etc, still hold. However, the results on controllability as given in Chapter 4 of [1] do no longer hold, see [3]. This implies that we have to treat controllability separately. This is done in the next section.

V. Exact Controllability

Consider the PDE (1) with initial and homogeneous boundary conditions as defined in equations (2) and (4), respectively. We assume that the control function is zero outside the set Ω_1, $\Omega_1 \subset \Omega$. Thus $s : \Omega_1 \times (0, \infty) \longrightarrow \mathbb{R}$. This implies that we replace $B = \begin{pmatrix} 0 \\ \frac{\alpha}{\tau_q} I \end{pmatrix}$ in (8) by

$$B = \begin{pmatrix} 0 \\ \frac{\alpha}{\tau_q} I_{\Omega_1} \end{pmatrix}. \tag{40}$$

We want to study the controllability properties of this system.

For infinite dimensional systems there are different notions of controllability. We list two of them.

Definition 5.1: The abstract system (8) is

- *Exactly controllable* if for any two states h_1, h_2 there exists a $t_f > 0$ and a control function s such that that solution of (8) with initial condition h_1 and state at time t_f equals h_2;
- *Null controllable* if for any state h_1 there exists a $t_f > 0$ and a control function s such that the solution of (8) with initial condition h_1 equals zero at time t_f;

We have the following remarks:

Remark 5.2: 1) If the system is exponentially stable, it is exactly controllable on finite-time if and only if it is exactly controllable on infinite-time.
2) If the semigroup can be extended to a group, then exact controllability is equivalent to null controllability.

Theorem 5.3: If $\Omega_1 \subset \Omega$, then the system (8) with input operator as given in (40) is not null controllable. Thus it is not exactly controllable either.

Proof: The system is null controllable if and only if the dual system is final state observable, i.e., there exists an m_f such that for all $f \in \mathcal{H}$

$$\int_0^{t_f} \|B^*T(t)^*f\|^2 dt \geq m_f \|T(t_f)^*f\|^2 \tag{41}$$

It is easy to see that

$$B^* = \begin{pmatrix} 0 & \frac{\alpha}{\tau_q} I_{\Omega_1} \end{pmatrix}$$

Substituting the eigenfunction $\tilde{\psi}_{+nmk}$ of A^* associated to λ_{+nmk}, in (41), we find that

$$\begin{aligned}\int_0^{t_f} \int_{\Omega_1} |e^{\lambda_{+nmk}t} \frac{\alpha\lambda_{+nmk}}{\tau_q} \sin(\frac{n\pi\cdot}{l}) \sin(\frac{m\pi\cdot}{h}) \\ \sin(\frac{k\pi\cdot}{\epsilon})|^2 dxdydzdt \\ \geq m_f \|e^{\lambda_{+nmk}t_f} \frac{1}{\beta_{+nmk}} \tilde{\psi}_{+nmk}\|^2.\end{aligned}$$

Since λ_{+nmk} is bounded, the left hand-side is bounded in n, m, and k. However, since $\|\frac{1}{\beta_{+nmk}}\tilde{\psi}_{+nmk}\| = \|\varphi_{+nmk}\|$, we conclude from (27) that the right hand-side is unbounded. Thus (41) cannot hold, and so the system is not null controllable. ∎

VI. Conclusion

The DPL equation is formulated as a first order differential equation. The closed analytical form solution of this equation is obtained by using semigroup theory. It is proved that heat conduction at micro scale is stable. Furthermore, the spectrum of the DPL equation contains an interval. It is shown that DPL equation is not null controllable. Thus it is not exactly controllable.

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