SIGNAL CONSTELLATIONS FOR MULTILEVEL CODED MODULATION WITH SPARSE GRAPH CODES

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A method to combine error-correction coding and spectral efficient modulation for transmission over channels with Gaussian noise is presented. The method of modulation leads to a signal constellation in which the constellation symbols have a nonuniform distribution. This gives a so-called shape gain which can be as high as $\frac{\pi e}{6}$ (1.5 dB). A sparse graph code is constructed which is based on a LDPC code and includes the method of modulation. An efficient decoding algorithm can be derived for this sparse graph code. Simulation results show that the performance of the code is quite good compared to other coded modulation schemes proposed in literature.

INTRODUCTION

In this paper we consider the transmission of information over Gaussian noise channels. Conventional modulation methods use signal constellations in which each constellation point is selected with equal probability. This approach maximizes the supported bitrate of the constellation, but does not take the energy cost of constellation points into account. If constellation points with small energy are chosen more often than constellation points with large energy, the loss in supported bitrate can be compensated and even gains can be realized. This gain is called a *shape gain* which can be as high as 1.5 dB [1].

Modulation can be combined with forward error-control coding to achieve a near capacity performance. Several coded modulation schemes are proposed in literature, of which examples are trellis-coded modulation [2], bit-interleaved coded modulation (BICM) [3], BICM with iterative detection [4] and multi-level coding [5]. There are methods to obtain shaping gains with some of these methods, but the required computational complexity for decoding is often high.

We investigate the use of signal constellations in which the constellation points have a nonuniform distribution. These constellations are easily generated and can be incorporated into a powerfull sparse graph code for which efficient decoding algorithms can be derived.

The outline of the paper is as follows. First we define the constellation parameters which are required throughout the paper. Second we describe a method for nonuniform signaling and show how to combine it with an error-correcting code. Simulation results are presented and the paper ends with conclusions.

DEFINITIONS

In this section we describe several parameters related to general signal constellations. Most of these parameters are defined precisely in [1] and [6]. Let Ω be a signal constellation and let $|\Omega|$ denote its size. The probability that the transmitter selects a constellation point $\mathbf{r} \in \Omega$ is denoted by $f(\mathbf{r})$. We limit ourselves to one dimensional (1D) and two dimensional (2D) signal constellations. Furthermore, the 2D signal constellations we describe, can be generated by a Cartesian product of 1D signal constellations.

Bit Rate and Transmitter Power

The transmitter can be viewed as a discrete memoryless source whose output symbols are drawn from Ω according to $f(\mathbf{r})$. The bitrate supported by a constellation is given by the entropy of the transmitter output symbols:

$$\beta = -\sum_{\mathbf{r}\in\Omega} f(\mathbf{r}) \log_2 [f(\mathbf{r})] \quad \textit{bit/symbol} \tag{1}$$

The average amount of energy expended per channel use is defined as:

$$E = \sum_{\mathbf{r}\in\Omega} f(\mathbf{r}) ||\mathbf{r}||^2 \tag{2}$$

Reliability and Gain

To illustrate the potential advantage of nonuniform signal constellations, we investigate the asymptotic advantage in signal to noise ratio for low symbol error rates. In [6] it is shown that for a certain 2D signal constellation the asymptotic gain with respect to a 2D QAM constellation is given by:

$$G = \frac{(2^{2\beta} - 1)d_{min}^2}{12E}, \quad \beta \geq 1 \tag{3}$$

where d_{min}^2 is the minimum squared Euclidian distance between constellation points.

NONUNIFORM SIGNAL CONSTELLATIONS

A nonuniform signal constellation can be generated as follows. Let X_i denote a discrete uniform distributed random variable with values -1 and 1. A second random variable Z_d is defined as follows:

$$Z_d = \sum_{i=1}^{d} X_i. \tag{4}$$

The set of possible outcomes of Z_d is denoted by $\mathcal{X}_d$ and Z_d has a binomial distribution which is given by:

$$f_d(z) = P(Z_d = z) = \binom{d}{\frac{1}{2}(z+d)} 2^{-d}, \quad z \in \mathcal{X}_d. \tag{5}$$

To generate a 2D signal constellation for use on a 2D Gaussian channel, a symbol for each dimension can be generated independently. Figure 1 shows two 2D constellations for $d = 2$ and $d = 4$. Each constellation point is labeled with its probability of occurrence.

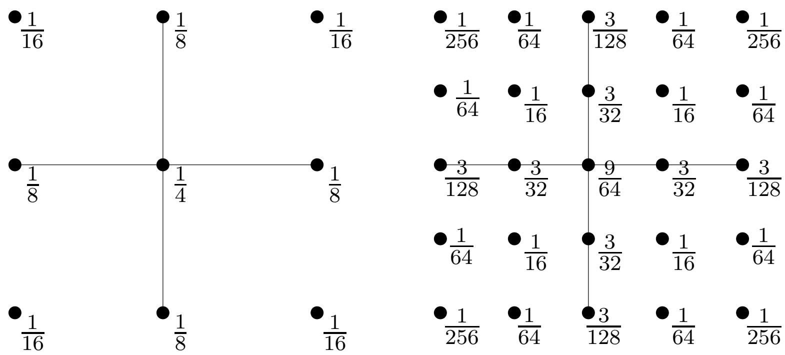

Figure 1: The signal constellations for $d = 2$ and $d = 4$.

With equation 5 the supported bitrate of the constellations can be expressed as:

$$\beta_d = -\sum_{i=0}^{d} 2^{-d}\binom{d}{i} \log_2\left[2^{-d}\binom{d}{i}\right] = d - 2^{-d}\sum_{i=0}^{d}\binom{d}{i}\log_2\binom{d}{i}, \tag{6}$$

where the right handside of this equation corresponds to the loss in supported bitrate compared to an uniform distribution on the constellation symbols.

The average amount of energy expended per channel can be computed as:

$$E_d = \sum_{i=0}^{d}\binom{d}{i}2^{-d}(2i-d)^2 = d \tag{7}$$

Figure 2 gives a table with numerical values for the parameters described in this section for several values of d. Note that β_d and E_d are given for 1D constellations and G_d for the corresponding 2D constellations.

Figure 2: Numerical values of the constellation parameters for several d.

d	β_d	E_d	G_d [dB]	d	β_d	E_d	G_d [dB]
1	1.00	1.00	0.00	5	2.20	5.00	1.26
2	1.50	2.00	0.67	6	2.33	6.00	1.32
3	1.81	3.00	0.99	7	2.45	7.00	1.36
4	2.03	4.00	1.17	8	2.54	8.00	1.39

Next we show that the proposed constellations can achieve the ultimate shape gain of $\frac{\pi e}{6}$. With an approximation of the entropy of the binomial distribution [7], equation 6 can be written as:

$$\beta_d \approx \frac{1}{2}\log_2\left[\frac{1}{2}\pi e d\right] + \frac{\ln 2}{12d^2} + O(\frac{1}{d^3}), \tag{8}$$

and the gain is given by:

$$G_d = \frac{(2^{2\beta_d}-1)d_{min}^2}{12E_d} = \frac{2^{\log_2\left[\frac{1}{2}\pi e d\right]+\frac{\ln 2}{12d^2}+O(\frac{1}{d^3})}-1}{3d} = \frac{\pi e}{6}\cdot 2^{\frac{ln2}{6d^2}+O(\frac{1}{d^3})} - \frac{1}{3d}, \tag{9}$$

from which follows that:

$$G_\infty = lim_{d\to\infty} G_d = \frac{\pi e}{6} \tag{10}$$

Finally, we illustrate the potential advantage of these constellations when used on the additive white gaussian noise (AWGN) channel. Let X and Y denote the input and output of the 1D-AWGN channel, respectively. We can compute

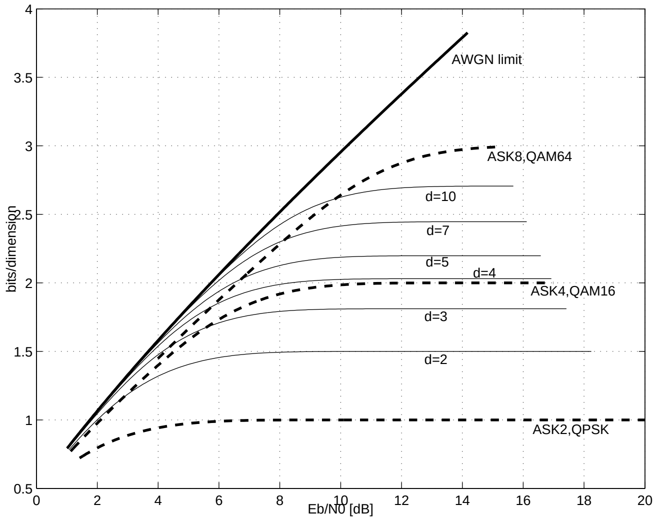

Figure 3: The capacity of several signal constellations.

the capacity of the channel with an input distribution that is given by equation 5. The mutual information between the channel input X and channel output Y is:

$$I(X;Y) = \sum_{z \in \mathcal{X}_d} \int_{-\infty}^{\infty} f(y|x=z) P(Z_d = z) \log \frac{f(y|x=z)}{f(y)} dy, \tag{11}$$

where

$$f(y|x=z) = \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(y-z)^2}{2\sigma_n^2}} \tag{12}$$

and

$$f(y) = \sum_{z \in \mathcal{X}_d} f(y|x=z) P(Z_d = z). \tag{13}$$

We have evaluated this integral numerically for several values of d. The results are shown in figure 3 together with capacity curves for conventional signal constellations. From the figure it is clear that the capacity curves of the proposed signal constellations follow the AWGN curve much longer.

SPARSE GRAPH CODES

The signal constellations described in the previous section are generated by a summation of d uniform i.i.d. distributed random variables. To construct an error-correcting code, we take a low-density parity-check (LDPC) code as a starting point. Let $\mathbf{H}$ denote the parity-check matrix of a LDPC code $\mathcal{C}$ and $\mathbf{x} \in \mathcal{C}$ one of its codewords. A symbol z_i from the proposed signal constellations is generated as follows. d bits of a codeword $\mathbf{x}$ are selected at random and mapped to the real numbers by a BPSK mapping $(0 \rightarrow 1, 1 \rightarrow -1)$. Next, the result is summed:

$$z_i = \sum_{i=1}^{d} \varphi(x_{j(i)}), \tag{14}$$

where $j(i)$ is the ith element from a set which contains the d randomly chosen bit node indices. The BPSK mapping is denoted by $\varphi(\ldots)$. The Tanner graph of the LDPC code is extended with additional variable nodes which represent the modulated bits. We refer to these nodes as *z nodes* and to the bit variable nodes as *bit nodes*. To enforce equation 14, the graph is extended with an additional constraint node for each z node. Figure 4 shows the graph representation of the resulting code. An iterative version of the sum-product algorithm is applied to perform decoding. More details are described in [8] and [9].

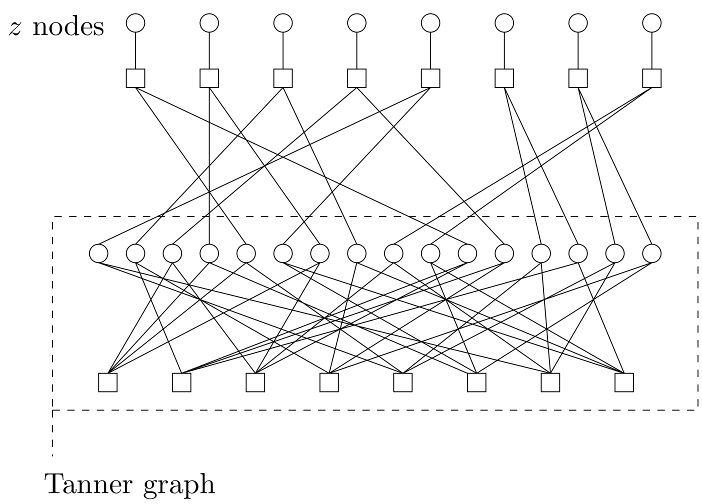

Figure 4: Graph representation of the code.

We conclude this section with a definition of some of the parameters of the code. Let r denote the design rate of the underlying LDPC code. The rate R of the code is defined as:

$$R = \frac{rN}{N_z}, \tag{15}$$

where N and N_z are the number of bit nodes and z nodes respectively.

The energy expended per channel use is defined as:

$$E_s = \mathcal{E}\{z_i^2\} = E_d, \tag{16}$$

where $\mathcal{E}\{\cdot\}$ denotes the expectation operator. The energy expended per information bit is given by:

$$E_b = \frac{N_z E_s}{rN}. \tag{17}$$

PRACTICAL RESULTS

In this section we present simulation results for two different code structures. The degree distributions [10] and other parameters of the code are summarized in the following table:

Code	d	r	R	$L(x)$	$R(x)$
I	2	0.5	1	$0.35x^2 + 0.65x^3$	$0.82x^4 + 0.18x^{11}$
II	4	0.5	1.5	$0.45x^2 + 0.55x^3$	$0.82x^3 + 0.18x^{13}$

Figure 5 shows the performance of the code for transmission over the AWGN channel. The number of bit nodes (N) is chosen as 32000, 16000 and 8000 for code I and as 32000 for code II. Code I requires an E_b/N_0 of 3.3 dB to achieve a BER of 10^{-5} for $N = 32000$. At this BER and E_b/N_0, the distance to the capacity of the AWGN channel constained to the proposed constellation is 1.3 dB. Code II transmit at a higher rate and requires a E_b/N_0 of 5.9 dB to achieve a BER of 10^{-5}.

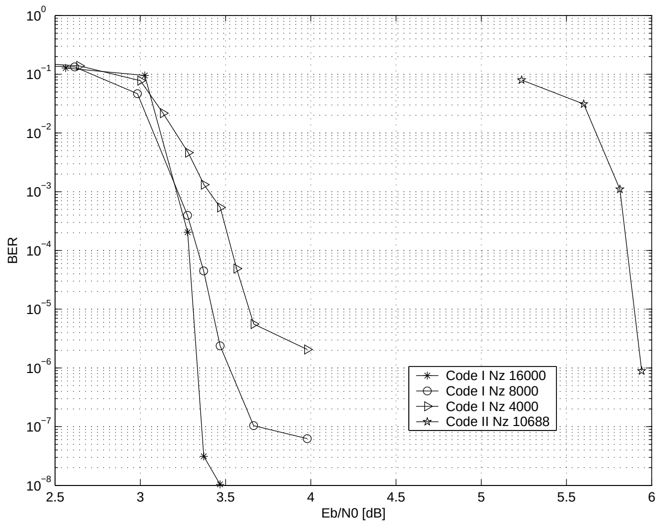

Figure 5: The BER for transmission over the AWGN channel.

CONCLUSIONS AND DISCUSSION

We have presented a method to combine modulation with error correction. The Tanner graph of a LDPC code is extended to include modulation. Simulation results show a performance which compares well against other coded modulation schemes. We expect that the performance of the code can be improved by an optimization of the structure of the graph.

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