

Stochastic almost regulated output synchronization for heterogeneous time-varying networks with non-introspective agents and without exchange of controller states

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Abstract—We consider almost regulated output synchronization for heterogeneous directed networks with external stochastic disturbances where agents are non-introspective (i.e. agents have no access to their own states or outputs). A purely decentralized time-invariant protocol based on a low-and-high gain method is designed for each agent to achieve almost regulated output synchronization while reducing the impact of disturbances on the synchronization error. It is also shown that this protocol can work in the case of time-varying graphs.

I. INTRODUCTION

In the last decade, the topic of synchronization in a multi-agent system has received considerable attention. Its potential applications can be seen in cooperative control on autonomous vehicles, distributed sensor network, swarming and flocking and others. The objective of synchronization is to secure an asymptotic agreement on a common state or output trajectory through decentralized control protocols (see [1], [12], [18], [28]). Research has mainly focused on the state synchronization based on full-state/partial-state coupling in a homogenous network (i.e. agents have identical dynamics), where the agent dynamics progress from single- and double-integrator dynamics to more general dynamics (e.g., [7], [14], [15], [21], [24], [25], [26], [29]). The counterpart of state synchronization is output synchronization, which is mostly done on heterogeneous networks (i.e., agents are non-identical). When the agents have access to part of their own states it is frequently referred to as introspective and, otherwise, non-introspective. Quite a few of the recent works on output synchronization have assumed agents are introspective (e.g., [3], [6], [27], [30]) while few have considered non-introspective agents. For non-introspective agents, the paper [5] addressed the output synchronization for heterogeneous networks.

In [7] for homogeneous networks a controller structure was introduced which included not only sharing the relative outputs over the network but also sharing the relative states of the protocol over the network. This was also used in our earlier work such as [5], [16], [17]. This type of additional communication is not always natural. Some papers such as [21] (homogeneous network) and [6] (heterogeneous network

but introspective) already avoided this additional communication of controller states.

Almost synchronization is a notion that was brought up by Peymani and his coworkers in [17] (introspective) and [16] (homogeneous, non-introspective), where it deals with agents that are affected by external disturbances. The goal of their work is to reduce the impact of disturbances on the synchronization error to an arbitrary degree of accuracy (expressed in the \mathcal{H}_∞ norm). But they assume availability of an additional communication channel to exchange information about internal controller or observer states between neighbouring agents. The earlier work on almost synchronization for introspective, heterogeneous networks was extended in [31] to design a dynamic protocol to avoid exchange of controller states.

The majority of the works assume the topology associated with the network is fixed. Extensions to time-varying topologies are done in the framework of switching topologies. Synchronization with time-varying topologies is studied utilizing concepts of dwell-time and average dwell-time (e.g., [22], [23], [11]). It is assumed that time-varying topologies switch among a finite set of topologies. In [33], switching laws are designed to achieve synchronization.

This paper also aims to solve the almost regulated output synchronization problem for heterogeneous networks of non-introspective agents under switching graphs. However, instead of deterministic disturbances with finite power, we consider stochastic disturbances with bounded variance. We name this problem as stochastic almost regulated output synchronization.

A. Notations and definitions

Given a matrix $A \in \mathbb{C}^{m \times n}$, A' denotes its conjugate transpose, $\|A\|$ is the induced 2-norm, and $\lambda_i(A)$ denotes its i 'th eigenvalue if $m = n$. A square matrix A is said to be Hurwitz stable if all its eigenvalues are in the open left half complex plane. We denote by $\text{blkdiag}\{A_i\}$, a block-diagonal matrix with A_1, \dots, A_N as the diagonal elements, and by $\text{col}\{x_i\}$, a column vector with x_1, \dots, x_N stacked together, where the range of index i can be identified from the context. $A \otimes B$ depicts the Kronecker product between A and B . I_n denotes the n -dimensional identity matrix and 0_n denotes $n \times n$ zero matrix; sometimes we drop the subscript if the dimension is clear from the context. $\mathbf{1}$ is the column vector with each element being equal to 1.

A *weighted directed graph* \mathcal{G} is defined by a triple $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, \dots, N\}$ is a node set, \mathcal{E} is a set of pairs of nodes indicating connections among nodes, and

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$\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is the weighting matrix, and $a_{ij} > 0$ iff $(i, j) \in \mathcal{E}$. Each pair in \mathcal{E} is called an *edge*. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1, \dots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, \dots, k-1$. A *directed tree* with root r is a subset of nodes of the graph \mathcal{G} such that a path exists between r and every other node in this subset. A *directed spanning tree* is a directed tree containing all the nodes of the graph. For a weighted graph \mathcal{G} , a matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^N a_{ik}, & i = j, \\ -a_{ij}, & i \neq j, \end{cases}$$

is called the *Laplacian matrix* associated with the graph \mathcal{G} . In the case where \mathcal{G} has non-negative weights, L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $\mathbf{1}$.

Definition 1: Let $\mathcal{L}_N \subset \mathbb{R}^{N \times N}$ be the family of all possible Laplacian matrices associated to a graph with N agents. We denote by \mathcal{G}_L the graph associated with a Laplacian matrix $L \in \mathcal{L}_N$. Then, a time-varying graph $\mathcal{G}(t)$ with N agents has such a definition as

$$\mathcal{G}(t) = \mathcal{G}_{\sigma(t)},$$

where $\sigma : \mathbb{R} \rightarrow \mathcal{L}_N$ is a piecewise constant, right-continuous function with minimal dwell-time τ (see [8]), i.e. $\sigma(t)$ remains fixed for $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}$ and switches at $t = t_k$, $k = 1, 2, \dots$ where $t_{k+1} - t_k \geq \tau$ for $k = 0, 1, \dots$. For ease of presentation we assume $t_0 = 0$.

Definition 2: A matrix pair (A, C) is said to contain the matrix pair (S, R) if there exists a matrix Π such that $\Pi S = A\Pi$ and $C\Pi = R$.

Remark 1: Definition 2 implies that for any initial condition $\omega(0)$ of the system $\dot{\omega} = S\omega$, $y_r = R\omega$, there exists an initial condition $x(0)$ of the system $\dot{x} = Ax$, $y = Cx$, such that $y(t) = y_r(t)$ for all $t \geq 0$ ([10]).

II. HETEROGENEOUS MULTI-AGENT SYSTEMS

We consider a multi-agent system/network consisting of N non-identical non-introspective agents $\tilde{\Sigma}_i$ with $i \in \{1, \dots, N\}$ described by the stochastic differential equation:

$$\tilde{\Sigma}_i : \begin{cases} d\tilde{x}_i = \tilde{A}_i \tilde{x}_i dt + \tilde{B}_i \tilde{u}_i dt + \tilde{G}_i dw_i, & \tilde{x}_i(0) = \tilde{x}_{i0}, \\ y_i = \tilde{C}_i \tilde{x}_i, \end{cases} \quad (1)$$

with the order of the infinite zeros at most equal to $\tilde{\rho}_i$, where $\tilde{x}_i \in \mathbb{R}^{\tilde{n}_i}$, $\tilde{u}_i \in \mathbb{R}^{\tilde{m}_i}$, and $y_i \in \mathbb{R}^p$ are the state, input and output of agent i , and $w = \text{col}\{w_i\}$ is a Wiener process (a Brownian motion) with mean 0 and rate Q_0 , that is, $\text{Cov}[w(t)] = tQ_0$ and the initial condition \tilde{x}_{i0} of (1) is a Gaussian random vector which is independent of w_i . Its solution \tilde{x}_i is rigorously defined through Wiener integrals and is a Gauss-Markov process. See, for instance, [13].

Remark 2: If we have an agent described by:

$$\check{\Sigma}_i : \begin{cases} \dot{\check{x}}_i = \check{A}_i \check{x}_i + \check{B}_i \check{u}_i + \check{G}_i \check{w}_i, \\ y_i = \check{C}_i \check{x}_i, \end{cases} \quad (2)$$

with \check{w}_i colored stochastic noise, and we assume that \check{w}_i can be modeled as being generated by a linear model:

$$\check{\Sigma}_{w_i} : \begin{cases} d\check{p}_i = \check{A}_{w_i} \check{p}_i dt + \check{G}_{w_i} dw_i, \\ \check{w}_i = \check{C}_{w_i} \check{p}_i, \end{cases} \quad (3)$$

Then combining (2) and (3) we get a model of the form (1).

The topology of time-varying networks can be described by a time-varying graph $\mathcal{G}(t)$, which is defined by a triple $(\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$, where $\mathcal{V} = \{1, \dots, N\}$ is a node set (each node denotes an agent in the network), $\mathcal{E}(t)$ is a time-varying set for a pair of nodes, and $\mathcal{A}(t) = [a_{ij}(t)]$ is the weighted time-varying adjacency matrix. The Laplacian matrix of $\mathcal{G}(t)$ is defined as $L(t) = [\ell_{ij}(t)]$. With the definition of the time-varying graph $\mathcal{G}(t)$, $\mathcal{A}(t)$ is a piecewise constant matrix and right-continuous in time, and so is $L(t)$.

The network provides each agent with a linear combination of its own output relative to those of other neighboring agents, that is, agent $i \in \mathcal{V}$, has access to the quantity

$$\zeta_i(t) = \sum_{j=1}^N a_{ij}(t)(y_i(t) - y_j(t)), \quad (4)$$

which is equivalent to

$$\zeta_i(t) = \sum_{j=1}^N \ell_{ij}(t)y_j(t). \quad (5)$$

We make the following assumption on the agent dynamics.

Assumption 1: For each agent $i \in \mathcal{V}$, we have:

- $(\tilde{A}_i, \tilde{B}_i, \tilde{C}_i)$ is right-invertible and minimum-phase;
- $(\tilde{A}_i, \tilde{B}_i)$ is stabilizable, and $(\tilde{A}_i, \tilde{C}_i)$ is detectable;

Remark 3: Right-invertibility of a triple $(\tilde{A}_i, \tilde{B}_i, \tilde{C}_i)$ means that, given a reference output $y_r(t)$, there exist an initial condition $\tilde{x}_i(0)$ and an input $\tilde{u}_i(t)$ such that $y_i(t) = y_r(t)$ for all $t \geq 0$.

III. STOCHASTIC ALMOST REGULATED OUTPUT SYNCHRONIZATION UNDER SWITCHING TOPOLOGY

In this section, we consider the almost regulated output synchronization problem for heterogeneous multi-agents systems/networks defined in (1) under switching topologies, where the goal is to make the agents asymptotically converge to a reference trajectory in the presence of external stochastic disturbances. The reference trajectory in this paper is generated by an autonomous system

$$\Sigma_0 : \begin{cases} \dot{x}_r = Sx_r, & x_r(0) = x_{r0}, \\ y_r = Rx_r, \end{cases} \quad (6)$$

where $x_r \in \mathbb{R}^{n_r}$, $y_r \in \mathbb{R}^p$. Moreover, we assume that (S, R) is observable and all eigenvalues of S are in the closed right half complex plane.

Define $e_i := y_i - y_r$ as the regulated output synchronization error for agent $i \in \mathcal{V}$ and $\mathbf{e} = \text{col}\{e_i\}$. In order to achieve our goal, it is clear that a non-empty subset of agents must have knowledge of their outputs relative to the reference trajectory y_r generated by the reference system. Specially, each agent has access to the quantity

$$\psi_i = \iota_i(y_i - y_r), \quad \iota_i = \begin{cases} 1, & i \in \pi, \\ 0, & i \notin \pi, \end{cases} \quad (7)$$

where π is a subset of \mathcal{V} .

Assumption 2: Every node of the network graph \mathcal{G} is a member of a directed tree with the root contained in π .

In the following, we will refer to the node set π as *root set* in view of Assumption 2 (A special case is when π consists of a single element corresponding to the root of a directed spanning tree of \mathcal{G}).

Based on the Laplacian matrix $L(t)$ of our time-varying network graph $\mathcal{G}(t)$, we define the expanded Laplacian matrix as

$$\bar{L}(t) = L(t) + \text{blkdiag}\{\iota_i\} = [\bar{\ell}_{ij}(t)].$$

Note that $\bar{L}(t)$ is also written as \bar{L}_t , and it is clearly not a Laplacian matrix associated to some graph since it does not have a zero row sum. From [5, Lemma 7], all eigenvalues of $\bar{L}(t)$ are in the open right-half complex plane for all $t \in \mathbb{R}$.

It should be noted that, in practice, perfect information of the communication topology is usually not available for controller design and only some rough characterization of the network can be obtained. Next we will define a set of time-varying graphs based on some rough information of the graph. Before doing so, we first define a set of fixed graphs, based on which the set of time-varying graphs is defined.

Definition 3: For given root set π , $\alpha, \beta, \varphi > 0$ and N , $\mathbb{G}_{\alpha, \beta, \pi}^{\varphi, N}$ is the set of directed graphs \mathcal{G} composed of N agents satisfying the following properties:

- The eigenvalues of the associated expanded Laplacian \bar{L} (Here $\bar{L} = L + \text{blkdiag}\{\iota_i\}$, and L is the Laplacian matrix for the graph \mathcal{G}), denoted by $\lambda_1, \dots, \lambda_N$, satisfy $\text{Re}\{\lambda_i\} > \beta$ and $|\lambda_i| < \alpha$.
- The condition number¹ of the expanded Laplacian \bar{L} is less than φ .

Remark 4: In order to achieve regulated output synchronization for all agents, the first condition is obviously necessary.

Note that for undirected graphs the condition number of the Laplacian matrix is always bounded. Moreover, if we have a *finite* set of possible directed graphs each of which has a spanning tree then there always exists a set of the form $\mathbb{G}_{\alpha, \beta, \pi}^{\varphi, N}$ for suitable $\alpha, \beta, \varphi > 0$ and N containing these graphs. The only limitation is that we cannot find **one** protocol for a sequence of graphs converging to a graph without a spanning tree or whose Laplacian matrix either diverges or approaches some ill-conditioned matrix.

Definition 4: Given a root set π , $\alpha, \beta, \varphi, \tau > 0$ and positive integer N , we define the set of time-varying network graphs $\tilde{\mathbb{G}}_{\alpha, \beta, \pi}^{\varphi, \tau, N}$ as the set of all time-varying graphs \mathcal{G} for which

$$\mathcal{G}(t) = \mathcal{G}_{\sigma(t)} \in \mathbb{G}_{\alpha, \beta, \pi}^{\varphi, \tau, N}$$

for all $t \in \mathbb{R}$, where $\sigma : \mathbb{R} \rightarrow \mathcal{L}_N$ is a piecewise constant, right-continuous function with minimal dwell-time τ .

Remark 5: Note that the minimal dwell-time is assumed to avoid chattering problems. However, it can be arbitrarily small.

We will define the stochastic almost regulated output synchronization problem under switching graphs as follows.

¹In this context, we mean by condition number the minimum of $\|U\| \|U^{-1}\|$ over all possible matrices U whose columns are the (generalized) eigenvectors of the matrix L .

Problem 1: Consider a multi-agent system (1), (4) under Assumption 1, and reference system (6), (7) under Assumption 2. For any given root set π , $\alpha, \beta, \varphi, \tau > 0$ and positive integer N defining a set of time-varying network graphs $\tilde{\mathbb{G}}_{\alpha, \beta, \pi}^{\varphi, \tau, N}$, the *stochastic almost regulated output synchronization* problem is to find, if possible, for any $\gamma > 0$, for any upper bound for the rate of the stochastic disturbance \bar{Q}_0 , a linear time-invariant dynamic protocol such that, for any $\mathcal{G} \in \tilde{\mathbb{G}}_{\alpha, \beta, \pi}^{\varphi, \tau, N}$, for all initial conditions of agents and reference system, the stochastic almost regulated output synchronization error satisfies

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{E} \mathbf{e}(t) &= 0, \\ \limsup_{t \rightarrow \infty} \text{Var}[\mathbf{e}(t)] &= \limsup_{t \rightarrow \infty} \mathbb{E} \mathbf{e}'(t) \mathbf{e}(t) < \gamma, \end{aligned} \quad (8)$$

for any $Q_0 \leq \bar{Q}_0$.

Remark 6: Clearly, we can also define (8) via the expectation of the RMS ([2]) as:

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \int_0^T \mathbf{e}(t)' \mathbf{e}(t) dt < \gamma.$$

Remark 7: Note that because of the time-varying graph the complete system is time-variant and hence the variance of the error signal might not converge as time tends to infinity. Hence we use in the above a lim sup instead of a regular limit.

The main result in this section is given in the following theorem.

Theorem 1: Consider a multi-agent system (1), (4), and reference system (6), (7). Let a root set π , $\alpha, \beta, \varphi, \tau > 0$ and positive integer N be given, and hence a set of network graphs $\tilde{\mathbb{G}}_{\alpha, \beta, \pi}^{\varphi, \tau, N}$ be defined.

Under Assumption 1 and 2, the stochastic almost regulated output synchronization problem is solvable, i.e., for any given $\gamma > 0$, for any upper bound for the rate of the disturbance \bar{Q}_0 , there exists a family of distributed dynamic protocols, parametrized in terms of low-and-high gain parameters δ, ε , of the form

$$\begin{cases} \dot{\chi}_i = \mathcal{A}_i(\delta, \varepsilon) \chi_i + \mathcal{B}_i(\delta, \varepsilon) \begin{pmatrix} \zeta_i \\ \psi_i \end{pmatrix}, \\ \ddot{u}_i = \mathcal{C}_i(\delta, \varepsilon) \chi_i + \mathcal{D}_i(\delta, \varepsilon) \begin{pmatrix} \zeta_i \\ \psi_i \end{pmatrix}, \end{cases} \quad i \in \mathcal{V} \quad (9)$$

where $\chi_i \in \mathbb{R}^{q_i}$, such that for any time-varying graph $\mathcal{G} \in \tilde{\mathbb{G}}_{\alpha, \beta, \pi}^{\varphi, \tau, N}$, for all initial conditions of agents, the stochastic almost regulated output synchronization error satisfies (8).

Specifically, there exists a $\delta^* \in (0, 1]$ such that for each $\delta \in (0, \delta^*]$, there exists an $\varepsilon^* \in (0, 1]$ such that for any $\varepsilon \in (0, \varepsilon^*]$, the protocol (9) achieves stochastic almost regulated output synchronization.

Remark 8: In the above, we would like to stress that the initial condition of the reference system is deterministic while the initial conditions of the agents are stochastic. Our protocol yields (8) independent of the initial condition of the reference system and independent of the stochastic properties for the agents, i.e. we do not need to impose bounds on the second order moments.

The proof will be presented in a constructive way in the following subsection.

A. The proof of Theorem 1

In this section, we will present the constructive proof in three steps.

Step 1: In this step, we augment agent (1) with a pre-compensator in such a way that the interconnection of agent (1) and the pre-compensator is of uniform rank and contains the reference system (6).

With the method presented in the appendix, we can find pre-compensators

$$\begin{cases} \dot{z}_i = A_{ip}z_i + B_{ip}u_i, \\ \tilde{u}_i = C_{ip}z_i, \end{cases} \quad (10)$$

for each agent $i = 1, \dots, N$, such that agent (1) plus pre-compensator (10) can be represented as:

$$\begin{cases} dx_i = A_i x_i dt + B_i u_i dt + G_i dw_i, \\ y_i = C_i x_i, \end{cases} \quad (11)$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^p$, $y_i \in \mathbb{R}^p$ are states, inputs and outputs of the interconnection of agent (1) and pre-compensator (10). Moreover (A_i, C_i) contains (S, R) while (A_i, B_i, C_i) has uniform rank $\rho \geq 1$.

It is shown in the appendix that, (A_i, B_i, C_i) is already in the *special coordinate basis* (SCB) [19], which can be written in another form where $x_i = [x_{ia}; x_{id}]$, with $x_{ia} \in \mathbb{R}^{n_i - p\rho}$ representing the finite zero structure and $x_{id} \in \mathbb{R}^{p\rho}$ the infinite zero structure, and

$$\begin{cases} dx_{ia} = A_{ia}x_{ia}dt + L_{iad}y_idt + G_{iad}dw_i, \\ dx_{id} = A_d x_{id}dt + B_d(u_i + E_{ida}x_{ia} + E_{idd}x_{id})dt + G_{id}dw_i, \\ y_i = C_d x_{id}, \end{cases} \quad (12)$$

for $i = 1, \dots, N$, where

$$A_d = \begin{pmatrix} 0 & I_{p(\rho-1)} \\ 0 & 0 \end{pmatrix}, \quad B_d = \begin{pmatrix} 0 \\ I_p \end{pmatrix}, \quad C_d = (I_p \quad 0). \quad (13)$$

Furthermore, the eigenvalues of A_{ia} are the invariant zeros of (A_i, B_i, C_i) , which are all in the open left-half complex plane (OLHP) due to the minimum phase property in Assumption 1 and possibly additional stable invariant zeros added during including the dynamics of reference system.

Step 2: For each interconnection dynamics (12), we will design a purely decentralized controller based on a low-and-high gain method. Let $\delta \in (0, 1]$ be the low-gain parameter and $\varepsilon \in (0, 1]$ be the high-gain parameter [4]. First, select K such that $A_d - KC_d$ is Hurwitz stable. Next, choose $F_\delta = -B'_d P_d$ where $P'_d = P_d > 0$ is uniquely determined by the following algebraic Riccati equation:

$$P_d A_d + A'_d P_d - \beta P_d B_d B'_d P_d + \delta I = 0, \quad (14)$$

where $\beta > 0$ is the lower bound on the real parts of all eigenvalues of expanded Laplacian matrices $\tilde{L}(t)$ for all t . Next, define $S_\varepsilon = \text{blkdiag}\{I_p, \varepsilon I_p, \dots, \varepsilon^{\rho-1} I_p\}$, $K_\varepsilon = \varepsilon^{-1} S_\varepsilon^{-1} K$ and $F_{\delta\varepsilon} = \varepsilon^{-\rho} F_\delta S_\varepsilon$.

Then, we define a dynamic controller for each agent $i \in \mathcal{V}$:

$$\begin{aligned} \dot{\hat{x}}_{id} &= A_d \hat{x}_{id} + K_\varepsilon (\zeta_i + \psi_i - C_d \hat{x}_{id}), \\ u_i &= F_{\delta\varepsilon} \hat{x}_{id}, \end{aligned} \quad (15)$$

where ψ_i is defined in (7).

The state \hat{x}_{id} is an estimator for a linear combination of other agents' relative state with the same weights as in the measurement of $\zeta_i + \psi_i$.

In the proof we need the following lemma [32],

Lemma 1: Consider the matrix $\tilde{A}_{\delta,t}$ defined by:

$$\tilde{A}_{\delta,t} = I_N \otimes \begin{pmatrix} A_d & 0 \\ 0 & A_d - KC_d \end{pmatrix} + J_t \otimes \begin{pmatrix} B_d F_\delta & -B_d F_\delta \\ B_d F_\delta & -B_d F_\delta \end{pmatrix}, \quad (16)$$

For any δ small enough the matrix $\tilde{A}_{\delta,t}$ is asymptotically stable for any Jordan matrix J_t whose eigenvalues satisfy $\text{Re}\{\lambda_i\} > \beta$ and $|\lambda_i| < \alpha$. Moreover, there exists $P'_\delta = P_\delta > 0$ and $\nu > 0$ such that

$$\tilde{A}_{\delta,t} P_\delta + P_\delta \tilde{A}'_{\delta,t} \leq -\nu P_\delta - 4I \quad (17)$$

is satisfied for all possible Jordan matrices J_t and such that there exists $P_a > 0$ for which

$$P_a A_a + A'_a P_a = -\nu P_a - I, \quad (18)$$

with $A_a = \text{blkdiag}\{A_{ia}\}$.

Proof: For each fixed t , eigenvalues of J_t satisfy $\text{Re}\{\lambda_i\} > \beta$ and $|\lambda_i| < \alpha$. Hence the arguments from [32] apply. \blacksquare

The following lemma then provides a constructive proof of Theorem 1:

Lemma 2: For any given $\gamma > 0$, there exists a $\delta^* \in (0, 1]$ such that, for each $\delta \in (0, \delta^*]$, there exists an $\varepsilon^* \in (0, 1]$ such that for any $\varepsilon \in (0, \varepsilon^*]$, the protocol (15) solves stochastic almost regulated output synchronization problem for any time-varying graph $\mathcal{G} \in \tilde{\mathcal{G}}_{\alpha, \beta, \pi}^{\varphi, \tau, N}$, for all initial conditions, and for any $Q_0 \leq \tilde{Q}_0$.

Proof: Using similar arguments as in [32], we can assure that the complete network system can be brought in the form:

$$\begin{aligned} d\eta_a &= A_a \eta_a dt + \tilde{W}_{ad,t} \eta_a dt + G_a dw, \\ \varepsilon d\eta_d &= \tilde{A}_{\delta,t} \eta_d dt + \tilde{W}_{da,t}^\varepsilon \eta_a dt + \tilde{W}_{dd,t}^\varepsilon \eta_d dt + \varepsilon \tilde{G}_{d,t}^\varepsilon dw. \end{aligned} \quad (19)$$

Note that η_d has discontinuous jumps when the network changes.

Define $V_a = \varepsilon^2 \eta'_a P_a \eta_a$ as a Lyapunov function for the dynamics of η_a in (19). Similarly, we define $V_d = \varepsilon \eta'_d P_\delta \eta_d$ as a Lyapunov function for the η_d dynamics in (19). It is easy to find that V_d also has discontinuous jumps when the network changes. The derivative of V_a is bounded by:

$$\begin{aligned} dV_a &= -\nu V_a dt - \varepsilon^2 \|\eta_a\|^2 dt + 2\varepsilon^2 \text{Re}(\eta'_a P_a \tilde{W}_{ad,t} \eta_a) dt \\ &\quad + \varepsilon^2 \text{trace}(P_a G_a Q_0 G'_a) dt + 2\varepsilon^2 \text{Re}(\eta'_a P_a G_a) dw \\ &\leq -\nu V_a dt + \varepsilon c_3 V_d dt \\ &\quad + \varepsilon^2 r_5 \text{trace}(Q_0) dt + 2\varepsilon^2 \text{Re}(\eta'_a P_a G_a) dw, \end{aligned} \quad (20)$$

where r_5 and c_3 are such that:

$$\text{trace}(P_a G_a Q_0 G'_a) \leq r_5 \text{trace} Q_0,$$

and

$$\begin{aligned} 2 \operatorname{Re}(\eta'_a P_a \tilde{W}_{ad,t} \eta_d) &\leq 2r_4 \|\eta_a\| \|\eta_d\| \\ &\leq \frac{1}{2} \|\eta_a\|^2 + 2r_4^2 \|\eta_d\|^2 \leq \frac{1}{2} \|\eta_a\|^2 + \varepsilon^{-1} c_3 V_d \end{aligned}$$

Note that we can choose r_4 , r_5 and c_3 independent of the network graph but only depending on our bounds on the eigenvalues and condition number of our expand Laplacian $\tilde{L}(t)$. Taking the expectation, we get:

$$d \mathbb{E} V_a \leq -\nu \mathbb{E} V_a dt + \varepsilon c_3 \mathbb{E} V_d dt + \varepsilon^2 r_5 \operatorname{trace}(Q_0) dt$$

Next, the derivative of V_d is bounded by

$$\begin{aligned} dV_d &= -\nu \varepsilon^{-1} V_d dt - 4 \|\eta_d\|^2 dt + 2 \operatorname{Re}(\eta'_d P_\delta \tilde{W}_{da,t}^\varepsilon \eta_a) dt \\ &\quad + 2 \operatorname{Re}(\eta'_d P_\delta \tilde{W}_{da,t}^\varepsilon \eta_d) dt + \varepsilon \operatorname{trace}(P_\delta \tilde{G}_{d,t}^\varepsilon Q_0 (\tilde{G}_{d,t}^\varepsilon)') dt \\ &\quad + 2\varepsilon \operatorname{Re}(\eta'_d P_\delta \tilde{G}_{d,t}^\varepsilon) dw \\ &\leq c_2 V_a dt - (\nu \varepsilon^{-1} + \nu - \varepsilon^2 \frac{c_2 c_3}{\nu}) V_d dt - \|\eta_d\|^2 dt \\ &\quad + \varepsilon r_3 \operatorname{trace}(Q_0) dt + 2\varepsilon \operatorname{Re}(\eta'_d P_\delta \tilde{G}_{d,t}^\varepsilon) dw, \end{aligned} \quad (21)$$

where $2 \operatorname{Re}(\eta'_d P_\delta \tilde{W}_{da,t}^\varepsilon \eta_a) \leq \|\eta_d\|^2$ for small ε and

$$\begin{aligned} 2 \operatorname{Re}(\eta'_d P_\delta \tilde{W}_{da,t}^\varepsilon \eta_a) &\leq \varepsilon r_1 \|\eta_a\| \|\eta_d\| \\ &\leq \varepsilon^2 r_1^2 \|\eta_a\|^2 + \|\eta_d\|^2 \leq c_2 V_a + \|\eta_d\|^2, \end{aligned}$$

provided r_1 is such that we have $\varepsilon r_1 \geq \|P_\delta \tilde{W}_{da,t}^\varepsilon\|$ and c_2 sufficiently large. Finally

$$\operatorname{trace}(P_\delta \tilde{G}_{d,t}^\varepsilon Q_0 (\tilde{G}_{d,t}^\varepsilon)') \leq r_3 \operatorname{trace} Q_0$$

for suitably chosen r_3 . Taking the expectation, we get:

$$\begin{aligned} d \mathbb{E} V_d &\leq c_2 \mathbb{E} V_a dt - (\nu \varepsilon^{-1} + \nu - \varepsilon^2 \frac{c_2 c_3}{\nu}) \mathbb{E} V_d dt \\ &\quad - \mathbb{E} \|\eta_d\|^2 dt + \varepsilon r_3 \operatorname{trace}(Q_0) dt \end{aligned}$$

We get:

$$\frac{d}{dt} \begin{pmatrix} \mathbb{E} V_a \\ \mathbb{E} V_d \end{pmatrix} \leq A_e \begin{pmatrix} \mathbb{E} V_a \\ \mathbb{E} V_d \end{pmatrix} + \begin{pmatrix} \varepsilon^2 r_5 \operatorname{trace}(Q_0) \\ \varepsilon r_3 \operatorname{trace}(Q_0) \end{pmatrix}$$

where

$$A_e = \begin{pmatrix} -\nu & \varepsilon c_3 \\ c_2 & -\varepsilon^{-1} \nu - \nu + \varepsilon^2 \frac{c_2 c_3}{\nu} \end{pmatrix}.$$

Note that the inequality here is componentwise. We find by integration that:

$$\begin{pmatrix} \mathbb{E} V_a \\ \mathbb{E} V_d \end{pmatrix} (t_k^-) \leq e^{A_e(t_k - t_{k-1})} \begin{pmatrix} \mathbb{E} V_a \\ \mathbb{E} V_d \end{pmatrix} (t_{k-1}^+) + \int_{t_{k-1}}^{t_k} e^{A_e(t_k - s)} \begin{pmatrix} \varepsilon^2 r_5 \operatorname{trace}(Q_0) \\ \varepsilon r_3 \operatorname{trace}(Q_0) \end{pmatrix} ds$$

componentwise. We have a potential jump at time t_{k-1} in V_d . However, there exists m such that

$$V_d(t_{k-1}^+) \leq m V_d(t_{k-1}^-)$$

while V_a is continuous. Using our explicit expression for $e^{A_e t}$ and the fact that $t_k - t_{k-1} > \tau$ we find:

$$\begin{aligned} \begin{pmatrix} \mathbb{E} V_a(t_k^-) + \mathbb{E} V_d(t_k^-) \end{pmatrix} &\leq \\ e^{\lambda_3(t_k - t_{k-1})} \begin{pmatrix} \mathbb{E} V_a(t_{k-1}^-) + \mathbb{E} V_d(t_{k-1}^-) \end{pmatrix} &+ r \varepsilon^2 \operatorname{trace}(Q_0) \end{aligned}$$

where $\lambda_3 = -\nu/2$ and r is a sufficiently large constant. Combining these time-intervals, we get:

$$\begin{aligned} \begin{pmatrix} \mathbb{E} V_a(t_k^-) + \mathbb{E} V_d(t_k^-) \end{pmatrix} &\leq \\ e^{\lambda_3 t_k} \begin{pmatrix} \mathbb{E} V_a(0) + \mathbb{E} V_d(0) \end{pmatrix} &+ \frac{r \varepsilon^2}{1 - \mu} \operatorname{trace}(Q_0) \end{aligned}$$

where $\mu < 1$ is such that $e^{\lambda_3(t_k - t_{k-1})} \leq e^{\lambda_3 \tau} \leq \mu$ for all k . Assume $t_{k+1} > t > t_k$. Since we do not necessarily have that $t - t_k > \tau$ we use the bound:

$$\begin{pmatrix} 1 & 1 \end{pmatrix} e^{A_e(t-t_k)} \begin{pmatrix} \mathbb{E} V_a \\ \mathbb{E} V_d \end{pmatrix} (t_k^+) \leq 2m e^{\lambda_3(t-t_k)} [\mathbb{E} V_a + \mathbb{E} V_d](t_k^-)$$

where the factor m is due to the potential discontinuous jump. Combining all together, we get:

$$\begin{aligned} \begin{pmatrix} \mathbb{E} V_a(t) + \mathbb{E} V_d(t) \end{pmatrix} &\leq \\ 2m e^{\lambda_3 t} \begin{pmatrix} \mathbb{E} V_a(0) + \mathbb{E} V_d(0) \end{pmatrix} &+ (2m + 1) \frac{r \varepsilon^2}{1 - \mu} \operatorname{trace}(Q_0) \end{aligned}$$

This implies:

$$\limsup_{t \rightarrow \infty} \mathbb{E} \eta'_d(t) \eta_d(t) \leq \frac{2m + 1}{\sigma_{\min}(P_\delta)} \frac{r \varepsilon}{1 - \mu} \operatorname{trace}(Q_0)$$

Following the proof in [32], we find that

$$\begin{aligned} \mathbf{e} &= (I_N \otimes C_d)(I_N \otimes S_\varepsilon^{-1})(U_t J_t^{-1} \otimes I_{p\rho}) \begin{pmatrix} I_{N p\rho} & 0 \end{pmatrix} N_d^{-1} \eta_d \\ &= (U_t J_t^{-1} \otimes C_d) \begin{pmatrix} I_{N p\rho} & 0 \end{pmatrix} N_d^{-1} \eta_d = \Theta_t \eta_d, \end{aligned}$$

for suitably chosen matrix Θ_t , which is bounded because of the boundedness of U_t, J_t for any graph in $\mathbb{G}_{\alpha, \beta, \pi}^{\varphi, N}$. The fact that we can make the asymptotic variance of η_d arbitrarily small then immediately implies that the asymptotic variance of \mathbf{e} can be made arbitrarily small. Because of all agents and protocols are linear it is obvious that the expectation of \mathbf{e} is equal to zero. ■

Step 3: Combining the pre-compensator (10) in step 1 and the controller (15) in Step 2, we obtain the protocol in the form of (9) as:

$$\begin{aligned} \mathcal{A}_i &= \begin{pmatrix} A_d - K_\varepsilon C_d & 0 \\ B_{ip} F_{\delta \varepsilon} & A_{ip} \end{pmatrix}, & \mathcal{B}_i &= \begin{pmatrix} K_\varepsilon & K_\varepsilon \\ 0 & 0 \end{pmatrix}, \\ \mathcal{C}_i &= \begin{pmatrix} 0 & C_{ip} \end{pmatrix}, & \mathcal{D}_i &= \begin{pmatrix} 0 \end{pmatrix}. \end{aligned} \quad (22)$$

APPENDIX

In this section, we will design pre-compensators such that agent model (1) plus pre-compensators can be represented in (11) that we considered in section III-A. To fulfil this target, we need a pre-compensator for each agent. This pre-compensator is designed in two steps.

Step A: Design a pre-compensator such that the interconnection of agent model and pre-compensator contains the dynamics of the reference system

The design of this pre-compensator is quite straightforward if, a priori, the agent has no dynamics in common with the reference system. In that case, Π_i and Γ_i are uniquely determined by the so-called regulator equations:

$$\tilde{A}_i \Pi_i + \tilde{B}_i \Gamma_i = \Pi_i S, \quad \tilde{C}_i \Pi_i = R. \quad (23)$$

Then the pre-compensator is given by:

$$\dot{p}_{i,1} = Sp_{i,1} + B_{i,1}u_i^1, \quad \tilde{u}_i = \Gamma_i p_{i,1}, \quad (24)$$

where $B_{i,1}$ is chosen according to the technique presented in [9] to guarantee that the interconnection of (1) and (24), indicated by system $\hat{\Sigma}_i(\hat{A}_i, \hat{B}_i, \hat{C}_i, \hat{D}_i)$ ($\hat{D}_i = 0$), contains the reference system, and is minimum-phase, right-invertible and with its highest order of infinite zeros equal to ρ_i . However, if \tilde{A}_i and S have common eigenvalues the design is a bit more involved. For details we refer to [4].

Step B: Design another pre-compensator such that the cascade of system $\hat{\Sigma}_i$ and pre-compensator is invertible and of uniform rank

Let $\rho = \max\{\rho_i\}$, $i = 1, \dots, N$. According to [20, Theorem 1], a pre-compensator of this form

$$\begin{aligned} \dot{p}_{i2} &= A_{ip2}p_{i2} + B_{ip2}u_i^2, \\ u_i^1 &= C_{ip2}p_{i2} + D_{ip2}u_i^2, \end{aligned} \quad (25)$$

is designed for system $\hat{\Sigma}_i$ to square down it to an inevitable uniform rank system, denoted by system $\Sigma_i(A_i, B_i, C_i, D_i)$, with its order of infinite zeros equal to ρ , and moreover Σ_i contains the invariant zeros of system $\hat{\Sigma}_i$ and possibly additional invariant zeros that can be freely assigned in the OLHP.

From the above steps, we can see that the cascade interconnection of agent (1), pre-compensator (24) and pre-compensator (25) yields system (11), and pre-compensator (24) plus pre-compensator (25) can be represented in pre-compensator (10).

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