2015 American Control Conference
Palmer House Hilton
July 1-3, 2015. Chicago, IL, USA

Almost regulated output synchronization for heterogeneous time-varying networks of non-introspective agents and without exchange of controller states

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Abstract— **We consider almost regulated output synchronization for heterogeneous directed networks with external disturbances where agents are non-introspective (i.e. agents have no access to their own states or outputs). A purely decentralized time-invariant protocol based on a low-and-high gain method is designed for each agent to achieve almost regulated output synchronization while reducing the impact of disturbances on the regulated output synchronization error. It is also shown that this protocol also works in the case of time-varying graphs.**

I. INTRODUCTION

In the last decade, the topic of synchronization in a multi-agent system has received considerable attention. Its potential applications can be seen in cooperative control of autonomous vehicles, distributed sensor network, swarming and flocking and others. The objective of synchronization is to guarantee an asymptotic agreement on a common state or output trajectory through decentralized control protocols (see [1], [10], [15], [23]). Most work has focused on state synchronization based on full-state/partial-state coupling in the homogenous network (i.e. agents have identical dynamics), where the agent dynamics progress from single-and double-integrator dynamics to more general dynamics (e.g., [6], [11], [12], [16], [19], [20], [21], [25]). The counterpart of state synchronization is output synchronization, which is mostly done in heterogeneous networks (i.e., agents are non-identical). When the agent has access to part of its own state it is frequently referred to as introspective and, otherwise, it is referred to as non-introspective. Quite a few of the recent works have assumed agents are introspective (e.g., [2], [5], [22], [26]) while others have considered non-introspective agents. For non-introspective agents, the paper [4] addressed the output synchronization for heterogeneous networks.

In [6] for homogeneous networks a controller structure was introduced which included not only sharing the relative outputs over the network but also sharing the relative states of the protocol over the network. This was also used in our earlier work such as [4], [14], [13] mentioned above. This type of additional communication is not always natural. Some papers such as [16] (homogeneous network) and [5] (heterogeneous network but introspective) already avoided this additional communication of controller states. The earlier work on almost synchronization for introspective, heterogeneous networks was extended in [27] to design a dynamic protocol to avoid exchange of controller states.

Almost synchronization is a notion that was brought up by Peymani and his coworkers in [14] (introspective) and [13] (homogeneous, non-introspective), where it deals with agents that are affected by external disturbances. The goal of this work is to reduce the impact of disturbances on the synchronization error to an arbitrarily degree of accuracy (expressed in the $\mathcal{H}_\infty$ norm). But they assume availability of an additional communication channel to exchange information about internal controller or observer states between neighboring agents.

Most papers in this area assume the topology associated with the network is fixed. Extensions to time-varying topologies are done in the framework of switching topologies. Synchronization with time-varying topologies is studied utilizing concepts of dwell-time and average dwell-time (e.g., [17], [18], [9]). It is assumed that time-varying topologies switch among a finite set of topologies. In [30], switching laws are designed to achieve synchronization. For heterogeneous networks, it is always assumed that the agents are introspective. In [24] synchronization was considered for heterogeneous networks of non-introspective agents without sharing of controller states and under switching topologies. In [29], almost output synchronization is considered for heterogeneous networks of introspective agents without sharing of controller states and under switching topologies.

The main focus of this paper is to solve the almost output synchronization problem for heterogeneous networks with non-introspective agents as studied earlier in [13]. Our paper has three main contributions over this earlier work:

- We allow heterogeneous networks with non-introspective agents,
- We allow the presence of external disturbances,
- We use time-varying graphs.

A. Notations and definitions

Given a matrix $A \in \mathbb{C}^{m\times n}$, A' denotes its conjugate transpose, $\|A\|$ is the induced 2-norm, and $\lambda_i(A)$ denotes

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978-1-4799-8686-6/$31.00 ©2015 AACC

its $i'th$ eigenvalue if $m = n$. A square matrix A is said to be Hurwitz stable if all its eigenvalues are in the open left half complex plane. We denote by blkdiag$\{A_i\}$, a block-diagonal matrix with $A_1, \ldots, A_N$ as the diagonal elements, and by col$\{x_i\}$, a column vector with $x_1, \ldots, x_N$ stacked together, where the range of index i can be identified from the context. $A \otimes B$ depicts the Kronecker product between A and B. I_n denotes the n-dimensional identity matrix and 0_n denotes $n \times n$ zero matrix; sometimes we drop the subscript if the dimension is clear from the context.

A *weighted directed graph* $\mathcal{G}$ is defined by a triple $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, \ldots, N\}$ is a node set, $\mathcal{E}$ is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the weighting matrix, with $a_{ij} > 0$ iff $(i, j) \in \mathcal{E}$ and $a_{ii} = 0$. Each pair in $\mathcal{E}$ is called an *edge*. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, \ldots, k-1$. A *directed tree* with *root* r is a subset of nodes of the graph $\mathcal{G}$ such that a path exists between r and every other node in this subset. A *directed spanning tree* is a directed tree containing all the nodes of the graph. For a weighted graph $\mathcal{G}$, a matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^{N} a_{ik}, & i = j, \\ -a_{ij}, & i \neq j, \end{cases}$$

is called the *Laplacian matrix* associated with the graph $\mathcal{G}$. In the case where $\mathcal{G}$ has non-negative weights, L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $\mathbf{1}$.

Definition 1: A matrix pair (A, C) is said to contain the matrix pair (S, R) if there exists a matrix Π such that $\Pi S = A\Pi$ and $C\Pi = R$.

Remark 1: Definition 1 implies that for any initial condition $\omega(0)$ of the system $\dot{\omega} = S\omega$, $y_r = R\omega$, there exists an initial condition $x(0)$ of the system $\dot{x} = Ax$, $y = Cx$, such that $y(t) = y_r(t)$ for all $t \geq 0$ ([8]).

Definition 2: Let $\mathcal{L}_N \subset \mathbb{R}^{N\times N}$ be the family of all possible Laplacian matrices associated to a graph with N agents. We denote by $\mathcal{G}_L$ the graph associated with a Laplacian matrix $L \in \mathcal{L}_N$. Then, a time-varying graph $\mathcal{G}(t)$ with N agents has such a definition as

$$\mathcal{G}(t) = \mathcal{G}_{\sigma(t)},$$

where $\sigma : \mathbb{R} \to \mathcal{L}_N$ is a piecewise constant, right-continuous function with minimal dwell-time τ (see [7]), i.e. $\sigma(t)$ remains fixed for $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}$ and switches at $t = t_k$, $k = 1, 2, \ldots$ where $t_{k+1} - t_k \geq \tau$ for $k = 0, 1, \ldots$. For ease of presentation we assume $t_0 = 0$.

II. HETEROGENEOUS MULTI-AGENT SYSTEM

We consider a multi-agent system/network consisting of N non-identical non-introspective agents $\tilde{\Sigma}_i$ with $i \in \{1, \ldots, N\} \triangleq \mathcal{V}$ described by

$$\tilde{\Sigma}_i : \begin{cases} \dot{\tilde{x}}_i = \tilde{A}_i \tilde{x}_i + \tilde{B}_i \tilde{u}_i + \tilde{G}_i w_i, \\ y_i = \tilde{C}_i \tilde{x}_i, \end{cases} \tag{1}$$

where $\tilde{x}_i \in \mathbb{R}^{\tilde{n}_i}$, $\tilde{u}_i \in \mathbb{R}^{m_i}$, and $y_i \in \mathbb{R}^p$ are the state, input and output of agent i and with the order of the infinite zeros at most $\tilde{\rho}_i$. Finally, $w_i \in \mathbb{R}^{m_{w_i}}$ is the external disturbance which is either in the set $\Gamma_\kappa^{\text{rms}}$ or in the set Γ_κ^∞ for given κ as defined below:

Definition 3: The set of disturbances with power less than κ is defined as

$$\Gamma_\kappa^{\text{rms}} = \{\, w \in L_{2,\text{loc}} : \|w\|_{\text{rms}} \triangleq \limsup_{T\to\infty} \frac{1}{T} \int_0^T w(t)' w(t) \mathrm{d}t < \kappa \,\}.$$

The set of disturbances which are bounded by κ is defined as

$$\Gamma_\kappa^\infty = \{\, w \in L_\infty : \|w\|_\infty < \kappa \,\}.$$

The topology of a time-varying networks can be described by a time-varying graph $\mathcal{G}(t)$, which is defined by a triple $(\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$, where $\mathcal{V} = \{1, \ldots, N\}$ is a node set (each node denotes an agent in the network), $\mathcal{E}(t)$ is a time-varying set of pairs of nodes, and $\mathcal{A}(t) = [a_{ij}(t)]$ is the weighted time-varying adjacency matrix. The Laplacian matrix of $\mathcal{G}(t)$ is defined as $L(t) = [\ell_{ij}(t)]$. With the definition of the time-varying graph $\mathcal{G}(t)$, $\mathcal{A}(t)$ is a piecewise constant matrix and right-continuous in time, and so is $L(t)$.

The network provides each agent with a linear combination of its own output relative to those of other neighboring agents, that is, agent $i \in \mathcal{V}$, has access to the quantity

$$\zeta_i(t) = \sum_{j=1}^{N} a_{ij}(t)(y_i(t) - y_j(t)) = \sum_{j=1}^{N} \ell_{ij}(t) y_j(t). \tag{2}$$

We make the following assumption on the agent dynamics.

Assumption 1: For each agent $i \in \mathcal{V}$, we have:

- $(\tilde{A}_i, \tilde{B}_i, \tilde{C}_i)$ is right-invertible and minimum-phase;
- $(\tilde{A}_i, \tilde{B}_i)$ is stabilizable, and $(\tilde{A}_i, \tilde{C}_i)$ is detectable.

III. ALMOST REGULATED OUTPUT SYNCHRONIZATION UNDER SWITCHING TOPOLOGIES

In this section, we consider the almost regulated output synchronization problem for heterogeneous multi-agent systems/networks defined in Section II, where the goal is to make the outputs of the agents asymptotically converge to a reference trajectory in the presence of external disturbances. The reference trajectory in this paper is generated by an autonomous exosystem

$$\begin{cases} \dot{x}_r = S x_r, & x_r(0) = x_{r0}, \\ y_r = R x_r, \end{cases} \tag{3}$$

where $x_r \in \mathbb{R}^{n_r}$, $y_r \in \mathbb{R}^p$. Moreover, we assume that (S, R) is observable, all eigenvalues of S are in the closed right half complex plane, and finally, R has full row rank.

Define $e_i \triangleq y_i - y_r$ as the regulated output synchronization error for agent $i \in \mathcal{V}$ and $\mathbf{e} = \text{col}\{e_i\}$. In order to achieve our goal, it is clear that a non-empty subset of agents must have knowledge of their output relative to the reference trajectory

y_r generated by the reference system. Specially, each agent has access to the quantity

$$\psi_i = \iota_i(y_i - y_r), \qquad \iota_i = \begin{cases} 1, & i \in \pi, \\ 0, & i \notin \pi, \end{cases} \tag{4}$$

where π is a subset of $\mathcal{V}$. In order to achieve regulated output synchronization for all agents, the following assumption is clearly necessary:

Assumption 2: Every node of the network graph $\mathcal{G}$ is a member of a directed tree which has a root contained in the set π.

In the following, we will refer to the node set π as *root set* in view of Assumption 2 (when the network graph $\mathcal{G}$ has a directed spanning tree, the set π may contain one node which is the root of such a spanning tree).

Based on the Laplacian matrix $L(t)$ of our time-varying network graph $\mathcal{G}(t)$, we define the expanded Laplacian matrix as

$$\bar{L}(t) = L(t) + \text{blkdiag}\{\iota_i\} = [\bar{\ell}_{ij}(t)].$$

Note that $\bar{L}(t)$ is also written as $\bar{L}_t$, and it is clearly not a Laplacian matrix associated to some graph since it does not have a zero row sum for any fixed t. From [4, Lemma 7], all eigenvalues of $\bar{L}(t)$ are in the open right-half complex plane for any $t \in \mathbb{R}$.

It should be noted that, in practice, perfect information of the communication topology is usually not available for controller design and only some rough characterization of the network can be obtained. Next we will define a set of time-varying graphs based on some rough information of the graph. Before doing so, we first define a set of fixed graphs, based on which the set of time-varying graphs is defined.

Definition 4: For given root set π, $\alpha, \beta, \varphi > 0$ and N, the set $\mathbb{G}^{\varphi,N}_{\alpha,\beta,\pi}$ is the set of directed graphs composed of N nodes satisfying the following properties:

- The eigenvalues of the associated expanded Laplacian $\bar{L}$, denoted by $\lambda_1, \ldots, \lambda_N$, satisfy $\text{Re}\{\lambda_i\} > \beta$ and $|\lambda_i| < \alpha$.
- The condition number[1] of the expanded Laplacian matrix $\bar{L}$ is less than φ.

Remark 2: Note that for undirected graphs the condition number of the Laplacian matrix is always bounded. Moreover, if we have a *finite* set of possible graphs each of which has a directed spanning tree then there always exists a set of the form $\mathbb{G}^{\varphi,N}_{\alpha,\beta,\pi}$ for suitable $\alpha, \beta, \varphi > 0$ and N containing these graphs. The only limitation is that we cannot find **one** protocol for a sequence of graphs converging to a graph without a spanning tree or whose Laplacian either diverges or approaches some ill-conditioned matrix.

Definition 5: Given a root set π, $\alpha, \beta, \varphi, \tau > 0$ and positive integer N, we define the set of time-varying network graphs $\tilde{\mathbb{G}}^{\varphi,\tau,N}_{\alpha,\beta,\pi}$ as the set of all time-varying graphs $\mathcal{G}$ for which

$$\mathcal{G}(t) = \mathcal{G}_{\sigma(t)} \in \mathbb{G}^{\varphi,N}_{\alpha,\beta,\pi}$$

[1]In this context, we mean by condition number the minimum of $\|U\|\|U^{-1}\|$ over all possible matrices U whose columns are the (generalized) eigenvectors of the expanded Laplacian matrix $\bar{L}$.

for all $t \in \mathbb{R}$, where $\sigma : \mathbb{R} \to \mathcal{L}_N$ is a piecewise constant, right-continuous function with minimal dwell-time τ.

Remark 3: Note that the minimal dwell-time is assumed to avoid chattering problems. However, it can be arbitrarily small.

We will define the almost regulated output synchronization problem as follows.

Problem 1: Consider a multi-agent system (1), (2) under Assumption 1, and reference system (3), (4) under Assumption 2. For any given root set π, $\alpha, \beta, \varphi, \tau > 0$ and positive integer N defining a set of time-varying network graphs $\tilde{\mathbb{G}}^{\varphi,\tau,N}_{\alpha,\beta,\pi}$, the *almost regulated output synchronization* problem is to find, if possible, for any $\gamma > 0$, and for any disturbance bound κ, a linear time-invariant dynamic protocol such that, for any time-varying graph $\mathcal{G} \in \tilde{\mathbb{G}}^{\varphi,N}_{\alpha,\beta,\pi}$, for all initial conditions of agents and reference system, the almost regulated output synchronization error satisfies

- For all $w_i \in \Gamma^{\infty}_{\kappa}$, $i = 1, \ldots, N$,
$$\limsup_{t\to\infty} \|\mathbf{e}(t)\| < \gamma; \tag{5}$$
- For all $w_i \in \Gamma^{\text{rms}}_{\kappa}$, $i = 1, \ldots, N$,
$$\|\mathbf{e}\|_{\text{rms}} < \gamma. \tag{6}$$

The main result in this section is presented in the following theorem:

Theorem 1: Consider a multi-agent system (1), (2), and reference system (3), (4). Let a root set π, $\alpha, \beta, \varphi, \tau > 0$ and positive integer N be given, and hence a set of network graphs $\tilde{\mathbb{G}}^{\varphi,\tau,N}_{\alpha,\beta,\pi}$ be defined.

Under Assumptions 1 and 2, the almost regulated output synchronization problem is solvable, i.e., for any given $\gamma > 0$, and for any disturbance bound κ, there exists a family of distributed dynamic protocols, parametrized in terms of low-and-high gain parameters δ, ε, of the form:

$$\begin{cases} \dot{\chi}_i = \mathcal{A}_i(\delta,\varepsilon)\chi_i + \mathcal{B}_i(\delta,\varepsilon)\begin{pmatrix}\zeta_i \\ \psi_i\end{pmatrix} \\ \tilde{u}_i = \mathcal{C}_i(\delta,\varepsilon)\chi_i + \mathcal{D}_i(\delta,\varepsilon)\begin{pmatrix}\zeta_i \\ \psi_i\end{pmatrix} \end{cases}, \qquad i \in \mathcal{V} \tag{7}$$

where $\chi_i \in \mathbb{R}^{q_i}$, such that for any time-varying graph $\mathcal{G} \in \tilde{\mathbb{G}}^{\varphi,\tau,N}_{\alpha,\beta,\pi}$, for all initial conditions, the almost regulated output synchronization error satisfies (5) and (6).

In particular, there exits a $\delta^* \in (0,1]$ such that, for each $\delta \in (0,\delta^*]$, there exists an $\varepsilon^* \in (0,1]$ such that for any $\varepsilon \in (0,\varepsilon^*]$, the protocol (7) solves the almost regulated output synchronization problem.

The proof will be presented in a constructive way in the following subsection.

A. The proof of Theorem 1

In this section, we will present the constructive proof in three steps.

Step 1: In this step, we augment agent (1) with a pre-compensator in such a way that the interconnection of agent (1) and the pre-compensator is square, of uniform rank and contains the reference system (3).

With exactly the same method presented in the appendix of [28], we can find a pre-compensator

$$\begin{cases} \dot{z}_i = A_{ip} z_i + B_{ip} u_i, \\ \bar{u}_i = C_{ip} z_i, \end{cases} \tag{8}$$

for $i \in \mathcal{V}$, such that the interconnection of agent (1) and pre-compensator (8) can be represented in this form:

$$\begin{cases} \dot{x}_i = A_i x_i + B_i u_i + G_i w_i, \\ y_i = C_i x_i, \end{cases} \tag{9}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^p$, $y_i \in \mathbb{R}^p$ are states, inputs, and outputs of the interconnection system. Moreover, we have (A_i, C_i) contains (S, R), and (A_i, B_i, C_i) is of uniform rank $\rho \geq 1$.

As shown in [28], the interconnection system (9) has another representation form that is called the *special coordinate basis* (SCB) form

$$\begin{cases} \dot{x}_{ia} = A_{ia} x_{ia} + L_{iad} y_i + G_{ia} w_i, \\ \dot{x}_{id} = A_d x_{id} + B_d (u_i + E_{ida} x_{ia} + E_{idd} x_{id}) + G_{id} w_i, \\ y_i = C_d x_{id}, \end{cases} \tag{10}$$

for $i \in \mathcal{V}$, where $x_{ia} \in \mathbb{R}^{n_i - p\rho}$ represents the finite zero structure, $x_{id} \in \mathbb{R}^{p\rho}$ represents the infinite zero structure, $u_i, y_i \in \mathbb{R}^p$, $w_i \in \mathbb{R}^{m_{w_i}}$, and

$$A_d = \begin{pmatrix} 0 & I_{p(\rho-1)} \\ 0 & 0 \end{pmatrix}, \qquad B_d = \begin{pmatrix} 0 \\ I_p \end{pmatrix}, \qquad C_d = \begin{pmatrix} I_p & 0 \end{pmatrix}.$$

Step 2: For each interconnection system (10), we will design a purely decentralized controller based on a low-and-high gain method. Let $\delta \in (0,1]$ be the low-gain parameter and $\varepsilon \in (0,1]$ be the high-gain parameter. First select K such that $A_d - KC_d$ is Hurwitz stable. Next, choose $F_\delta = -B_d' P_d$ where $P_d > 0$ is uniquely determined by the following algebraic Riccati equation:

$$P_d A_d + A_d' P_d - \beta P_d B_d B_d' P_d + \delta I = 0, \tag{11}$$

where $\beta > 0$ is the given lower bound on the real parts of the non-zero eigenvalues of all the expanded Laplacian matrices $\bar{L}$. Next, define $S_\varepsilon = \text{blkdiag}\{I_p, \varepsilon I_p, \ldots, \varepsilon^{\rho-1} I_p\}$,, $K_\varepsilon = \varepsilon^{-1} S_\varepsilon^{-1} K$ and $F_{\delta\varepsilon} = \varepsilon^{-\rho} F_\delta S_\varepsilon$.

Then, we define the dynamic controller for each agent $i \in \mathcal{V}$:

$$\begin{aligned} \dot{\hat{x}}_{id} &= A_d \hat{x}_{id} + K_\varepsilon (\zeta_i + \psi_i - C_d \hat{x}_{id}), \\ u_i &= F_{\delta\varepsilon} \hat{x}_{id}, \end{aligned} \tag{12}$$

where ψ_i is defined in (4).

The state $\hat{x}_{id}$ is an estimator for a linear combination of the relative states of agent i to other agents' with the same weights as in the measurement $\zeta_i + \psi_i$. The following lemma then provides a constructive proof of Theorem 1:

Lemma 1: For any given $\gamma > 0$, there exists a $\delta^* \in (0,1]$ such that, for each $\delta \in (0, \delta^*]$, there exists an $\varepsilon^* \in (0,1]$ such that for any $\varepsilon \in (0, \varepsilon^*]$, the dynamic protocol (12) achieves (5) and (6) for any time-varying graph $\mathcal{G} \in \tilde{\mathbb{G}}_{\alpha,\beta,\pi}^{\varphi,\tau,N}$, for any initial conditions, and for any disturbance bound κ.

Proof: Recall that $x_i = [x_{ia}; x_{id}]$ and that (9) is a shorthand notation for (10). For each $i \in \mathcal{V}$, let $\bar{x}_i = x_i - \Pi_i x_r$, where Π_i is defined to satisfy $\Pi_i S = A_i \Pi_i$, $C_i \Pi_i = R$. Then

$$\dot{\bar{x}}_i = A_i x_i - \Pi_i S x_r + B_i u_i + G_i w_i = A_i \bar{x}_i + B_i u_i + G_i w_i$$

and

$$e_i = y_i - y_r = C_i x_i - R x_r = C_i x_i - C_i \Pi_i x_r = C_i \bar{x}_i.$$

Since the dynamics of the $\bar{x}_i$ system with output e_i is governed by the same dynamics as the dynamics of agent i, we can present $\bar{x}_i$ in the same form as (10), with $\bar{x}_i = [\bar{x}_{ia}; \bar{x}_{id}]$, where

$$\begin{aligned} \dot{\bar{x}}_{ia} &= A_{ia} \bar{x}_{ia} + L_{iad} e_i + G_{ia} w_i, \\ \dot{\bar{x}}_{id} &= A_d \bar{x}_{id} + B_d (u_i + E_{ida} \bar{x}_{ia} + E_{idd} \bar{x}_{id}) + G_{id} w_i, \\ e_i &= C_d \bar{x}_{id}. \end{aligned}$$

Define $\xi_{ia} = \bar{x}_{ia}$, $\xi_{id} = S_\varepsilon \bar{x}_{id}$ and $\hat{\xi}_{id} = S_\varepsilon \hat{x}_{id}$. Then

$$\begin{aligned} \dot{\xi}_{ia} &= A_{ia} \xi_{ia} + V_{iad} \xi_{id} + G_{ia} w_i, \\ \varepsilon \dot{\xi}_{id} &= A_d \xi_{id} + B_d F_\delta \hat{\xi}_{id} + V_{ida}^\varepsilon \xi_{ia} + V_{idd}^\varepsilon \xi_{id} + \varepsilon G_{id}^\varepsilon w_i, \\ e_i &= C_d \xi_{id}, \end{aligned}$$

where $V_{iad} = L_{iad} C_d$, $V_{ida}^\varepsilon = \varepsilon^\rho B_d E_{ida}$, $V_{idd}^\varepsilon = \varepsilon^\rho B_d E_{idd} S_\varepsilon^{-1}$ and $G_{id}^\varepsilon = S_\varepsilon G_{id}$.

Similarly, the controller (12) can be rewritten as

$$\varepsilon \dot{\hat{\xi}}_{id} = A_d \hat{\xi}_{id} + K \sum_{j=1}^{N} \bar{\ell}_{ij}(t) C_d \xi_{jd} - K C_d \hat{\xi}_{id},$$

for $\zeta_i + \psi_i = \sum_{j=1}^N \ell_{ij}(t)(y_i - y_j) + \iota_i(y_i - y_r) = \sum_{j=1}^N \bar{\ell}_{ij}(t) e_j$. Let $\xi_a = \text{col}\{\xi_{ia}\}, \xi_d = \text{col}\{\xi_{id}\}, \hat{\xi}_d = \text{col}\{\hat{\xi}_{id}\}, w = \text{col}\{w_i\}$. Then we have,

$$\begin{aligned} \dot{\xi}_a &= A_a \xi_a + V_{ad} \xi_d + G_a w, \\ \varepsilon \dot{\xi}_d &= (I_N \otimes A_d) \xi_d + (I_N \otimes B_d F_\delta) \hat{\xi}_d \\ &\qquad + V_{da}^\varepsilon \xi_a + V_{dd}^\varepsilon \xi_d + \varepsilon G_d^\varepsilon w, \\ \varepsilon \dot{\hat{\xi}}_d &= (I_N \otimes A_d - K C_d) \hat{\xi}_d + (\bar{L}(t) \otimes K C_d) \xi_d, \end{aligned}$$

where $A_a = \text{blkdiag}\{A_{ia}\}$, and V_{ad}, V_{da}^ε, V_{dd}^ε, G_a, G_d^ε are similarly defined.

Define $U_t^{-1} \bar{L}(t) U_t = J_t$, where J_t is the Jordan form of $\bar{L}(t)$, and let $v_a = \xi_a$, $v_d = (J_t U_t^{-1} \otimes I_{p\rho}) \xi_d$, $\tilde{v}_d = v_d - (U_t^{-1} \otimes I_{p\rho}) \hat{\xi}_d$. Then,

$$\begin{aligned} \dot{v}_a &= A_a v_a + W_{ad,t} v_d + G_a w, \\ \varepsilon \dot{v}_d &= (I_N \otimes A_d) v_d + (J_t \otimes B_d F_\delta)(v_d - \tilde{v}_d) \\ &\qquad + W_{da,t}^\varepsilon v_a + W_{dd,t}^\varepsilon v_d + \varepsilon \bar{G}_{d,t}^\varepsilon w, \\ \varepsilon \dot{\tilde{v}}_d &= (I_N \otimes (A_d - K C_d)) \tilde{v}_d + (J_t \otimes B_d F_\delta)(v_d - \tilde{v}_d) \\ &\qquad + W_{da,t}^\varepsilon v_a + W_{dd,t}^\varepsilon v_d + \varepsilon \bar{G}_{d,t}^\varepsilon w, \end{aligned} \tag{13}$$

where $W_{ad,t} = V_{ad}(U_t J_t^{-1} \otimes I_{p\rho})$, $W_{da,t}^\varepsilon = (J_t U_t^{-1} \otimes I_{p\rho}) V_{da}^\varepsilon$, $W_{dd,t}^\varepsilon = (J_t U_t^{-1} \otimes I_{p\rho}) V_{dd}^\varepsilon (U_t J_t^{-1} \otimes I_{p\rho})$, and $\bar{G}_{d,t}^\varepsilon = (J_t U_t^{-1} \otimes I_{p\rho}) G_d^\varepsilon$. Note that v_d and $\tilde{v}_d$ exhibit discontinuous jumps when the network changes.

Finally, let $\eta_a = v_a$, and define N_d such that

$$\eta_d \triangleq N_d \begin{pmatrix} v_d \\ \tilde{v}_d \end{pmatrix} = \begin{pmatrix} v_{1d} \\ \tilde{v}_{1d} \\ \vdots \\ v_{(N-1)d} \\ \tilde{v}_{(N-1)d} \end{pmatrix} \text{ where } N_d = \begin{pmatrix} e_1 & 0 \\ 0 & e_1 \\ \vdots & \vdots \\ e_{N-1} & 0 \\ 0 & e_{N-1} \end{pmatrix} \otimes I_{p\rho},$$

where $e_i \in \mathbb{R}^{N-1}$ is the i'th standard basis vector whose elements are all zero except for the i'th element which is equal to 1. Then (13) can be written as:

$$\begin{aligned}\dot{\eta}_a &= A_a \eta_a + \tilde{W}_{ad,t}\eta_d + G_a w,\\ \varepsilon\dot{\eta}_d &= \tilde{A}_{\delta,t}\eta_d + \tilde{W}^{\varepsilon}_{da,t}\eta_a + \tilde{W}^{\varepsilon}_{dd,t}\eta_d + \varepsilon\tilde{G}^{\varepsilon}_{d,t}w,\end{aligned} \quad (14)$$

where

$$\tilde{A}_{\delta,t} = I_N \otimes \begin{pmatrix} A_d & 0 \\ 0 & A_d - KC_d \end{pmatrix} + J_t \otimes \begin{pmatrix} B_d F_\delta & -B_d F_\delta \\ B_d F_\delta & -B_d F_\delta \end{pmatrix}, \quad (15)$$

and

$$\tilde{W}_{ad,t} = \begin{pmatrix} W_{ad,t} & 0 \end{pmatrix} N_d^{-1}, \qquad \tilde{G}^{\varepsilon}_{d,t} = N_d \begin{pmatrix} \bar{G}^{\varepsilon}_{d,t} \\ \bar{G}^{\varepsilon}_{d,t} \end{pmatrix},$$

$$\tilde{W}^{\varepsilon}_{da,t} = N_d \begin{pmatrix} W^{\varepsilon}_{da,t} \\ W^{\varepsilon}_{da,t} \end{pmatrix}, \qquad \tilde{W}^{\varepsilon}_{dd,t} = N_d \begin{pmatrix} W^{\varepsilon}_{dd,t} & 0 \\ W^{\varepsilon}_{dd,t} & 0 \end{pmatrix} N_d^{-1}.$$

Lemma 2: Consider the matrix $\tilde{A}_{\delta,t}$ defined in (15). According to [3], for any δ small enough the matrix $\tilde{A}_{\delta,t}$ is asymptotically stable for any Jordan matrix J_t whose eigenvalues satisfy $\mathrm{Re}\{\lambda_i\} > \beta$ and $|\lambda_i| < \alpha$ for any time t. Moreover, there exists $P'_\delta = P_\delta > 0$ and $\nu > 0$ such that

$$\tilde{A}_{\delta,t}P_\delta + P_\delta \tilde{A}'_{\delta,t} \le -\nu P_\delta - 4I \quad (16)$$

is satisfied for all possible Jordan matrices J_t and such that there exists $P_a > 0$ for which

$$P_a A_a + A'_a P_a = -\nu P_a - I. \quad (17)$$

Define $V_a = \varepsilon^2 \eta'_a P_a \eta_a$ as a Lyapunov function for the dynamics of η_a in (14). Similarly, we define $V_d = \varepsilon \eta'_d P_\delta \eta_d$ as a Lyapunov function for the dynamics of η_d in (14). It is easy to find that V_d also has discontinuous jumps when the network changes. The derivative of V_a is bounded by:

$$\begin{aligned}\dot{V}_a &= -\nu V_a - \varepsilon^2\|\eta_a\|^2 + 2\varepsilon^2\,\mathrm{Re}(\eta'_a P_a \tilde{W}_{ad,t}\eta_d)\\ &\quad + 2\varepsilon^2\,\mathrm{Re}(\eta'_a P_a G_a w)\\ &\le -\nu V_a \mathrm{d}t + \varepsilon c_3 V_d + 2\varepsilon^2 r_5^2\|w\|^2,\end{aligned} \quad (18)$$

where r_5 and c_3 are such that:

$$\begin{aligned}2\,\mathrm{Re}(\eta'_a P_a \tilde{W}_{ad,t}\eta_d) &\le 2r_4\|\eta_a\|\|\eta_d\|\\ &\le \tfrac{1}{2}\|\eta_a\|^2 + 2r_4^2\|\eta_d\|^2 \le \tfrac{1}{2}\|\eta_a\|^2 + \varepsilon^{-1}c_3 V_d,\end{aligned}$$

$$2\,\mathrm{Re}(\eta'_a P_a G_a w) \le 2r_5\|\eta_a\|\|w\| \le \tfrac{1}{2}\|\eta_a\|^2 + 2r_5^2\|w\|^2.$$

Note that we can choose r_4, r_5 and c_3 independent of the network graph but only depending on our bounds on the eigenvalues and condition number of our expand Laplacian $\tilde{L}(t)$.

Next, the derivative of V_d is bounded by

$$\begin{aligned}\dot{V}_d &= -\nu\varepsilon^{-1}V_d - 4\|\eta_d\|^2 + 2\,\mathrm{Re}(\eta'_d P_\delta \tilde{W}^{\varepsilon}_{da,t}\eta_a)\\ &\quad + 2\,\mathrm{Re}(\eta'_d P_\delta \tilde{W}^{\varepsilon}_{dd,t}\eta_d) + 2\varepsilon\,\mathrm{Re}(\eta'_d P_\delta \tilde{G}^{\varepsilon}_{d,t}w)\\ &\le c_2 V_a - (\nu\varepsilon^{-1} + \nu - \varepsilon^2\frac{c_2c_3}{\nu})V_d + \varepsilon^2 r_3^2\|w\|^2,\end{aligned} \quad (19)$$

where $2\,\mathrm{Re}(\eta'_d P_\delta \tilde{W}^{\varepsilon}_{dd,t}\eta_d) \le \|\eta_d\|^2$ for small ε, and

$$\begin{aligned}2\varepsilon\,\mathrm{Re}(\eta'_d P_\delta \tilde{G}^{\varepsilon}_{d,t}w) &\le 2\varepsilon r_3\|\eta_d\|\|w\| \le \|\eta_d\|^2 + \varepsilon^2 r_3^2\|w\|^2,\\ 2\,\mathrm{Re}(\eta'_d P_\delta \tilde{W}^{\varepsilon}_{da,t}\eta_a) &\le 2\varepsilon r_1\|\eta_a\|\|\eta_d\| \le \varepsilon^2 r_1^2\|\eta_a\|^2 + \|\eta_d\|^2\\ &\le c_2 V_a + \|\eta_d\|^2,\end{aligned}$$

provided r_3, r_1 is such that we have $\varepsilon r_3 \ge \varepsilon\|P_\delta \tilde{G}^{\varepsilon}_{d,t}\|$, $\varepsilon r_1 \ge \|P_\delta \tilde{W}^{\varepsilon}_{da,t}\|$, and c_2 sufficiently large. Then, we get:

$$\begin{pmatrix}\dot{V}_a \\ \dot{V}_d\end{pmatrix} \le A_e \begin{pmatrix} V_a \\ V_d \end{pmatrix} + \begin{pmatrix} 2\varepsilon^2 r_5^2\|w\|^2 \\ \varepsilon^2 r_3^2\|w\|^2 \end{pmatrix},$$

where

$$A_e = \begin{pmatrix} -\nu & \varepsilon c_3 \\ c_2 & -\varepsilon^{-1}\nu - \nu + \varepsilon^2\frac{c_2c_3}{\nu} \end{pmatrix}.$$

Note that the inequality here is componentwise. We find by integration that:

$$\begin{aligned}&\begin{pmatrix} V_a \\ V_d \end{pmatrix}(t_k^-) \le\\ &e^{A_e(t_k - t_{k-1})}\begin{pmatrix} V_a \\ V_d \end{pmatrix}(t_{k-1}^+) + \int_{t_{k-1}}^{t_k} e^{A_e(t_k - s)}\begin{pmatrix} 2\varepsilon^2 r_5^2\|w\|^2 \\ \varepsilon^2 r_3^2\|w\|^2 \end{pmatrix}\mathrm{d}s\end{aligned} \quad (20)$$

componentwise. We have a potential jump at time t_{k-1} in V_d. However, there exists m such that $V_d(t_{k-1}^+) \le mV_d(t_{k-1}^-)$, while V_a is continuous. Using the explicit expression for $e^{A_e t}$ and the fact that $t_k - t_{k-1} > \tau$ we find:

$$\begin{aligned}&\begin{pmatrix} 1 & 1 \end{pmatrix} e^{A_e(t_k - t_{k-1})}\begin{pmatrix} V_a \\ V_d \end{pmatrix}(t_{k-1}^+) \le\\ &\qquad e^{\lambda_3(t_k - t_{k-1})}\left[V_a(t_{k-1}^-) + V_d(t_{k-1}^-)\right],\end{aligned}$$

where $\lambda_3 = -\nu/2$.

When $w_i \in \Gamma^{\infty}_{\kappa}$, it can be easily verified that:

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\int_{t_{k-1}}^{t_k} e^{A_e(t_k - s)}\begin{pmatrix} 2\varepsilon^2 r_5\|w\|^2 \\ \varepsilon^2 r_3^2\|w\|^2 \end{pmatrix}\mathrm{d}s \le r\varepsilon^2\|w\|^2_\infty,$$

where r is a sufficiently large constant. We find

$$\begin{aligned}&\left[V_a(t_k^-) + V_d(t_k^-)\right] \le\\ &\qquad e^{\lambda_3(t_k - t_{k-1})}\left[V_a(t_{k-1}^-) + V_d(t_{k-1}^-)\right] + r\varepsilon^2\|w\|^2_\infty.\end{aligned} \quad (21)$$

Combining these time-intervals, we get:

$$\left[V_a(t_k^-) + V_d(t_k^-)\right] \le e^{\lambda_3 t_k}\left[V_a(0) + V_d(0)\right] + \frac{r\varepsilon^2}{1-\mu}\|w\|^2_\infty,$$

where $\mu < 1$ is such that $e^{\lambda_3(t_k - t_{k-1})} \le e^{\lambda_3\tau} \le \mu$ for all k. Assume $t_{k+1} > t > t_k$. Since we do not necessarily have that $t - t_k > \tau$ we use the bound:

$$\begin{pmatrix} 1 & 1 \end{pmatrix} e^{A_e(t - t_k)}\begin{pmatrix} V_a \\ V_d \end{pmatrix}(t_k^+) \le 2me^{\lambda_3(t - t_k)}(V_a + V_d)(t_k^-),$$

where the factor m is due to the potential discontinuous jump. Combining all together, we get:

$$V(t) \le 2me^{\lambda_3 t}V(0) + (2m+1)\frac{r\varepsilon^2}{1-\mu}\|w\|^2_\infty,$$

where $V = V_a + V_d$. Hence,

$$\limsup_{t\to\infty}\|\eta_d(t)\|^2 \le (2m+1)\frac{r\varepsilon^2}{(1-\mu)\sigma_{\min}(P_\delta)}\kappa. \quad (22)$$

On the other hand, for $w \in \Gamma_\kappa^{\rm rms}$ we note that (20) implies:

$$\int_{t_{k-1}}^{t_k} V_d(s)\mathrm{d}s \le r_6\varepsilon(V_a(t_{k-1}^+)+V_d(t_{k-1}^+))+r_7\varepsilon^2\int_{t_{k-1}}^{t_k}\|w(\tau)\|^2\mathrm{d}\tau \tag{23}$$

for some large enough r_6, r_7.

Similar to (21), we get

$$\begin{aligned}&\left[V_a(t_k^-)+V_d(t_k^-)\right] \\ &\le e^{\lambda_3\nu/2}\left[V_a(t_{k-1}^-)+V_d(t_{k-1}^-)\right]+r\varepsilon^2\int_{t_{k-1}}^{t_k}\|w(\tau)\|^2\mathrm{d}\tau.\end{aligned} \tag{24}$$

We have

$$\frac{1}{T}\int_0^T \eta_d'(t)\eta_d(t)\,\mathrm{d}t \le \frac{1}{\varepsilon\sigma_{\min}(P_\delta)}\frac{1}{T}\int_0^T V_d(t)\,\mathrm{d}t. \tag{25}$$

Combine (23), (24), and (25), and taking the limit as $T\to\infty$, we find:

$$\|\eta_d\|_{\rm rms} \le \varepsilon r_8\|w\|_{\rm rms}. \tag{26}$$

Following the proof above, we find that

$$\begin{aligned}\mathbf{e} &= (I_N\otimes C_d)(I_N\otimes S_\varepsilon^{-1})(U_tJ_t^{-1}\otimes I_{p\rho})\begin{pmatrix}I_{Np\rho} & 0\end{pmatrix}N_d^{-1}\eta_d \\ &= \Theta_t\eta_d,\end{aligned}$$

for suitably chosen matrix Θ_t. Although Θ_t is time-varying it is uniformly bounded, because for graphs in $\mathbb{G}_{\alpha,\beta,\pi}^{\varphi,N}$ the matrices U_t and J_t are bounded. Therefore, we have

$$\|\mathbf{e}(t)\| = \|\Theta_t\eta_d(t)\| \le \|\Theta_t\|\|\eta_d(t)\|.$$

Using this and (22) we can conclude that for $w\in\Gamma_\kappa^\infty$ we have (5) for any fixed $\gamma>0$ provided we choose ε small enough. Similarly, using this and (26) we can conclude that for $w\in\Gamma_\kappa^{\rm rms}$ we have (6) for any fixed $\gamma>0$ provided we choose ε small enough. ■

Step 3: Combining the pre-compensator (8) in Step 1 and the controller (12) in Step 2, we obtain the protocol in the form of (7) in Theorem 1 as:

$$\begin{aligned}\mathcal{A}_i &= \begin{pmatrix}A_d-K_\varepsilon C_d & 0\\ B_{ip}F_{\delta\varepsilon} & A_{ip}\end{pmatrix}, & \mathcal{B}_i &= \begin{pmatrix}K_\varepsilon & K_\varepsilon\\ 0 & 0\end{pmatrix},\\ C_i &= \begin{pmatrix}0 & C_{ip}\end{pmatrix}, & \mathcal{D}_i &= \begin{pmatrix}0\end{pmatrix}.\end{aligned} \tag{27}$$

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