YET ANOTHER DISCRETE-TIME $\mathcal{H}^\infty$ FIXED-LAG SMOOTHING SOLUTION

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Abstract. A solution of the discrete $\mathcal{H}^\infty$ smoothing problem is presented that lends itself well for generalization to sampled-data problems. The solution is complete and requires two sign-definite spectral factorizations and one Nehari extension problem. A state-space equivalent is derived that is believed to be minimal.

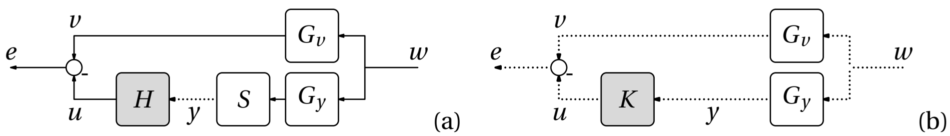
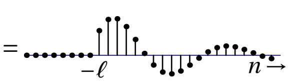


FIG. 1.1. *(a) Hold design. (b) Discrete filter design*

1. Introduction. The $\mathcal{H}^\infty$ problem hardly needs an introduction, and one might argue that it definitely needs no other solution. However, for certain sampled data problems, matters are still not settled. One case (and the motivation for this work) is shown in Fig. 1.1(a). Here the game is to design a stable hold H that renders the $\mathcal{L}^\infty$-norm of the error system $G_v - HSG_y$ less than some given bound. For causal holds this problem is elegantly solved in [10] using the machinery of *systems with jumps* but if the hold H is allowed a given amount of preview then [8] is the first solution. The sampled-data solution put forward in [8] is, largely, a careful translation to sampled-data systems of a pure discrete $\mathcal{H}^\infty$ fixed-lag smoothing solution. It is this discrete solution that we present in this note.

The discrete fixed-lag smoothing problem that we consider is depicted in Fig. 1.1(b). Here G_v and G_y are given discrete and causal LTI systems and the problem is to find a filter K that is stable and causal up to some given degree of preview ℓ and that renders the $\mathcal{L}^\infty$-norm of the error system $G_e := G_v - KG_y$ smaller than a given bound γ. A preview of ℓ means that the impulse response $k[n]$ of K is zero for discrete time less than $-\ell$, say

$k[n] =$  (1.1)

It is well known that a smoothing problem can be cast as a standard filtering, $\ell = 0$, problem by incorporating the delay $z^{-\ell}$ in the v and u channel (see Fig. 1.2). This approach, however, increases the problem dimension and might blur properties of the resulting solution. In the $\mathcal{H}^2$ (Kalman smoothing) case, the structure of the filtering solution can be exploited to derive a solution that is based on a fixed-size (independent of ℓ) Riccati equation and whose computational burden is $\mathcal{O}(\ell)$, see [1, Sec. 7.3]. A similar approach does not work so smoothly in the $\mathcal{H}^\infty$ optimization because the corresponding Riccati equation in this case is more involved. Indeed the solutions put forward

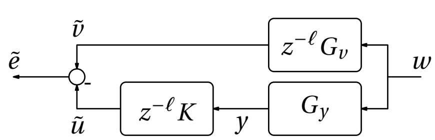


FIG. 1.2. *standard filtering*

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in [3, 7, 13] along this route provide sufficient but not necessary solvability conditions, see [12, §III-B] for a discussion about their limitations. Two solutions do exist that offer *necessary and sufficient* solvability conditions yet do not suffer from state inflation: [4] and [12]. The former, however, requires restrictive assumptions that prevent extension to sampled-data problems and the latter is not readily extendible either because of the complexity of its final formulae. In this paper we present a solution that overcomes these obstacles. It is a solution that

- does not suffer from state inflation and that works well as $\ell \to \infty$;
- can be generalized to sampled data systems, see Fig. 1.1(a), where the to-be-designed system is a hold, see [8];
- can handle unstable signal generators G_v, G_y. (This, actually, does not complicate matters; also the standard $\mathscr{H}^\infty$ filtering problem has been generalized to deal with unstable G_v, G_y.)

And while doing that, we tried to connect the solution with known, related, problems like classic Kalman filtering and fixed-interval ($\ell = \infty$) solutions. In addition, we have tried to keep the dependency on the smoothing lag ℓ and performance level γ simple, so as to be able to analyze their effect on the performance and solution K. First we present the frequency domain solution and then translate the steps in state space. This translation is not entirely straightforward as the simple dependencies on smoothing lag ℓ and performance level γ appear to come at the cost of state inflation. However, on second thought these inflations can be circumvented, leaving a solution whose state space representation we believe to be minimal.

Notation.. All signals are discrete and we denote them with lower case symbols such as $y : \mathbb{Z} \to \mathbb{C}^m$, and then $y[n]$ denotes its value at time n. Systems and transfer functions (or matrices) are denoted by capital letters. For any system P, the lower case p denotes its impulse response. We assume throughout that systems are LTI, i.e. that $y = P(u)$ is a convolution $y = p * u$. The conjugate $P^\sim$ of a system P is defined as having impulse response $p^\sim[n] := p'[-n]$ or, in terms of its z-transform, by $P^\sim(z) = [P(1/\overline{z})]'$. The $'$ here means complex conjugate transpose. The $\mathscr{L}^2$ system norm $\|\cdot\|_2$ can be defined by either impulse response or transfer matrix as

$$\|P\|_2^2 = \operatorname{tr}\sum_{n\in\mathbb{Z}} p[n]p'[n] = \operatorname{tr}\sum_{k\in\mathbb{Z}} (p * p^\sim)[0] = \frac{1}{2\pi}\operatorname{tr}\int_{-\pi}^{\pi} P(e^{i\theta})P^\sim(e^{i\theta})\,d\theta$$

and its $\mathscr{L}^\infty$-norm is

$$\|P\|_\infty = \sup_{u\in l^2}\frac{\|Pu\|_2}{\|u\|_2} = \sup_{\theta\in[-\pi,\pi]} \sigma_{\max} P(e^{i\theta}).$$

We say that a system is *stable* if its $\mathscr{L}^\infty$-norm is finite. This allows noncausal systems. A system whose impulse response is zero for time less than $-\ell$ is called *ℓ-causal.* Then 0-causal is the same causal and 1-causal means a preview of one sample. The *ℓ-causal part* $\{P\}_{[-\ell,\infty)}$ of a system P is obtained by truncating its impulse response to $[-\ell,\infty)$. For example, the impulse response of Eqn. (1.1) could be an ℓ-causal part. Likewise $\{P\}_{[a,b]}$ denotes system whose impulse response is truncated to $[a,b]$. The set of stable LTI ℓ-causal systems is denoted as

$$z^\ell \mathscr{H}^\infty.$$

For $\ell = 0$ this is just $\mathscr{H}^\infty$ (the set of stable, causal LTI systems) and for $\ell = \infty$ we take this to mean $\mathscr{L}^\infty$.

2. Problem formulation and frequency domain solution. Fix a smoothing lag $\ell \geq 0$, possibly $\ell = \infty$. We are given two discrete LTI systems G_v and G_y, both causal but possibly not stable, and we are after a stable and ℓ-causal filter K that renders the error mapping

$$G_e := G_v - KG_y$$

in $z^\ell \mathcal{H}^\infty$ and makes it γ-contractive, that is,

$$\|G_e\|_\infty < \gamma.$$

The so defined error system G_e is the mapping from w to e in Fig. 1.1(b) and the idea is that the smaller γ is the better solutions K reconstruct the signal v from y. For $\ell = 0$ this is the standard $\mathcal{H}^\infty$ *filtering* problem, and for $\ell > 0$ this is *fixed-lag* smoothing. The case $\ell = \infty$ is sometimes called *fixed interval* smoothing.

In smoothing, $\ell > 0$, we have more design freedom than in filtering, $\ell = 0$, and therefore not all the solvability conditions for the filtering problem will be met in the smoothing problem. In particular the classic $\mathcal{H}^\infty$ filtering Riccati equation need not have a solution for the level of γ one can achieve with smoothing. It might therefore be beneficial to connect the smoothing problem with the fixed interval problem, $\ell = \infty$, which clearly is solvable for level γ if so is the fixed-lag smoothing problem.

Now for the smoothing problem to have a solution at all, we need that the error system G_e can be stabilized. Loosely speaking this means that any unstable pole in G_v must be present in G_y as well because otherwise this pole will reappear in the error system $G_e = G_v - KG_y$. Stability is equivalent to a special coprime factorization of the joint system $\left[\begin{smallmatrix} G_v \\ G_y \end{smallmatrix}\right]$.

LEMMA 2.1 (Stabilization & normalization). *Fix $0 \leq \ell < \infty$ and suppose that G_v, G_y are rational. Then a $K \in z^\ell \mathcal{H}^\infty$ exists that renders $G_e \in z^\ell \mathcal{H}^\infty$ if and only if $\left[\begin{smallmatrix} G_v \\ G_y \end{smallmatrix}\right]$ has a coprime factorization over $\mathcal{H}^\infty$ of the form*

$$\begin{bmatrix} G_v \\ G_y \end{bmatrix} = \begin{bmatrix} I & M_v \\ 0 & M_y \end{bmatrix}^{-1} \begin{bmatrix} N_v \\ N_y \end{bmatrix}. \tag{2.1}$$

In that case all $K \in z^\ell \mathcal{H}^\infty$ that achieve $G_e \in \mathcal{L}^\infty$ are parameterized by

$$K = M_v - QM_y, \qquad Q \in z^\ell \mathcal{H}^\infty, \tag{2.2}$$

and for this parameterization the error system becomes

$$G_e = N_v - QN_y. \tag{2.3}$$

Moreover, if G_y has full row rank on the unit circle, then there exist such factorizations that are normalized in the sense that

$$V := N_v N_y^\sim \text{ is strictly anticausal and } N_y N_y^\sim = I. \tag{2.4}$$

Proof. Essentially from [9]. □

Notice that the above stabilizability conditions are independent of ℓ. Owing to the normalization (2.4) we have

$$\begin{aligned} G_e G_e^\sim = (N_v - QN_y)(N_v - QN_y)^\sim &= N_v N_v^\sim - QV^\sim - VQ^\sim + QQ^\sim \\ &= N_v N_v^\sim + (Q - V)(Q - V)^\sim - VV^\sim \end{aligned} \tag{2.5}$$

so G_e is strictly γ-contractive (i.e. $\|G_e\|_\infty < \gamma$) iff

$$(Q-V)(Q-V)^\sim < \gamma^2 I - (N_v N_v^\sim - VV^\sim).$$

Since the left-hand side is nonnegative, this suggests to factorize the right-hand side

$$W_\gamma W_\gamma^\sim = \gamma^2 I - (N_v N_v^\sim - VV^\sim) \tag{2.6}$$

with W_γ bistable. For $\gamma > \gamma_\ell := \inf_{K\in z^\ell \mathscr{H}^\infty} \|G_e\|_\infty$ this spectral factorization exists. Now $\|G_e\|_\infty < \gamma$ holds iff

$$\|W_\gamma^{-1}(Q-V)\|_\infty < 1. \tag{2.7}$$

The search for $Q \in z^\ell \mathscr{H}^\infty$ that satisfy (2.7) is a classic Nehari extension problem with "symbol" $W_\gamma^{-1}V$ and it is well known that a solution $Q \in z^\ell \mathscr{H}^\infty$ exists iff the Hankel operator $H_{W_\gamma^{-1}V}$ is a contraction

$$\|H_{W_\gamma^{-1}V}\| < 1.$$

(This Hankel operator here maps from $z^\ell \mathscr{H}^2$ to $(z^\ell \mathscr{H}^2)^\perp$.) That the Hankel operator $H_{W_\gamma^{-1}V}$ is a contraction may be hard to verify in the general case but for state space systems this is well known and explicit solutions Q_H exist that achieve (2.7) [5, Theorem XXXV.7.2]. Given such a $Q_\mathrm{H} \in z^\ell \mathscr{H}^\infty$ the filter (2.2) becomes $K = M_v - Q_\mathrm{H} M_y$ and it is a solution to our problem (i.e. it is in $z^\ell \mathscr{H}^\infty$ and achieves $\|G_e\|_\infty < \gamma$.)

A couple of observations can be made:

- For $Q = 0$ the filter (2.2) becomes M_v and it is the Kalman filter (i.e. the causal filter that minimizes the $\mathscr{L}^2$-norm of G_e). This can for instance be seen as follows: since V is strictly anticausal and Q causal, their impulse response have disjoint support. Therefore $(Q-V)(Q-V)^\sim \geq VV^\sim$ for every causal Q and equality is holds iff $Q = 0$. Hence $Q = 0$ minimizes (2.5) over all stable, causal Q and so also minimizes the $\mathscr{L}^2$-norm of G_e. For $Q = 0$ the error system (2.3) becomes $G_e = N_v$.
- By the same reasoning, the FIR system $Q_\ell := \{V\}_{[-\ell,-1]}$, defined by its impulse response q_ℓ as the truncation of V's impulse response to the interval $[-\ell,-1]$ is the one that minimizes the $\mathscr{L}^2$-norm $\|G_e\|_2$ over all ℓ-causal stable filters Q (and hence over all ℓ-causal stable filters K by Lemma 2.1) and then

  $$\|G_e\|_2^2 = \|N_v\|_2^2 - \|Q_\ell\|_2^2 = \|N_v\|_2^2 - \sum_{n=-\ell}^{-1} \|v[n]\|_{\mathrm{Frob}}^2.$$

  The term V thus tells us quite explicitly how much the $\mathscr{L}^2$-norm can be improved by increasing the smoothing lag ℓ.
- For $Q = V$ the filter (2.2) is the one that minimizes both the $\mathscr{L}^2$-norm and $\mathscr{L}^\infty$-norm of G_e over all stable filters filters (causal or noncausal). This is a consequence of the fact that $Q = V$ in (2.5) minimizes $G_e G_e^\sim$.
- As for the $\mathscr{L}^\infty$-norm, the dependence of γ only shows up in the second spectral factorization problem (2.6) (and, therefore, also in the ensuing Nehari extension problem (2.7).)
- The preview lag ℓ does not show up in either of the two spectral factorization problems (2.4), (2.6). It only plays a role in the, final, Nehari extension problem (2.7).

- The two spectral factorization problems needed—(2.4) and (2.6)—are classic (sign definite) factorization problems, meaning that they always exist if γ exceeds $\gamma_\ell := \min_{K\in z^\ell \mathscr{H}^\infty} \|G_e\|_\infty$. Notice that the classic $\mathscr{H}^\infty$ (indefinite) spectral factorization problems need not have a solution because with smoothing the optimal level γ_ℓ may be less than the $\mathscr{H}^\infty$-optimal γ_0.

In the next section we see that another benefit of this solution is that the state dimension of the optimal filter K is not higher than that of G save a FIR correction term. Explicitly, if n is the state dimension of G then $K = M_v - Q_{\mathrm{H}} M_y$ can be rearranged to a sum of an LTI system of state dimension n and a FIR system of order ℓ obtained by truncating another system of state dimension n. For $\ell = 0$ this FIR system is void, and for $\ell \to \infty$ it converges to an LTI system of dimension n.

3. State space solution. In principle the state space solution is easy because every step of the frequency domain solution has a state space counterpart. The nuisance comes from the technicalities, in particular that some realizations can be reduced in state dimension. In this section we document the state space solution. Detailed derivations are mostly absent due to space limitations. The final algorithmic result is:

THEOREM 3.1 (Sate space solution). *Let $\gamma > 0$ and let*

$$G(z) := \begin{bmatrix} G_v(z) \\ G_y(z) \end{bmatrix} = \left(\begin{array}{c|c} A - zI & B \\ \hline C_v & D_v \\ C_y & D_y \end{array}\right)$$

be a realization of G and assume that (A, C_y, B) is stabilizable and detectable and that $\left[\begin{smallmatrix} A-zI & B \\ C_y & D_y \end{smallmatrix}\right]$ has full row rank on the unit circle $z = \mathrm{e}^{\mathrm{i}\theta}$. Then the Kalman Filtering DARE

$$Y_\kappa = AY_\kappa A' + BB' \underbrace{-(AY_\kappa C_y' + BD_y')(D_y D_y' + C_y Y_\kappa C_y')^{-1}}_{=L_\kappa}(C_y Y_\kappa A' + D_y B') \tag{3.1}$$

has a solution Y_κ for which $\tilde{A} := A + L_\kappa C_y$ is Schur stable. In fact $Y_\kappa \geq 0$ and $D_y D_y' + C_y Y_\kappa C_y' > 0$ so the latter has an inverted square root, $\Xi := (D_y D_y' + C_y Y_\kappa C_y')^{-1/2}$. With it define $\Omega := -(D_v D_y' + C_v Y_\kappa C_y')(D_y D_y' + C_y Y_\kappa C_y')^{-1}$ and

$$\begin{bmatrix} \tilde{C}_v & \tilde{D}_v \\ \tilde{C}_y & \tilde{D}_y \end{bmatrix} := \begin{bmatrix} I & \Omega \\ 0 & \Xi \end{bmatrix} \begin{bmatrix} C_v & D_v \\ C_y & D_y \end{bmatrix}, \qquad \tilde{B} := B + L_\kappa D_y.$$

Then V defined in (2.4) has realization

$$V(z) = \left(\begin{array}{c|c} \tilde{A}' - \frac{1}{z} I & \tilde{C}_y' \\ \hline -\tilde{L}' & 0 \end{array}\right) \quad \text{in which} \quad \tilde{L} = -(\tilde{B}\tilde{D}_v' + \tilde{A} Y_\kappa \tilde{C}_v). \tag{3.2}$$

Let X be the observability gramian defined by $X = \tilde{A}' X \tilde{A} + \tilde{C}_y' \tilde{C}_y$. Now the fixed-interval ($\ell = \infty$) problem has a solution (i.e. there is a $K \in \mathscr{L}^\infty$ that renders $\|G_e\|_\infty < \gamma$) if and only if the DARE

$$Y_\gamma = \tilde{A} Y_\gamma \tilde{A}' + \underbrace{(\tilde{L} - \tilde{A} Y_\gamma (\tilde{C}_v' + \tilde{A}' X \tilde{L}))}_{\tilde{B}_\gamma} \tilde{R}_\gamma^{-1} (\tilde{L} - \tilde{A} Y_\gamma (\tilde{C}_v' + \tilde{A} X \tilde{L}))'$$

has solution Y_γ for which $A_\gamma := \tilde{A} - \tilde{B}_\gamma \tilde{R}_\gamma^{-1}(\tilde{C}_v + \tilde{L}' X \tilde{A})$ is Schur stable and $\tilde{R}_\gamma$ defined as $\tilde{R}_\gamma = \gamma^2 I - \tilde{D}_v \tilde{D}_v' - \tilde{C}_v Y_\kappa \tilde{C}_v' + \tilde{L}' X \tilde{L} - (\tilde{C}_v + \tilde{L}' X \tilde{A}) Y_\gamma (\tilde{C}_v + \tilde{L}' X \tilde{A})'$ is positive definite. There exist $K \in z^\ell \mathscr{H}^\infty$ that render $\|G_e\|_\infty < \gamma$ if-and-only if in addition to the above we have

$$\rho(Y_\gamma (\tilde{A}')^\ell X \tilde{A}^\ell) < 1$$

where ρ denotes spectral radius. If this radius is less than one, then $K := M_v - Q_H M_y$ is in $z^\ell \mathscr{H}^\infty$ and renders $G_e \in z^\ell \mathscr{H}^\infty$ with $\|G_e\|_\infty < \gamma$ for

$$z^{-\ell} Q_H(z) = \underbrace{z\left(\begin{array}{c|c} \tilde{A}(I + Z_\ell \tilde{C}_y' \tilde{C}_y \tilde{A}^\ell)^{-1} - zI & Z_\ell \tilde{C}_y' \\ \hline (\tilde{C}_v + \tilde{L}'(X - (\tilde{A}')^\ell X \tilde{A}^\ell)(I + Z_\ell \tilde{C}_y' \tilde{C}_y \tilde{A}^\ell)^{-1} & 0 \end{array}\right)}_{\text{causal system}} - \underbrace{\sum_{i=1}^{\ell} z^{-i} \tilde{L}'(\tilde{A}')^{\ell - i} \tilde{C}_y'}_{\text{strictly causal FIR system}}$$

where $Z_\ell := (I - Y_\gamma (\tilde{A}')^\ell X \tilde{A}^\ell)^{-1} Y_\gamma (\tilde{A}')^\ell$. $\triangle$

For $\ell = 0$ the FIR part is void in the above Q_{H}. For $\ell = \infty$ the first part of Q_{H} is void (because $Z_\infty = 0$) and the "FIR" block becomes the nth order system V.

The rest of this note is a proof of this theorem. Notice that we added a z in the realizations of G and K. This way the transfer matrix is just the Schur complement of the realization with respect to its upper-left block and it allows us to realize non-causal systems as well. We step-by-step translate the frequency domain solution to state space equivalents. First we have to find a coprime factorization (2.1). By detectability and stabilizability of (A, C_y, B) one has that

$$\begin{bmatrix} I & M_v(z) & N_v(z) \\ 0 & M_y(z) & N_y(z) \end{bmatrix} = \left(\begin{array}{c|ccc} \tilde{A} - zI & 0 & L_\kappa & \tilde{B} \\ \hline \tilde{C}_v & I & \Omega & \tilde{D}_v \\ \tilde{C}_y & 0 & \Xi & \tilde{D}_y \end{array}\right) := \left(\begin{array}{c|ccc} A + L_\kappa C_y - zI & 0 & L_\kappa & B + L_\kappa D_y \\ \hline C_v + \Omega C_y & I & \Omega & D_v + \Omega D_y \\ \Xi C_y & 0 & \Xi & \Xi D_y \end{array}\right) \tag{3.3}$$

is such a left coprime factorization for every triple of matrices L_κ, Ω, Ξ provided that Ξ is invertible and $\tilde{A} := A + L_\kappa C_y$ is Schur stable [11]. Since we are after a co-inner N_y we bring in the (Kalman filtering) DARE associated with $G_y G_y^\sim$. This DARE is (3.1). By assumption, $\left[\begin{smallmatrix} A - zI & B \\ C_y & D_y \end{smallmatrix}\right]$ has no unit circle zeros so this DARE has a unique symmetric solution that renders $\tilde{A} := A + L_\kappa C_y$ Schur-stable. It is standard result (it also follows from Thm. 4.1) that for L_κ as defined by this DARE (3.1) we have

$$D_y D_y' + C_y Y_\kappa C_y' > 0.$$

The Ξ defined in Thm. 3.1 makes N_y defined in (3.3) co-inner and Y_κ its controllability gramian: $Y_\kappa = \tilde{A} Y_\kappa \tilde{A}' + \tilde{B}\tilde{B}'$. This controllability gramian constitutes an additive causal-anticausal split of $NN^\sim$ for $N := \left[\begin{smallmatrix} N_v \\ N_y \end{smallmatrix}\right]$ as defined in (3.3),

$$NN^\sim = \begin{bmatrix} N_v \\ N_y \end{bmatrix} \begin{bmatrix} N_v^\sim & N_y^\sim \end{bmatrix} = \left(\begin{array}{cc|cc} \tilde{A} - zI & 0 & -\tilde{L} & 0 \\ 0 & \tilde{A}' - \frac{1}{z} I & \tilde{C}_v' & \tilde{C}_y' \\ \hline \tilde{C}_v & -\tilde{L}' & \tilde{D}_v \tilde{D}_v' + \tilde{C}_v Y_\kappa \tilde{C}_v' & \tilde{D}_v \tilde{D}_y' + \tilde{C}_y Y_\kappa \tilde{C}_v' \\ \tilde{C}_y & 0 & \tilde{D}_y \tilde{D}_y' + \tilde{C}_y Y_\kappa \tilde{C}_y' & I \end{array}\right)$$

where we used the short hand $\tilde{L} := -(\tilde{B}\tilde{D}_v' + \tilde{A} Y_\kappa \tilde{C}_v)$. In particular V defined in (2.4) as $V = N_v N_y^\sim$ (i.e. the upper-right block of $NN^\sim$) equals

$$V(z) = \left(\begin{array}{c|c} \tilde{A}' - \frac{1}{z} I & \tilde{C}_y' \\ \hline -\tilde{L} & \tilde{D}_v \tilde{D}_y' + \tilde{C}_v Y_\kappa \tilde{C}_y' \end{array}\right).$$

For this V to be strictly anti-causal (as Lemma 2.1 requires) we use the freedom of Ω to make the direct feedthrough term of V equal to zero. The direct feedthrough term is linear in Ω,

$$\tilde{D}_v \tilde{D}_y' + \tilde{C}_v Y_\kappa \tilde{C}_y' = (D_v + \Omega D_y) D_y' \Xi' + (C_v + \Omega C_y) Y_\kappa C_y \Xi'$$

and to make it zero we hence take $\Omega = -(D_v D_y' + C_v Y_\kappa C_y')(D_y D_y' + C_y Y_\kappa C_y')^{-1}$ and the required inverse exists. By construction now V has zero direct feedthrough term and is anti-causal, see (3.2) (and it is in $\mathscr{L}^\infty$ because $\tilde{A}$ is Schur-stable). It thus satisfies the conditions of Lemma 2.1.

For the solution of the $\mathscr{L}^2$ smoothing problem the algorithm is complete. For the $\mathscr{L}^\infty$ problem we continue. The next step is to find a spectral co-factor W_γ of (2.6). For that we first additively split the spectrum of $VV^\sim$ to obtain

$$VV^\sim = \left(\begin{array}{cc|c} \tilde{A}' - \frac{1}{z}I & \tilde{C}_y'\tilde{C}_y & 0 \\ 0 & \tilde{A} - zI & \tilde{L} \\ \hline \tilde{L}' & 0 & 0 \end{array}\right) = \left(\begin{array}{cc|c} \tilde{A}' - \frac{1}{z}I & 0 & \tilde{A}'X\tilde{L} \\ 0 & \tilde{A} - zI & \tilde{L} \\ \hline \tilde{L}' & \tilde{L}'X\tilde{A} & \tilde{L}'X\tilde{L} \end{array}\right) = \left(\begin{array}{cc|c} \tilde{A} - zI & 0 & \tilde{L} \\ 0 & \tilde{A}' - \frac{1}{z}I & \tilde{A}'X\tilde{L} \\ \hline \tilde{L}'X\tilde{A} & \tilde{L}' & \tilde{L}'X\tilde{L} \end{array}\right)$$

where we used the observability gramian X defined as $X = \tilde{A}'X\tilde{A} + \tilde{C}_y'\tilde{C}_y$. Now we can combine the realization of $N_v N_v^\sim$ (the upper left part of $NN^\sim$) with that of $VV^\sim$ to the term that we need to factor,

$$\gamma^2 I - (N_v N_v^\sim - VV^\sim) = \left(\begin{array}{cc|c} \tilde{A} - zI & 0 & \tilde{L} \\ 0 & \tilde{A}' - \frac{1}{z}I & \tilde{C}_v' + \tilde{A}'X\tilde{L} \\ \hline \tilde{C}_v + \tilde{L}'X\tilde{A} & \tilde{L}' & \gamma^2 I - \tilde{D}_v\tilde{D}_v' - \tilde{C}_v Y_\kappa \tilde{C}_v' + \tilde{L}'X\tilde{L} \end{array}\right). \qquad (3.4)$$

For the fixed-interval ($\ell = \infty$) problem to have a solution we need that this is positive definite everywhere on the unit circle. From spectral factorization theory (see Thm. 4.1) we know that positivity on the unit circle is equivalent to the existence of a solution of an appropriate DARE. According to Thm. 4.1 the DARE is

$$Y_\gamma = \tilde{A}Y_\gamma\tilde{A}' + \underbrace{(\tilde{L} - \tilde{A}Y_\gamma(\tilde{C}_v' + \tilde{A}'X\tilde{L}))}_{\tilde{B}_\gamma :=} \tilde{R}_\gamma^{-1}(\tilde{L} - \tilde{A}Y_\gamma(\tilde{C}_v' + \tilde{A}X\tilde{L}))'$$

where $\tilde{R}_\gamma$ is as defined in Thm. 3.1. (The Y_γ is actually the negation of Y in Thm. 4.1.) The DARE solution needs to be stabilizing, meaning that $A_\gamma := A - B_\gamma\tilde{R}_\gamma^{-1}(\tilde{C}_v + \tilde{L}'X\tilde{A})$ is Schur-stable. According to Thm. 4.1 the positivity of (3.4) on the unit circle is equivalent to the existence of such Y_γ and that $\tilde{R}_\gamma > 0$. Then (see Thm. 4.1) the spectral co-factor W_γ of (2.6) equals

$$W_\gamma(z) = \left(\begin{array}{c|c} \tilde{A} - zI & \tilde{B}_\gamma\tilde{R}_\gamma^{-1} \\ \hline \tilde{C}_v + \tilde{L}'X\tilde{A} & I \end{array}\right)\tilde{R}_\gamma^{1/2}, \qquad W_\gamma^{-1}(z) = \tilde{R}_\gamma^{-1/2}\left(\begin{array}{c|c} \tilde{A}_\gamma - zI & \tilde{B}_\gamma\tilde{R}_\gamma^{-1} \\ \hline -\tilde{C}_v + \tilde{L}'X\tilde{A} & I \end{array}\right).$$

Incidentally, the DARE can be rewritten as

$$\tilde{Y}_\gamma = \tilde{A}_\gamma\tilde{Y}_\gamma\tilde{A}' + \tilde{B}_\gamma\tilde{R}_\gamma^{-1}\tilde{L}_\gamma'. \qquad (3.5)$$

Due to space limitations the rest of the formulae are given without any explanation. Exploiting the connection between $\tilde{A}$ and $\tilde{A}_\gamma$—see (3.5)—one can find a nth-order realization of $W_\gamma^{-1}V$ needed in the Nehari extension problem (2.7). The Nehari extension problem (2.7) has a solution $Q_\mathrm{H} \in z^\ell\mathscr{H}^\infty$ if-and-only-if some spectral radius is less than one, $\rho(Y_\gamma(\tilde{A}')^\ell X\tilde{A}^\ell) < 1$ (essentially from [5, Theorem XXXV.7.2]). If that is the case then the matrix $Z_\ell := (I - Y_\gamma(\tilde{A}')^\ell X\tilde{A}^\ell)^{-1}Y_\gamma(\tilde{A}')^\ell$ is well defined and then Q_H defined in Thm. 3.1 achieves (2.7).

4. Appendix: spectral factorization. In the derivation we need a spectral factorization $\Phi(z) = W(z)W^{\sim}(z)$ of a Φ that has a given "symmetric" realization of the form

$$\Phi(z) = \left(\begin{array}{cc|c} A - zI & Q & E \\ 0 & A' - \frac{1}{z}I & C' \\ \hline C & E' & R \end{array}\right)$$

with A Schur stable. The result is essentially known and can be found in various arrangements such as in [6, Thm. 12.5.2] and [2, Thm. 1.2 & 14.15]:

THEOREM 4.1. *Suppose Φ has the above realization with $Q = Q', R = R'$ and that A is Schur-stable. The following four statements are equivalent:*

1. *$\Phi(z)$ has no unit circle zeros and $(u, \Phi u) \in (H_2^{\perp}, H_2)$ iff $u = 0$;*
2. *The DARE*
$$Y = AYA' + Q \underbrace{-(E + AYC')(R + CYC')^{-1}}_{L}(E' + CYA')$$
has a solution for which $R + CYC'$ is invertible and $A + LC$ is Schur-stable.
3. *Symmetric nonsingular J and bistable $W(z)$ exist such that $\Phi(z) = W(z)JW^{\sim}(z)$.*

In that case Y is unique and

$$\Phi(z) = \hat{W}(z)(R + CYC')\hat{W}^{\sim}(z)$$

for the bistable system

$$\hat{W}(z) = \left(\begin{array}{c|c} A - zI & -L \\ \hline C & I \end{array}\right).$$

If in addition we have that $\Phi(z) > 0$ for all unit-circle z then the conditions of the three equivalent statements are satisfied and $R + CYC' > 0$.

REFERENCES

[1] B. D. O. Anderson and J. B. Moore. *Optimal Filtering.* Prentice-Hall, Englewood Cliffs, NJ, 1979.

[2] H Bart, I. Gohberg, M.A. Kaashoek, and A.C.M. Ran. *A state space approach to canonical factorization: convolution equations and mathematical systems.* Number 200 in Operator Theory, Adv. Appl. Birkhäuser, 2010.

[3] P. Bolzern, P. Colaneri, and G. De Nicolao. On discrete-time H^∞ fixed-lag smoothing. *IEEE Trans. Signal Processing*, 52:132–141, 2004.

[4] P. Colaneri and A. Ferrante. Algebraic Riccati equation and J-spectral factorization for $\mathscr{H}_\infty$ smoothing and deconvolution. *SIAM J. Control Optim.*, 45(1):123–145, 2006.

[5] I. Gohberg, S. Goldberg, and M.A. Kaashoek. *Classes of Linear Operators*, volume II. Birkhäuser, Basel, 1993.

[6] B. Hassibi, A. H. Sayed, and T. Kailath. *Indefinite Quadratic Estimation and Control: A Unified Approach to H^2 and H^∞ Theories.* SIAM, Philadelphia, 1999.

[7] A. C. Kahane, L. Mirkin, and Z. J. Palmor. On the discrete-time H^∞ fixed-lag smoothing. In *Proc. 15th IFAC World Congress*, Barcelona, Spain, 2002.

[8] G. Meinsma and L. Mirkin. Sampled-data L^∞ smoothing: Fixed-size ARE solution with free hold function. In *Proc. of the 20th MTNS Symposium*, Melbourne, Australia, 2012. submitted for publication.

[9] L. Mirkin and G. Tadmor. On geometric and analytic constraints in the H^∞ fixed-lag smoothing. *IEEE Trans. Automat. Control*, 52(8):1514–1519, 2007.

[10] K. M. Nagpal and P. P. Khargonekar. Filtering and smoothing in an H^∞-setting. *IEEE Trans. Automat. Control*, 36(2):151–166, 1991.

[11] C. N. Nett, C. A. Jacobson, and M. J. Balas. A connection between state-space and doubly coprime fractional representations. *IEEE Trans. Automat. Control*, 29(9):831–832, 1984.

[12] G. Tadmor and L. Mirkin. H^∞ control and estimation with preview—Part II: Fixed-size ARE solutions in discrete time. *IEEE Trans. Automat. Control*, 50(1):29–40, 2005.

[13] H. Zhang, L. Xie, and Y. C. Soh. A unified approach to linear estimation for discrete-time systems. II. H^∞ estimation. In *Proc. 40th IEEE Conf. Decision and Control*, pages 2923–2928, Orlando, FL, 2001.