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Stability and Stabilization of a Class of Boundary Control Systems

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Abstract— **We study a class of partial differential equations on a one dimensional spatial domain with control and observation at the boundary. For this class of systems we describe how to obtain an impedance energy-preserving system, as well as scattering energy-preserving system. For the first type of systems we consider (static and dynamic) feedback stabilization by means of boundary control. For the scattering energy-preserving systems we give conditions for which the system is either asymptotically or exponentially stable.**

I. INTRODUCTION

In this paper we study the following partial differential equation (PDE) on the spatial interval $[a,b]$

$$\dot{x}(t) = \mathscr{J} x(t), \qquad x(0) = x_0 \tag{1a}$$

$$u(t) = \mathscr{B}x(t), \tag{1b}$$

$$y(t) = \mathscr{C}x(t), \tag{1c}$$

here

$$\mathscr{J} x = \sum_{i=0}^{N} P(i) \frac{d^i x}{dz^i}(z) \qquad z \in [a,b], \tag{2}$$

with the domain of $\mathscr{J}$ being $H^N((a,b);\mathbb{R}^n)$, i.e., the Sobolev space of N times differentiable functions on the interval (a,b). $P(i)$, $i = 0,\ldots,N$, is a $n\times n$ real matrix satisfying

$$P(i) = P(i)^T(-1)^{i+1}, \quad \text{and} \quad \ker P(N) = \{0\}. \tag{3}$$

For the PDE (1) we describe how to obtain impedance energy-preserving systems (systems that satisfy $\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = u(t)^T y(t)$), as well as scattering energy-preserving ($\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2$). Next we study feedback (static and dynamic) stabilization for impedance energy-preserving systems. Finally, we study (asymptotic and exponential) stability of scattering energy-preserving.

Here we use the notation $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ for $X\times Y$ and $F_{|\mathfrak{D}}$ denotes the restriction of an operator F to the subspace $\mathfrak{D}$. $\rho(F)$ denotes the resolvent set of F and ∂_z^N indicates the N-times partial derivative with respect to the variable z. I is the identity operator.

II. SOME BACKGROUND

Most of the results described in this section can be found in [1]. First we need to introduce some notation. The $nN\times nN$ matrix Q is defined as

$$Q = \begin{pmatrix} P(1) & P(2) & P(3) & \cdots & P(N-1) & P(N) \\ -P(2) & -P(3) & -P(4) & \cdots & P(N) & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ (-1)^{N-1}P(N) & 0 & \cdots & \cdots & & 0 \end{pmatrix}.$$

Using this Q, the matrix R_{ext} is defined as

$$R_{\text{ext}} = \frac{1}{\sqrt{2}} \begin{pmatrix} Q & -Q \\ I & I \end{pmatrix}. \tag{4}$$

Since $P(N)$ is invertible, we have that R_{ext} is invertible as well. It is easy to see that it satisfies

$$R_{\text{ext}}^T \Sigma R_{\text{ext}} = \begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix}, \text{ where } \Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{5}$$

Definition 2.1: The boundary port variables associated with the differential operator $\mathscr{J}$ and the function $x \in H^N((a,b);\mathbb{R}^n)$ are the vectors $e_\partial, f_\partial \in \mathbb{R}^{nN}$, defined by

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = R_{\text{ext}} \begin{pmatrix} x(b) \\ \vdots \\ \partial_z^{N-1}x(b) \\ x(a) \\ \vdots \\ \partial_z^{N-1}x(a) \end{pmatrix}, \tag{6}$$

where R_{ext} is defined by (4).

In [1], the authors prove the following theorem.

Theorem 2.2: Let

$$W = S\left[\begin{array}{cc} I+V, & I-V \end{array}\right], \tag{7}$$

with S invertible and $VV^T \leq I$, be a full rank matrix of size $nN\times 2nN$ (satisfying $W\Sigma W^T \geq 0$), and define $\mathscr{B}: H^N((a,b),\mathbb{R}^n) \to \mathbb{R}^{nN}$ as

$$\mathscr{B}x(t) := W \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}. \tag{8}$$

Then the system (1a)-(1b) is a boundary control system. $A_W := \mathscr{J}_{|\ker\mathscr{B}}$ is the generator of a contraction semigroup and

$$D(A_W) = \{x \in L^2((a,b),\mathbb{R}^n) \mid \left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right] \in \ker W\}.$$

Furthermore, if we define the output via the mapping $\mathscr{C}: H^N((a,b),\mathbb{R}^n) \to \mathbb{R}^{nN}$ as

$$y(t) = \mathscr{C}x(t) := S_2 \begin{pmatrix} I-V^T & -I-V^T \end{pmatrix} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \tag{9}$$

with S_2 invertible, then for $u \in C^2((0,\infty);\mathbb{R}^{nN})$, $x(0) \in D(\mathscr{J})$, and $\mathscr{B}x(0) = u(0)$ the following balance equation is satisfied:

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = \begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_W \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, \tag{10}$$

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where P_W is given by

$$\frac{1}{4}\begin{pmatrix} S^{-T}(\tilde{P}_1^2-\tilde{P}_1VV^T\tilde{P}_1)S^{-1} & -2S^{-T}\tilde{P}_1V\tilde{P}_2S_2^{-1} \\ -2S_2^{-T}\tilde{P}_2V^T\tilde{P}_1S^{-1} & S_2^{-T}(-\tilde{P}_2^2+\tilde{P}_2V^TV\tilde{P}_2)S_2^{-1} \end{pmatrix}, \quad (11)$$

and $\tilde{P}_1=(I+VV^T)^{-1}, \quad \tilde{P}_2=(I+V^TV)^{-1}$.

For more information see [1] and [2].

III. Impedance energy-preserving systems

Here we use the term 'impedance energy-preserving system' in the sense of [3]. In that paper the author shows that an impedance energy-preserving system satisfies the relation

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2=u(t)^T y(t) \quad (12)$$

for $u\in C^2((0,\infty);\mathbb{R}^{nN})$, $x(0)\in D(\mathscr{J})$ and, $\mathscr{B}x(0)=u(0)$. In [1] the authors show that for an impedance energy-preserving system we have $V^TV=VV^T=I$, $A_W^*=-A_W$ and $D(A_W)=D(A_W^*)$. In [2] it is shown that, in this case, the inputs can be described by

$$u=\frac{1}{4}S_2^{-T}[-I-V^T,\ I-V^T]\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}=W_{\text{imp}}\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \quad (13)$$

and the outputs by

$$y=S_2\left[\ I-V^T, \quad -I-V^T\ \right]\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}=C_{\text{imp}}\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}. \quad (14)$$

Here, the state space is $X=L_2((a,b);\mathbb{R}^n)$ and the input and output spaces are $U=Y=\mathbb{R}^n$. Furthermore, we know that $A_W=\mathscr{J}_{|D(A_W)}$ is the generator of a contraction semigroup (see Theorem 2.2) with $D(A_W)=\ker\mathscr{B}$.

IV. Static feedback of an impedance energy-preserving system

In this section we apply feedback (see Figure 1), i.e.,

$$u=r-\alpha y, \quad (15)$$

where $r,u,y\in\mathbb{R}^{nN}$ and $\alpha>0$ is a positive definite matrix.

We have that the plant is described by equations (1a)–(1c), where $\mathscr{B}x(t)$ is given by (13), $\mathscr{C}x(t)$ is given by (14) and the differential operator $\mathscr{J}$ when restricted to $D(\mathscr{J})\cap\ker(\mathscr{B})$ generates a C_0-semigroup.

Using the feedback control (15) we can see that the closed-loop system is now described by

$$\dot{x}(t)=\mathscr{J}x(t)$$
$$(W_{\text{imp}}+\alpha C_{\text{imp}})\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}=(\mathscr{B}+\alpha\mathscr{C})x(t)=r(t) \quad (16)$$
$$\mathscr{C}x(t)=y(t).$$

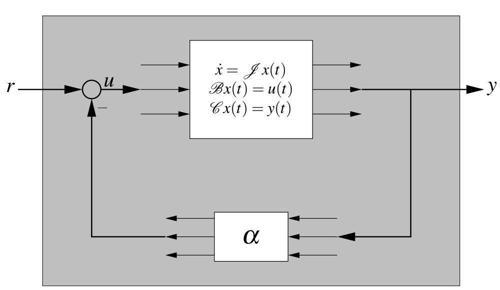

Fig. 1. General control system.

Lemma 4.1: The system described by (16) (with $VV^T=I$) is a boundary control system. Furthermore, the operator $A_s=\mathscr{J}_{|D(A_s)}$ generates a contraction semigroup on $X=L_2((a,b);\mathbb{R}^n)$, where

$$D(A_s)=\left\{x\in D(\mathscr{J})\,|\,\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}\in\ker\widetilde{W}\right\} \quad \text{and} \quad (17)$$

$$\widetilde{W}=(W_{\text{imp}}+\alpha C_{\text{imp}}) \quad (18)$$

is a full rank $nN\times 2nN$ matrix.

Proof: First observe that (18) follows from (16). From Theorem 2.2 we can see that if $\widetilde{W}$ satisfies $\widetilde{W}\Sigma\widetilde{W}^T\geq 0$ (Σ given by (5)), then we will be done. Since V is unitary we can use Lemma A-1 to show that $\widetilde{W}$ satisfies $\widetilde{W}\Sigma\widetilde{W}^T=\alpha+\alpha^T>0$ (since $\alpha>0$). ∎

Remark 4.2: Using Lemma A.1 of [1] we can see that (18) can be factorized as

$$\tilde{W}=\tilde{S}\left[\ I+\tilde{V}, \quad I-\tilde{V}\ \right]. \quad (19a)$$

Furthermore, from step 1 and 2 of the proof of Theorem 4.3 of [1] we can show that

$$\langle A_sx,x\rangle<0, \forall x\in D(A_s), \quad \langle A_s^*x,x\rangle<0, \forall x\in D(A_s^*). \quad (19b)$$

Remark 4.3: Following the same procedure used to prove (10)–(11), see [1, p.19], we can show that if $r(t)=0$, $x(0)\in D(\mathscr{J})$, and $(\mathscr{B}+\alpha\mathscr{C})x(0)=0$, then

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2=-y(t)^T\alpha y(t). \quad (20)$$

Next we study stability of the closed-loop system.

Theorem 4.4: Assume that $(\lambda-A_s)^{-1}:X\rightarrow X$ is a compact operator for $\lambda>0$. Then the system described by (16) (with $VV^T=I$ and $r=0$), is asymptotically stable.

Remark 4.5: It can be said that most of the examples encountered in the literature satisfy the assumption in the theorem above, see, e.g., [5, p. 269].

Proof: [Proof of Theorem 4.4] By Lemma 4.1 we know that A_s generates a contraction semigroup. Thus, for any $x(0)\in X$, the solution $x(t)=T(t)x(0)$ (classical or weak) is bounded in X. Since $(\lambda-A_s)^{-1}$ is a compact operator for $\lambda>0$, it follows that the trajectory of the solution $x(t)$, i.e. the set $\gamma(x(0))=\{x(t)\in X, t\geq 0\}$ is precompact in X, see Theorem 3.65 of [5]. It then follows that the w-limit set[1] $w(x(0))$ of the trajectory is nonempty, compact, and we have $x(t)\rightarrow w(x(0))$ as $t\rightarrow\infty$, see Theorem 3.61 of [5].

Next we show that $w(x(0))$ contains only the point zero. First we prove this for $x(0)\in D(A_s)$. In this case we have that $x(t)=T(t)x(0)\in D(A_s)$ for all $t\geq 0$, see Theorem 2.1.10 of [4]. Define the energy function

$$E(t)=\frac{1}{2}\|x(t)\|^2. \quad (21)$$

Since $x(0)\in D(A_s)$ we have that $x(t)$ is differentiable, see [4, §2.1]. Thus the derivative of $E(t)$ is given by (20) with $r(t)=0$, that is $\dot{E}(t)=\frac{1}{2}\frac{d}{dt}\|x(t)\|^2=-y(t)^T\alpha y(t)$. Since α is a positive definite matrix we have that $\dot{E}(t)<0$, which shows that $E(t)$ is a Lyapunov function, see Definition 3.62 of [5]. Observe that $\dot{E}(t)=0$ implies $y(t)=0$. Now consider the

[1] The w-limit set of x is given by $w(x)=\{y\in F\,|\,y=\lim_{n\rightarrow\infty}T(t_n)x$ with $t_n\rightarrow\infty$ as $n\rightarrow\infty\}$, where F is a closed subset of X.

set $\{x \in X \mid \dot{E}(t) = 0\}$ or equivalently $\mathscr{O} = \{x \in X \mid y(t) = 0\}$ and let $\mathscr{E}$ be its largest invariant subset. Since $\gamma(x(0))$ is precompact, it follows from LaSalle's principle that $x(t) \to \mathscr{E}$ as $t \to \infty$, see Theorem 3.64 of [5]. We show that $\mathscr{E} = \{0\}$. Since $x(0)$ and $x(t)$ belong to $D(A_s)$, we have that $\mathscr{E} \subset D(A_s)$, see [5, p.270]. Let $\tilde{x} \in \mathscr{E}$ and let $z(t) = T(t)\tilde{x}$ be the corresponding solution. Since $\mathscr{E}$ is invariant we have that $z(t) \in \mathscr{E}$ for $t \geq 0$. Recall that all $x \in \mathscr{O}$ satisfy $y(t) = 0$. It then follows that $z(t)$ is the solution of $\dot{z}(t) = \mathscr{J} z(t)$ satisfying $r(t) = y(t) = 0$. Since this is a PDE with all boundary variables set to zero, we must have that the only solution is $z(t) = 0$ for $t \geq 0$. Hence $\tilde{x} = 0$; and thus, $\mathscr{E} = \{0\}$. Altogether means that $x(t) \to 0$ as $t \to \infty$ for any $x(0) \in D(A_s)$.

The same statement holds for $x(0) \in X$, see [5, p.270]. ■

A. First order differential operator (case $N = 1$)

In this section we study the differential operator

$$\mathscr{J} x = P_0 x(z) + P_1 \frac{dx}{dz}(z). \tag{22}$$

This case includes the well-known beam and wave equations. We assume that the input and outputs have been chosen so that the resulting system is impedance energy-preserving, i.e., $\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = u(t)^T y(t)$ holds. We want to apply static feedback to the resulting BCS, see Figure 1. We already know that, in this case, the closed-loop system is a BCS and that the operator $A_s = \mathscr{J}_{|D(A_s)}$ generates a contraction semigroup, see Lemma 4.1. We want to check whether the closed-loop system is asymptotically stable. To do so we need to check that the resolvent $(\lambda - A_s)^{-1}$ is compact for $\lambda > 0$, see Theorem 4.4.

First we study the eigenvalues of A_s. From (19b) we know that $\langle A_s x, x \rangle < 0$ for all $x \in D(A_s)$. Let x_1 be any eigenvector of A_s with corresponding eigenvalue λ. Then we have that

$$\mathrm{Re}\, \langle A_s x_1, x_1 \rangle = \mathrm{Re}\, \langle \lambda x_1, x_1 \rangle = \mathrm{Re}\, \lambda \, \|x_1\|^2 < 0.$$

This implies that all eigenvalues of A_s satisfy $\mathrm{Re}\,\lambda < 0$. Moreover, we have that this eigenvector x_1 is the solution of

$$P_0 x_1 + P_1 \frac{dx_1}{dz} = \lambda x_1 \iff \frac{dx_1}{dz} = P_1^{-1}(\lambda - P_0) x_1.$$

The general solution of the equation above is given by

$$x_1(z) = \mathrm{e}^{P_1^{-1}(\lambda - P_0)(z-a)} c \tag{23}$$

where c is a constant vector. Using (19) and the boundary conditions on $D(A_s)$, see (17), we get

$$\tilde{W} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \tilde{W} R_{\mathrm{ext}} \begin{bmatrix} x_1(b) \\ x_1(a) \end{bmatrix} = 0$$

$$\iff \tilde{S}[I + \tilde{V}, \quad I - \tilde{V}] \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} x_1(b) \\ x_1(a) \end{bmatrix} = 0 \tag{24}$$

where we used Definition 2.1 and (19a). Using (23) in the above equation gives

$$\Big(\big[(I + \tilde{V})P_1 + (I - \tilde{V})\big] \mathrm{e}^{P_1^{-1}(\lambda - P_0)(b-a)} + \big[-(I + \tilde{V})P_1 + (I - \tilde{V})\big] \Big) c = 0. \tag{25}$$

We know that λ is an eigenvalue of A_s iff the matrix above is singular. At the beginning of this subsection we showed that the eigenvalues of A_s satisfy $\mathrm{Re}\,\lambda < 0$, which shows that the matrix above is nonsingular if $\mathrm{Re}\,\lambda \geq 0$, otherwise A_s would have eigenvalues with $\mathrm{Re}\,\lambda \geq 0$. In summary we have the following result.

Lemma 4.6: Consider the system described in Lemma 4.1 with $\mathscr{J}$ given by (22). Then the eigenvalues of A_s satisfy $\mathrm{Re}\,\lambda < 0$. Furthermore, if $\lambda \geq 0$ the matrix on (25) is nonsingular and $(\lambda - A_s)^{-1}$ is a compact operator.

Proof: That the eigenvalues of A_s satisfy $\mathrm{Re}\,\lambda < 0$ and that the matrix is nonsingular was proved above.

Next we study the resolvent of A_s. First we study the range of $(\lambda - A_s) : D(A_s) \to X$ for $\lambda \geq 0$. To see this, consider $(\lambda - A_s)x = y$, which is equivalent to solve

$$\frac{dx}{dz}(z) = P_1^{-1}(\lambda - P_0) x(z) - P_1^{-1} y(z) \tag{26}$$

for $x \in D(A_s)$. The general solution of (26) is given by

$$x(z) = \mathrm{e}^{P_1^{-1}(\lambda - P_0)(z-a)} c - \int_a^z \mathrm{e}^{P_1^{-1}(\lambda - P_0)(z-\tau)} P_1^{-1} y(\tau)\, d\tau \tag{27}$$

with c a constant vector. Since $x \in D(A_s)$ the boundary conditions (see (17)) are given by (24). Using (27) in (24) yields

$$\Big(\big[(I + \tilde{V})P_1 + (I - \tilde{V})\big] \mathrm{e}^{P_1^{-1}(\lambda - P_0)(b-a)} + \big[-(I + \tilde{V})P_1 + (I - \tilde{V})\big] \Big) c = \big[(I + \tilde{V})P_1 + (I - \tilde{V})\big] \int_a^b \mathrm{e}^{P_1^{-1}(\lambda - P_0)(b-\tau)} P_1^{-1} y(\tau)\, d\tau.$$

We already showed that when $\mathrm{Re}\,\lambda \geq 0$ the matrix on the LHS is nonsingular. In that case, c can be defined uniquely, which implies that (26) has a unique solution in $D(A_s)$. This solution is given by (27). Hence $(\lambda - A_s)^{-1}$ exists when $\mathrm{Re}\,\lambda \geq 0$. In order to prove that (27) defines the resolvent of A_s we need to show that it is bounded. First observe that the dimension of the domain of $\mathrm{e}^{P_1^{-1}(\lambda - P_0)(z-a)} c$ is finite, thus it is a compact operator (hence bounded), see [6, Th. 2.6-9 b, Th. 8.1-4, Th. 8.1-2 b]. Also, the integral operator on (27) is compact, see Theorem A.3.52 of [4]. Then, $(\lambda - A_s)^{-1}$ given by (27) is the sum of two compact operators (see see [6, p. 407]), hence $(\lambda - A_s)^{-1}$ is a compact operator when $\mathrm{Re}\,\lambda \geq 0$. ■

From the Lemma above and Theorem 4.4 we know that this type of closed-loop systems (when $N = 1$) are asymptotically stable. Also, if $P_0 = 0$ we can see that when λ is a pure imaginary number, the resolvent (27) is clearly a uniformly bounded operator since P_1 has only real eigenvalues and hence the magnitude of $\mathrm{e}^{\lambda P_1^{-1} z}$ is equal to one. Since A_s generates a contraction semigroup it follows that the conditions on Corollary 3.36 of [5] are satisfied, which proves that, in this case, the closed-loop is exponentially stable.

V. Dynamic feedback of an impedance energy-preserving system

In this section we generalize the class of static controllers described in the previous section. More precisely, we replace the static matrix α with a matrix transfer function $\alpha(s)$ where $s \in \mathbb{C}$ is a complex variable.

The state space representation of the controller is given by

$$\begin{aligned}\dot{v}(t) &= A_\alpha v(t) + B_\alpha y(t)\\ y_\alpha(t) &= C_\alpha v(t) + D_\alpha y(t)\end{aligned} \tag{28}$$

where $v \in \mathbb{R}^m$ is the state of the minimal realization. In this way, equation (15) becomes

$$u = r - y_\alpha = r - C_\alpha v - D_\alpha \mathscr{C} x, \tag{29}$$

where (1c) was used.

In the literature, an $n \times n$ rational matrix $H(s)$ is said to be **positive real (PR)** if: i) all elements of $H(s)$ are analytic in the open right-half plane $\mathrm{Re}(s) > 0$, ii) poles of any element of $H(s)$ on the jw-axis are distinct, and the associated residue matrix of $H(s)$ is ≥ 0, iii) $H(jw) + H^T(-jw) \geq 0 \;\forall w$ which is not a pole of any element of $H(jw)$.

Definition 5.1 (Tao and Ioannou [7]): A rational matrix $H(s)$ is **strictly positive real** (SPR) if $H(s-\varepsilon)$ is positive real (PR) for some $\varepsilon > 0$.

The next lemma is used in the stability analysis (see [7]).

Lemma 5.2: Assume that the transfer matrix $H(s)$ has all its poles in $\mathrm{Re}(s) < -\gamma$, where $\gamma > 0$ and (A,B,C,D) is a minimal realization of $H(s)$. Then $H(s-\gamma)$ is PR if and only if there exist matrices P, Q and K such that $P = P^T > 0$ and

$$PA + A^T P = -QQ^T - 2\gamma P;\; PB = C^T - QK;\; K^T K = D + D^T. \tag{30}$$

Throughout this section the controller is assumed to be SPR. Let $x \in X$ be the state of the plant, $v \in \mathbb{R}^m$ the state of the controller, and $w = \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right]$. Using the feedback control (29) and the fact that $u(t) = \mathscr{B}x(t)$, see (1b), we can see that the closed-loop system is now described by

$$\begin{aligned}\dot{w}(t) &= \mathscr{J}_c w(t), \quad w(0) \in \tilde{X}\\ \left[\begin{array}{cc} \mathscr{B} + D_\alpha \mathscr{C}, & C_\alpha \end{array}\right] w(t) &= r(t)\\ \left[\begin{array}{cc} \mathscr{C} & 0 \end{array}\right] w(t) &= y(t),\end{aligned} \tag{31}$$

where $\tilde{X} = \left[\begin{smallmatrix} X \\ \mathbb{R}^m \end{smallmatrix}\right]$ is the state space of the closed-loop system, $w = \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in \tilde{X}$, and $\mathscr{J}_c : \tilde{X} \to \tilde{X}$ is a linear operator defined as

$$\mathscr{J}_c w = \left[\begin{array}{cc} \mathscr{J} & 0 \\ B_\alpha \mathscr{C} & A_\alpha \end{array}\right] \left[\begin{array}{c} x \\ v \end{array}\right] \tag{32}$$

with $D(\mathscr{J}_c) = D(\mathscr{J}) \oplus \mathbb{R}^m$. The inner product on the space $\tilde{X}$ is defined as

$$\langle w_1, w_2 \rangle_{\tilde{X}} = \langle x_1, x_2 \rangle_X + \frac{1}{2} v_1^T P v_2 + \frac{1}{2} v_2^T P v_1, \tag{33}$$

where P is the positive definite matrix found in Lemma 5.2.

Lemma 5.3: Let the state of the open-loop system of Figure 1 satisfy $\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = u(t)^T y(t)$ and the controller $\alpha(s)$ be SPR. Then the system (31)-(32) is a boundary control system. Furthermore, the operator A_c defined by

$$A_c w = \left[\begin{array}{cc} \mathscr{J} & 0 \\ B_\alpha \mathscr{C} & A_\alpha \end{array}\right] \left[\begin{array}{c} x \\ v \end{array}\right] \tag{34a}$$

with

$$D(A_c) = \left\{ \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ \mathbb{R}^m \end{smallmatrix}\right] \;\middle|\; x \in D(\mathscr{J}), \text{ and } \left[\begin{smallmatrix} f_\partial \\ e_\partial \\ v \end{smallmatrix}\right] \in \ker \widetilde{W}_D \right\}, \tag{34b}$$

where

$$\widetilde{W}_D = \left[\begin{array}{cc} (W_{\mathrm{imp}} + D_\alpha C_{\mathrm{imp}}), & C_\alpha \end{array}\right], \tag{34c}$$

generates a contraction semigroup.

Proof: First we need to prove that there exists an operator $B \in \mathscr{L}(U, \tilde{X})$ such that for all $r \in U$, $Br \in D(\mathscr{J}) \times \mathbb{R}^m$, and $\left[\begin{array}{cc} \mathscr{B} + D_\alpha \mathscr{C}, & C_\alpha \end{array}\right] Br = r$. From the proof of Theorem 4.5 of [1] we know that if the matrix $\tilde{W}_D$ has full rank, then such operator B exists. Thus we need to prove that $\tilde{W}_D$ has full rank. Since the open-loop system is assumed to be impedance energy-preserving, we must have $VV^T = I$. From this and Lemma A-2 we can see that the matrix $(W_{\mathrm{imp}} + D_\alpha C_{\mathrm{imp}})$ has full row-rank. Hence, we can conclude that $\tilde{W}_D$ in (34c) has also full row-rank.

Equations (34) follow easily from (31) and the proof of Lemma 4.1. Next we need to prove that A_c generates a semigroup. We will use the Lumer-Phillips theorem (see Theorem 2.27 of [5]). First we prove that $\langle A_c w, w \rangle \leq 0$. Let $w = \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in D(A_c)$, then we have

$$\begin{aligned}\langle A_c w, w \rangle_{\tilde{X}} &= \langle \mathscr{J} x, x \rangle_X + \frac{1}{2}(A_\alpha v + B_\alpha y)^T P v\\ &\quad + \frac{1}{2} v^T P (A_\alpha v + B_\alpha y)\\ &= \langle \mathscr{J} x, x \rangle_X + \frac{1}{2} v^T (A_\alpha^T P + P A_\alpha) v + \frac{1}{2} y^T B_\alpha^T P v + \frac{1}{2} v^T P B_\alpha y.\end{aligned}$$

From Equation (4.8) of [1] and Lemma 5.2 we obtain

$$\begin{aligned}&= \frac{1}{2}\left[\begin{array}{cc} f_\partial^T & e_\partial^T \end{array}\right] \Sigma \left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right] + \frac{1}{2} v^T (-QQ^T - 2\gamma P) v\\ &\quad + \frac{1}{2} y^T (C_\alpha - K^T Q^T) v + \frac{1}{2} v^T (C_\alpha^T - QK) y.\end{aligned}$$

Using (A-1) together with (13) and (14) yields

$$\begin{aligned}&= \frac{1}{2} y^T u + \frac{1}{2} u^T y + \frac{1}{2} v^T (-QQ^T - 2\gamma P) v\\ &\quad + \frac{1}{2} y^T (C_\alpha - K^T Q^T) v + \frac{1}{2} v^T (C_\alpha^T - QK) y.\end{aligned}$$

Since $w = \left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in D(A_c)$ we have that $C_\alpha v = -(W_{\mathrm{imp}} + D_\alpha C_{\mathrm{imp}}) \left[\begin{smallmatrix} f_\partial \\ e_\partial \end{smallmatrix}\right]$, see (34), and using again (13) and (14) gives, after simplification

$$\begin{aligned}&= \frac{1}{2} v^T (-QQ^T - 2\gamma P) v - \frac{1}{2} y^T D_\alpha y - \frac{1}{2} y^T D_\alpha^T y\\ &\quad - \frac{1}{2} y^T K^T Q^T v - \frac{1}{2} v^T Q K y\end{aligned}$$

and using again Lemma 5.2 yields

$$\langle A_c w, w \rangle_{\tilde{X}} = -\gamma v^T P v - \frac{1}{2}(Ky + Q^T v)^T (Ky + Q^T v). \tag{35}$$

Since $\gamma > 0$ and P is positive definite it thus follows from the equation above that $\langle A_c w, w \rangle_{\tilde{X}} \leq 0$.

Next we need to prove that the range of $(I - A_c)$ is equal to $\tilde{X}$. In order to do so, we can show that for all $\left[\begin{smallmatrix} f \\ z \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ \mathbb{R}^m \end{smallmatrix}\right]$ there exists $\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in D(A_c)$ such that

$$\begin{bmatrix} f \\ z \end{bmatrix} = \begin{bmatrix} (I - \mathscr{J})x \\ -B_\alpha \mathscr{C} x + (I - A_\alpha)v \end{bmatrix}. \tag{36}$$

Observe that since $\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in D(A_c)$ we must have (see (31))

$$(\mathscr{B} + D_\alpha \mathscr{C})x + C_\alpha v = 0. \tag{37}$$

We need to solve (36) and (37) for $\left[\begin{smallmatrix} f \\ z \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ \mathbb{R}^m \end{smallmatrix}\right]$ given. Recall that A_α is assumed to have only negative eigenvalues, and hence $(I - A_\alpha)$ is a nonsingular matrix. Using the lower equation of (36) into (37) yields

$$(\mathscr{B} + \alpha(1)\mathscr{C})x = -C_\alpha (I - A_\alpha)^{-1} z, \tag{38}$$

where $\alpha(1) = C_\alpha (I - A_\alpha)^{-1} B_\alpha + D_\alpha$. We now need to find an x that satisfies (38) and the upper equation in (36). In these two equations let $\tilde{z} = -C_\alpha (I - A_\alpha)^{-1} z$ and $x = x_{\text{new}} + \tilde{B}\tilde{z}$ where $\tilde{B}$ is such that $(\mathscr{B} + \alpha(1)\mathscr{C})\tilde{B} = I$ (the existence of $\tilde{B}$ is proved in [1]). This gives

$$(I - \mathscr{J})x_{\text{new}} = f - (I - \mathscr{J})\tilde{B}\tilde{z} \tag{39}$$

$$(\mathscr{B} + \alpha(1)\mathscr{C})x_{\text{new}} = 0. \tag{40}$$

Following the proof of Theorem 4.1 it is not difficult to see that if (40) holds then $\mathscr{J}$ generates a contraction semigroup. This implies that $(I - \mathscr{J})$ has an inverse and hence x_{new} exists. Thus, for $\left[\begin{smallmatrix} f \\ z \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ \mathbb{R}^m \end{smallmatrix}\right]$ we can find $\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right] \in D(A_c)$ such that (36) and (37) holds. ■

In the rest of this section we denote by A_λ the operator A_s described in Lemma 4.1 with α replaced by $\alpha(\lambda) = C_\alpha(\lambda I - A_\alpha)^{-1} B_\alpha + D_\alpha$ in (18). Next we show that if the resolvent of A_λ is compact for $\lambda > 0$, then the associated closed-loop system will also have a compact resolvent.

Theorem 5.4: Consider the system described in Theorem 2.2. Let the energy of this system satisfy $\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = u(t)^T y(t)$. Assume that $\alpha(s)$ is a SPR function and that the resolvent of the operator A_λ described above is compact for $\lambda > 0$. Then $(\lambda I - A_c)^{-1}$ is also compact for $\lambda > 0$.

Proof: We will use Theorem 8.1-3 of [6], which states that an operator is compact iff it maps every bounded sequence onto a sequence which has a convergent subsequence. First we find the inverse of $(\lambda I - A_c)$ for $\lambda > 0$ by following the same procedure used to find (36)–(40). We know that this inverse exists since A_c generates a contraction semigroup. From (39)–(40) we see, in this case, that $x_{\text{new}} \in D(A_\lambda)$ and $x_{\text{new}} = (I - A_\lambda)^{-1} f - (I - A_\lambda)^{-1}(I - \mathscr{J})\tilde{B}\tilde{z}$, where $\tilde{z} = -C_\alpha (I - A_\alpha)^{-1} z$. Since $x = x_{\text{new}} + \tilde{B}\tilde{z}$ we obtain that

$$x = (I - A_\lambda)^{-1} f - (I - A_\lambda)^{-1}(I - \mathscr{J})\tilde{B}\tilde{z} + \tilde{B}\tilde{z}, \tag{41}$$

and from the lower equation of (36) we get

$$v = (I - A_\alpha)^{-1} B_\alpha \mathscr{C} x + (I - A_\alpha)^{-1} z. \tag{42}$$

Let $\{k_n\} = \left\{\left[\begin{smallmatrix} f_n \\ z_n \end{smallmatrix}\right]\right\} \in \tilde{X} = \left[\begin{smallmatrix} X \\ \mathbb{R}^m \end{smallmatrix}\right]$ be any bounded sequence in $\tilde{X}$ and let $w_n = \left[\begin{smallmatrix} x_n \\ v_n \end{smallmatrix}\right] \in D(A_c)$ such that $w_n = (\lambda I - A_c)^{-1} k_n$. By Theorem 5.3 we know that A_c generates a contraction semigroup and by the Hille-Yosida theorem it follows that $\left\|(\lambda I - A_c)^{-1}\right\| \leq \frac{1}{\lambda}$ for $\lambda > 0$. Hence the sequence $\{w_n\}$ is bounded too. Since we know that $(\lambda I - A_\lambda)^{-1}$ is compact and that $\mathscr{J}\tilde{B}$ is bounded (see Definition 3.3.2 of [4]), we have that $\{x_n\}$ has a convergent subsequence, see (41). Also, since $\{v_n\}$ is bounded and belongs to a finite dimensional subspace of $\tilde{X}$, it follows that $\{v_n\}$ has another convergent subsequence. Hence, $w_n = \left[\begin{smallmatrix} x_n \\ v_n \end{smallmatrix}\right]$ has a convergent subsequence and therefore $(\lambda I - A_c)^{-1}$ is compact for $\lambda > 0$. ■

Next we give an asymptotic stability result similar to the one in Theorem 4.4.

Theorem 5.5: Consider the system given by (31). Let the transfer function $\alpha(s)$ be a SPR function. Let the resolvent associated with A_λ be compact for $\lambda > 0$. Then the system described by (31) with $r = 0$, is asymptotically stable.

Proof: The proof is similar to that of Theorem 4.4. First we prove this for $w(0) \in D(A_c)$. By Lemma 4.1 we know that A_c generates a contraction semigroup. In this case we have that $w(t) = T_c(t)x(0) \in D(A_c)$ for all $t \geq 0$, see Theorem 2.1.10 of [4]. Define the energy function

$$E_c(t) = \frac{1}{2}\|w(t)\|_{\tilde{X}}^2 = \frac{1}{2}\langle w(t), w(t)\rangle_{\tilde{X}}. \tag{43}$$

Since $w(0) \in D(A_c)$ we have that $w(t)$ is differentiable, see [4, §2.1]. By differentiating the equation above and using (31) and (35) we obtain

$$\begin{aligned} \dot{E}_c(t) &= \langle \dot{w}(t), w(t)\rangle_{\tilde{X}} = \langle A_c w(t), w(t)\rangle_{\tilde{X}} \qquad (44) \\ &= -\gamma v(t)^T P v(t) - \frac{1}{2}(Ky(t) + Q^T v(t))^T (Ky(t) + Q^T v(t)), \end{aligned}$$

where $\gamma > 0$ and P is positive definite. Since $(\lambda I - A_\lambda)^{-1}$ is compact, it follows from Theorem 5.4 that $(\lambda I - A_c)^{-1}$ is also compact. Since $(\lambda I - A_c)^{-1}$ is compact and $T_c(t)$ is a contraction, it follows from LaSalle's principle that all solutions of (31) asymptotically tend to the maximal invariant set of $\mathscr{O}_c = \{x \in X \mid \dot{E}_c(t) = 0\}$. Let $\mathscr{E}$ be the largest invariant subset of $\mathscr{O}_c$. Next we show that $\mathscr{E} = \{0\}$. The condition $\dot{E}_c(t) = 0$ implies, from (44), that $v(t) = 0$; and hence, $\dot{v}(t) = 0$. Then by (28) we must have that $B_\alpha y(t) = 0$. Since $\alpha(s)$ is SPR, we have that $\alpha(jw) + \alpha^T(-jw) > 0$. This implies that

$$\begin{aligned} & y^T(t)[\alpha(jw) + \alpha^T(-jw)]y(t) > 0 \\ \Rightarrow \quad & y^T(t)[D_\alpha + D_\alpha^T]y(t) > 0 \quad \Rightarrow \quad K^T K > 0. \end{aligned}$$

In the second step the facts $\alpha(jw) = C_\alpha(jw - A_\alpha)^{-1} B_\alpha + D_\alpha$ and $B_\alpha y(t) = 0$ were used, and in the third step we used (30). Since $v(t) = 0$ and $K^T K > 0$ it follows from (44) that $y(t) = 0$, and hence by (28) we also obtain $y_\alpha(t) = 0$.

Therefore from (31) and (34) it follows that the invariant solution of (31) in $\mathscr{O}_c$ reduces to the invariant solution of the associated open-loop system in the domain (17)-(18). The rest of the proof follows from the proof of Theorem 4.4. ■

VI. Scattering energy-preserving systems

Here we use the term 'scattering energy-preserving system' in the sense of [3]. In that paper the author shows that a scattering energy-preserving system satisfies the relation

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2$$

for $u \in C^2((0,\infty);\mathbb{R}^{nN})$, $x(0) \in D(\mathscr{J})$ and, $\mathscr{B}x(0) = u(0)$. In [2] it is shown that, in this case, $V = 0$, $\tilde{P}_1 = \tilde{P}_2 = I$, $S^{-T}S^{-1} = 4I$, and $S_2^{-T}S_2^{-1} = 4I$. This means that W has the form $W = S\begin{bmatrix} I & I \end{bmatrix}$.

For this type of systems related to the operator $\mathscr{J}$ we have the following result.

Theorem 6.1: Consider the system described in Theorem 2.2 with $\mathscr{J}e = P_1\frac{de}{dz}(z)$ and $\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2$. Then the system is exponentially stable and exactly observable in finite time.

Proof: First observe that W has the form $W = S\begin{bmatrix} I & I \end{bmatrix}$. In order to check stability of the system we study the resolvent of $A_W = \mathscr{J}_{|D(A_W)}$. Let $x \in D(A_W)$ and observe that $y(z) = (\lambda - A_W)x(z) = \lambda x(z) - P_1\frac{dx}{dz}(z)$. Thus $\frac{dx}{dz}(z) = \lambda P_1^{-1}x(z) - P_1^{-1}y(z)$ need to be solved. The general solution is given by

$$x(z) = \mathrm{e}^{\lambda P_1^{-1}(z-a)}c - \int_a^z \mathrm{e}^{\lambda P_1^{-1}(z-\tau)}P_1^{-1}y(\tau)\,d\tau \qquad (45)$$

where c is a constant vector. Since $x \in D(A_W)$ the boundary conditions (see Theorem 2.2) are described by (see (4))

$$WR_{ext}\begin{bmatrix} x(b) \\ x(a) \end{bmatrix} = \begin{bmatrix} P_1 + I & I - P_1 \end{bmatrix}\begin{bmatrix} x(b) \\ x(a) \end{bmatrix} = 0.$$

Using (45) in the equation above yields

$$\begin{aligned}&(P_1+I)\left[\mathrm{e}^{\lambda P_1^{-1}(b-a)}c - \int_a^b \mathrm{e}^{\lambda P_1^{-1}(b-\tau)}P_1^{-1}y(\tau)\,d\tau\right]\\ &\quad + (I-P_1)c = 0\\ \iff &\left[(P_1+I)\mathrm{e}^{\lambda P_1^{-1}(b-a)} + (I-P_1)\right]c =\\ &(P_1+I)\int_a^b \mathrm{e}^{\lambda P_1^{-1}(b-\tau)}P_1^{-1}y(\tau)\,d\tau.\end{aligned}$$

It follows from Lemma 14 of [8] that when $\lambda \in \mathbb{C}\backslash\mathbb{R}^-$ the matrix on the left hand side is nonsingular. In that case, c can be defined uniquely, which implies that $(\lambda - A_W)^{-1}$ exists. Also, it is not difficult to show that $(\lambda - A_W)^{-1}$ is bounded, hence Equation (45) defines the resolvent operator of A_W.

It is easy to show that the resolvent (45) is uniformly bounded (see the last paragraph of Subsection IV-A). Since A_W generates a contraction semigroup it follows from Corollary 3.36 of [5] that the system is exponentially stable. Finally, from Theorem 11.3.8 of [9] one can see that the system is also exactly observable in finite time. ■

Lemma 6.2: Consider the system of Theorem 2.2 with $\mathscr{J}e = P_1\frac{de}{dz}(z)$ and $\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2$. Then the resolvent $(\lambda - A_W)^{-1}$, with $\lambda \geq 0$, is a compact operator, where $A_W = \mathscr{J}_{|D(A_W)}$. Furthermore, the operator $\tilde{\mathscr{A}}_{\mathscr{W}}e = P_0e(z) + P_1\frac{de}{dz}(z)$ has compact resolvent for $\lambda > 0$, i.e., $(\lambda - \tilde{A}_W)^{-1}$ is a compact operator.

Proof: From Theorem 6.1 we know that the resolvent of A_W is given by (45) for $\lambda \geq 0$. The integral operator on the RHS is clearly compact, see Theorem A.3.52 of [4]. The exponential operator $\mathrm{e}^{\lambda P_1^{-1}(z-a)}c$ is also compact, since it is a finite rank operator, see Theorem 8.1-4 of [6]. Since $(\lambda - A_W)^{-1}$ in (45) is the sum of two compact operators we have that it is also compact for $\lambda \geq 0$, see [6, p. 407].

Using Lemma A-3 we can conclude that $(\lambda - \tilde{A}_W)^{-1}$ is compact for $\lambda > 0$ since $\tilde{A}_W$ is also the generator of a contraction semigroup. ■

Theorem 6.3: Consider the system of Theorem 2.2 with $\mathscr{J}e = P_0e(z) + P_1\frac{de}{dz}(z)$ and $\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2$. Then the system is globally asymptotically stable.

Proof: From Lemma 6.2 we know that $(\lambda - A_W)^{-1}$, with $\lambda > 0$, is a compact operator, where $A_W = \mathscr{J}_{|D(A_W)}$. The rest of the proof follows similar to the proof of Theorem 4.4, noting that $\dot{E}(t) = -\|y(t)\|^2$. ■

APPENDIX

Lemma A-1: Let $S_2, V \in \mathbb{R}^{n\times n}$ and $R_{\text{ext}}, \Sigma \in \mathbb{R}^{2n\times 2n}$ where Σ is given by (5). Consider the matrices W_{imp} and C_{imp} given by (13) and (14), respectively. If V satisfies $VV^T = I = V^TV$ (impedance passive system), then we have

$$C_{\text{imp}}^T W_{\text{imp}} + W_{\text{imp}}^T C_{\text{imp}} = \Sigma, \qquad \text{(A-1)}$$

$$C_{\text{imp}}C_{\text{imp}}^T = 4S_2S_2^T, \quad W_{\text{imp}}W_{\text{imp}}^T = \frac{1}{4}S_2^{-T}S_2^{-1}, \qquad \text{(A-2)}$$

$$C_{\text{imp}}W_{\text{imp}}^T = 0, \quad \text{and} \quad W_{\text{imp}}C_{\text{imp}}^T = 0. \qquad \text{(A-3)}$$

Proof: It follows easily by using algebra and $VV^T = I$. ■

Lemma A-2: Consider the matrices W_{imp} and C_{imp} given by (13) and (14), respectively. Let V and D be $n \times n$ matrices, with $VV^T = I = V^TV$. Then, the matrix given by

$$\widetilde{W} = W_{\text{imp}} + DC_{\text{imp}}$$

has full row-rank.

Proof: It is easy to show that the matrix $\widetilde{W}\widetilde{W}^T$ is nonsingular. ■

The following lemma is not difficult to prove.

Lemma A-3: Let $A = A_u + A_b$ be a closed and densely defined operator on X, where A_b is a bounded operator on X and A_u is an unbounded operator with $D(A_u) = D(A)$. If A satisfies Re $\langle Az, z\rangle \leq 0$ and $(\lambda - A_u)^{-1}$ is a compact operator for $\lambda > 0$. Then the operator $(\lambda - A)^{-1}$ is a compact operator for $\lambda > 0$.

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