

# Stability and Stabilization of a Class of Boundary Control Systems

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**Abstract**—We study a class of partial differential equations on a one dimensional spatial domain with control and observation at the boundary. For this class of systems we describe how to obtain an impedance energy-preserving system, as well as scattering energy-preserving system. For the first type of systems we consider (static and dynamic) feedback stabilization by means of boundary control. For the scattering energy-preserving systems we give conditions for which the system is either asymptotically or exponentially stable.

## I. INTRODUCTION

In this paper we study the following partial differential equation (PDE) on the spatial interval  $[a, b]$

$$\dot{x}(t) = \mathcal{J}x(t), \quad x(0) = x_0 \quad (1a)$$

$$u(t) = \mathcal{B}x(t), \quad (1b)$$

$$y(t) = \mathcal{C}x(t), \quad (1c)$$

here

$$\mathcal{J}x = \sum_{i=0}^N P(i) \frac{d^i x}{dz^i}(z) \quad z \in [a, b], \quad (2)$$

with the domain of  $\mathcal{J}$  being  $H^N((a, b); \mathbb{R}^n)$ , i.e., the Sobolev space of  $N$  times differentiable functions on the interval  $(a, b)$ .  $P(i)$ ,  $i = 0, \dots, N$ , is a  $n \times n$  real matrix satisfying

$$P(i) = P(i)^T (-1)^{i+1}, \quad \text{and} \quad \ker P(N) = \{0\}. \quad (3)$$

For the PDE (1) we describe how to obtain impedance energy-preserving systems (systems that satisfy  $\frac{1}{2} \frac{d}{dt} \|x(t)\|^2 = u(t)^T y(t)$ ), as well as scattering energy-preserving ( $\frac{1}{2} \frac{d}{dt} \|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2$ ). Next we study feedback (static and dynamic) stabilization for impedance energy-preserving systems. Finally, we study (asymptotic and exponential) stability of scattering energy-preserving.

Here we use the notation  $\left[ \begin{smallmatrix} X \\ Y \end{smallmatrix} \right]$  for  $X \times Y$  and  $F|_{\mathcal{D}}$  denotes the restriction of an operator  $F$  to the subspace  $\mathcal{D}$ .  $\rho(F)$  denotes the resolvent set of  $F$  and  $\partial_z^N$  indicates the  $N$ -times partial derivative with respect to the variable  $z$ .  $I$  is the identity operator.

## II. SOME BACKGROUND

Most of the results described in this section can be found in [1]. First we need to introduce some notation. The  $nN \times nN$  matrix  $Q$  is defined as

$$Q = \begin{pmatrix} P(1) & P(2) & P(3) & \dots & P(N-1) & P(N) \\ -P(2) & -P(3) & -P(4) & \dots & P(N) & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ (-1)^{N-1} P(N) & 0 & \dots & \dots & 0 & 0 \end{pmatrix}.$$

Using this  $Q$ , the matrix  $R_{\text{ext}}$  is defined as

$$R_{\text{ext}} = \frac{1}{\sqrt{2}} \begin{pmatrix} Q & -Q \\ I & I \end{pmatrix}. \quad (4)$$

Since  $P(N)$  is invertible, we have that  $R_{\text{ext}}$  is invertible as well. It is easy to see that it satisfies

$$R_{\text{ext}}^T \Sigma R_{\text{ext}} = \begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix}, \quad \text{where} \quad \Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \quad (5)$$

*Definition 2.1:* The boundary port variables associated with the differential operator  $\mathcal{J}$  and the function  $x \in H^N((a, b); \mathbb{R}^n)$  are the vectors  $e_\partial, f_\partial \in \mathbb{R}^{nN}$ , defined by

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = R_{\text{ext}} \begin{pmatrix} x(b) \\ \vdots \\ \partial_z^{N-1} x(b) \\ x(a) \\ \vdots \\ \partial_z^{N-1} x(a) \end{pmatrix}, \quad (6)$$

where  $R_{\text{ext}}$  is defined by (4).

In [1], the authors prove the following theorem.

*Theorem 2.2:* Let

$$W = S \begin{bmatrix} I + V & I - V \end{bmatrix}, \quad (7)$$

with  $S$  invertible and  $V V^T \leq I$ , be a full rank matrix of size  $nN \times 2nN$  (satisfying  $W \Sigma W^T \geq 0$ ), and define  $\mathcal{B} : H^N((a, b), \mathbb{R}^n) \rightarrow \mathbb{R}^{nN}$  as

$$\mathcal{B}x(t) := W \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix}. \quad (8)$$

Then the system (1a)-(1b) is a boundary control system.  $A_W := \mathcal{J}|_{\ker \mathcal{B}}$  is the generator of a contraction semigroup and

$$D(A_W) = \{x \in L^2((a, b), \mathbb{R}^n) \mid \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \in \ker W\}.$$

Furthermore, if we define the output via the mapping  $\mathcal{C} : H^N((a, b), \mathbb{R}^n) \rightarrow \mathbb{R}^{nN}$  as

$$y(t) = \mathcal{C}x(t) := S_2 \begin{pmatrix} I - V^T & -I - V^T \end{pmatrix} \begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \quad (9)$$

with  $S_2$  invertible, then for  $u \in C^2((0, \infty); \mathbb{R}^{nN})$ ,  $x(0) \in D(\mathcal{J})$ , and  $\mathcal{B}x(0) = u(0)$  the following balance equation is satisfied:

$$\frac{1}{2} \frac{d}{dt} \|x(t)\|^2 = \begin{pmatrix} u^T(t) & y^T(t) \end{pmatrix} P_W \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, \quad (10)$$

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where  $P_W$  is given by

$$\frac{1}{4} \begin{pmatrix} S^{-T}(\tilde{P}_1^2 - \tilde{P}_1 V V^T \tilde{P}_1) S^{-1} & -2S^{-T} \tilde{P}_1 V \tilde{P}_2 S_2^{-1} \\ -2S_2^{-T} \tilde{P}_2 V^T \tilde{P}_1 S^{-1} & S_2^{-T}(-\tilde{P}_2^2 + \tilde{P}_2 V^T V \tilde{P}_2) S_2^{-1} \end{pmatrix}, \quad (11)$$

$$\text{and } \tilde{P}_1 = (I + V V^T)^{-1}, \quad \tilde{P}_2 = (I + V^T V)^{-1}.$$

For more information see [1] and [2].

### III. IMPEDANCE ENERGY-PRESERVING SYSTEMS

Here we use the term ‘impedance energy-preserving system’ in the sense of [3]. In that paper the author shows that an impedance energy-preserving system satisfies the relation

$$\frac{1}{2} \frac{d}{dt} \|x(t)\|^2 = u(t)^T y(t) \quad (12)$$

for  $u \in C^2((0, \infty); \mathbb{R}^{nN})$ ,  $x(0) \in D(\mathcal{J})$  and,  $\mathcal{B}x(0) = u(0)$ . In [1] the authors show that for an impedance energy-preserving system we have  $V^T V = V V^T = I$ ,  $A_W^* = -A_W$  and  $D(A_W) = D(A_W^*)$ . In [2] it is shown that, in this case, the inputs can be described by

$$u = \frac{1}{4} S_2^{-T} [-I - V^T, I - V^T] \begin{bmatrix} f_{\partial} \\ e_{\partial} \end{bmatrix} = W_{\text{imp}} \begin{bmatrix} f_{\partial} \\ e_{\partial} \end{bmatrix} \quad (13)$$

and the outputs by

$$y = S_2 \begin{bmatrix} I - V^T, & -I - V^T \end{bmatrix} \begin{bmatrix} f_{\partial} \\ e_{\partial} \end{bmatrix} = C_{\text{imp}} \begin{bmatrix} f_{\partial} \\ e_{\partial} \end{bmatrix}. \quad (14)$$

Here, the state space is  $X = L_2((a, b); \mathbb{R}^n)$  and the input and output spaces are  $U = Y = \mathbb{R}^n$ . Furthermore, we know that  $A_W = \mathcal{J}|_{D(A_W)}$  is the generator of a contraction semigroup (see Theorem 2.2) with  $D(A_W) = \ker \mathcal{B}$ .

### IV. STATIC FEEDBACK OF AN IMPEDANCE ENERGY-PRESERVING SYSTEM

In this section we apply feedback (see Figure 1), i.e.,

$$u = r - \alpha y, \quad (15)$$

where  $r, u, y \in \mathbb{R}^{nN}$  and  $\alpha > 0$  is a positive definite matrix.

We have that the plant is described by equations (1a)–(1c), where  $\mathcal{B}x(t)$  is given by (13),  $\mathcal{C}x(t)$  is given by (14) and the differential operator  $\mathcal{J}$  when restricted to  $D(\mathcal{J}) \cap \ker(\mathcal{B})$  generates a  $C_0$ -semigroup.

Using the feedback control (15) we can see that the closed-loop system is now described by

$$\begin{aligned} \dot{x}(t) &= \mathcal{J} x(t) \\ (W_{\text{imp}} + \alpha C_{\text{imp}}) \begin{bmatrix} f_{\partial}(t) \\ e_{\partial}(t) \end{bmatrix} &= (\mathcal{B} + \alpha \mathcal{C}) x(t) = r(t) \\ \mathcal{C} x(t) &= y(t). \end{aligned} \quad (16)$$

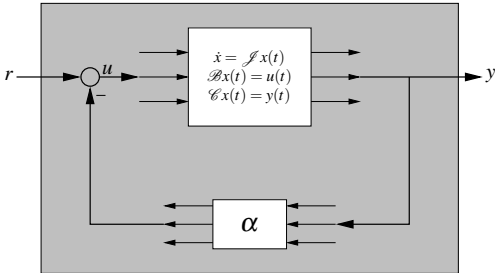


Fig. 1. General control system.

*Lemma 4.1:* The system described by (16) (with  $V V^T = I$ ) is a boundary control system. Furthermore, the operator  $A_s = \mathcal{J}|_{D(A_s)}$  generates a contraction semigroup on  $X = L_2((a, b); \mathbb{R}^n)$ , where

$$D(A_s) = \left\{ x \in D(\mathcal{J}) \mid \begin{bmatrix} f_{\partial} \\ e_{\partial} \end{bmatrix} \in \ker \tilde{W} \right\} \quad \text{and} \quad (17)$$

$$\tilde{W} = (W_{\text{imp}} + \alpha C_{\text{imp}}) \quad (18)$$

is a full rank  $nN \times 2nN$  matrix.

*Proof:* First observe that (18) follows from (16). From Theorem 2.2 we can see that if  $\tilde{W}$  satisfies  $\tilde{W} \Sigma \tilde{W}^T \geq 0$  ( $\Sigma$  given by (5)), then we will be done. Since  $V$  is unitary we can use Lemma A-1 to show that  $\tilde{W}$  satisfies  $\tilde{W} \Sigma \tilde{W}^T = \alpha + \alpha^T > 0$  (since  $\alpha > 0$ ). ■

*Remark 4.2:* Using Lemma A.1 of [1] we can see that (18) can be factorized as

$$\tilde{W} = \tilde{S} \begin{bmatrix} I + \tilde{V}, & I - \tilde{V} \end{bmatrix}. \quad (19a)$$

Furthermore, from step 1 and 2 of the proof of Theorem 4.3 of [1] we can show that

$$\langle A_s x, x \rangle < 0, \forall x \in D(A_s), \quad \langle A_s^* x, x \rangle < 0, \forall x \in D(A_s^*). \quad (19b)$$

*Remark 4.3:* Following the same procedure used to prove (10)–(11), see [1, p.19], we can show that if  $r(t) = 0$ ,  $x(0) \in D(\mathcal{J})$ , and  $(\mathcal{B} + \alpha \mathcal{C})x(0) = 0$ , then

$$\frac{1}{2} \frac{d}{dt} \|x(t)\|^2 = -y(t)^T \alpha y(t). \quad (20)$$

Next we study stability of the closed-loop system.

*Theorem 4.4:* Assume that  $(\lambda - A_s)^{-1} : X \rightarrow X$  is a compact operator for  $\lambda > 0$ . Then the system described by (16) (with  $V V^T = I$  and  $r = 0$ ), is asymptotically stable.

*Remark 4.5:* It can be said that most of the examples encountered in the literature satisfy the assumption in the theorem above, see, e.g., [5, p. 269].

*Proof:* [Proof of Theorem 4.4] By Lemma 4.1 we know that  $A_s$  generates a contraction semigroup. Thus, for any  $x(0) \in X$ , the solution  $x(t) = T(t)x(0)$  (classical or weak) is bounded in  $X$ . Since  $(\lambda - A_s)^{-1}$  is a compact operator for  $\lambda > 0$ , it follows that the trajectory of the solution  $x(t)$ , i.e. the set  $\gamma(x(0)) = \{x(t) \in X, t \geq 0\}$  is precompact in  $X$ , see Theorem 3.65 of [5]. It then follows that the  $w$ -limit set<sup>1</sup>  $w(x(0))$  of the trajectory is nonempty, compact, and we have  $x(t) \rightarrow w(x(0))$  as  $t \rightarrow \infty$ , see Theorem 3.61 of [5].

Next we show that  $w(x(0))$  contains only the point zero. First we prove this for  $x(0) \in D(A_s)$ . In this case we have that  $x(t) = T(t)x(0) \in D(A_s)$  for all  $t \geq 0$ , see Theorem 2.1.10 of [4]. Define the energy function

$$E(t) = \frac{1}{2} \|x(t)\|^2. \quad (21)$$

Since  $x(0) \in D(A_s)$  we have that  $x(t)$  is differentiable, see [4, §2.1]. Thus the derivative of  $E(t)$  is given by (20) with  $r(t) = 0$ , that is  $\dot{E}(t) = \frac{1}{2} \frac{d}{dt} \|x(t)\|^2 = -y(t)^T \alpha y(t)$ . Since  $\alpha$  is a positive definite matrix we have that  $\dot{E}(t) < 0$ , which shows that  $E(t)$  is a Lyapunov function, see Definition 3.62 of [5]. Observe that  $\dot{E}(t) = 0$  implies  $y(t) = 0$ . Now consider the

<sup>1</sup>The  $w$ -limit set of  $x$  is given by  $w(x) = \{y \in F \mid y = \lim_{n \rightarrow \infty} T(t_n)x \text{ with } t_n \rightarrow \infty \text{ as } n \rightarrow \infty\}$ , where  $F$  is a closed subset of  $X$ .

set  $\{x \in X \mid \dot{E}(t) = 0\}$  or equivalently  $\mathcal{O} = \{x \in X \mid y(t) = 0\}$  and let  $\mathcal{E}$  be its largest invariant subset. Since  $\gamma(x(0))$  is precompact, it follows from LaSalle's principle that  $x(t) \rightarrow \mathcal{E}$  as  $t \rightarrow \infty$ , see Theorem 3.64 of [5]. We show that  $\mathcal{E} = \{0\}$ . Since  $x(0)$  and  $x(t)$  belong to  $D(A_s)$ , we have that  $\mathcal{E} \subset D(A_s)$ , see [5, p.270]. Let  $\tilde{x} \in \mathcal{E}$  and let  $z(t) = T(t)\tilde{x}$  be the corresponding solution. Since  $\mathcal{E}$  is invariant we have that  $z(t) \in \mathcal{E}$  for  $t \geq 0$ . Recall that all  $x \in \mathcal{O}$  satisfy  $y(t) = 0$ . It then follows that  $z(t)$  is the solution of  $\dot{z}(t) = \mathcal{J}z(t)$  satisfying  $r(t) = y(t) = 0$ . Since this is a PDE with all boundary variables set to zero, we must have that the only solution is  $z(t) = 0$  for  $t \geq 0$ . Hence  $\tilde{x} = 0$ ; and thus,  $\mathcal{E} = \{0\}$ . Altogether means that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$  for any  $x(0) \in D(A_s)$ .

The same statement holds for  $x(0) \in X$ , see [5, p.270]. ■

#### A. First order differential operator (case $N = 1$ )

In this section we study the differential operator

$$\mathcal{J}x = P_0x(z) + P_1 \frac{dx}{dz}(z). \quad (22)$$

This case includes the well-known beam and wave equations. We assume that the input and outputs have been chosen so that the resulting system is impedance energy-preserving, i.e.,  $\frac{1}{2} \frac{d}{dt} \|x(t)\|^2 = u(t)^T y(t)$  holds. We want to apply static feedback to the resulting BCS, see Figure 1. We already know that, in this case, the closed-loop system is a BCS and that the operator  $A_s = \mathcal{J}|_{D(A_s)}$  generates a contraction semigroup, see Lemma 4.1. We want to check whether the closed-loop system is asymptotically stable. To do so we need to check that the resolvent  $(\lambda - A_s)^{-1}$  is compact for  $\lambda > 0$ , see Theorem 4.4.

First we study the eigenvalues of  $A_s$ . From (19b) we know that  $\langle A_s x, x \rangle < 0$  for all  $x \in D(A_s)$ . Let  $x_1$  be any eigenvector of  $A_s$  with corresponding eigenvalue  $\lambda$ . Then we have that

$$\operatorname{Re} \langle A_s x_1, x_1 \rangle = \operatorname{Re} \langle \lambda x_1, x_1 \rangle = \operatorname{Re} \lambda \|x_1\|^2 < 0.$$

This implies that all eigenvalues of  $A_s$  satisfy  $\operatorname{Re} \lambda < 0$ . Moreover, we have that this eigenvector  $x_1$  is the solution of

$$P_0 x_1 + P_1 \frac{dx_1}{dz} = \lambda x_1 \iff \frac{dx_1}{dz} = P_1^{-1}(\lambda - P_0)x_1.$$

The general solution of the equation above is given by

$$x_1(z) = e^{P_1^{-1}(\lambda - P_0)(z-a)} c \quad (23)$$

where  $c$  is a constant vector. Using (19) and the boundary conditions on  $D(A_s)$ , see (17), we get

$$\begin{aligned} \tilde{W} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} &= \tilde{W} R_{\text{ext}} \begin{bmatrix} x_1(b) \\ x_1(a) \end{bmatrix} = 0 \\ \iff \tilde{S} \begin{bmatrix} I + \tilde{V} & I - \tilde{V} \end{bmatrix} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} x_1(b) \\ x_1(a) \end{bmatrix} &= 0 \end{aligned} \quad (24)$$

where we used Definition 2.1 and (19a). Using (23) in the above equation gives

$$\begin{aligned} \left( \begin{bmatrix} (I + \tilde{V})P_1 + (I - \tilde{V}) \end{bmatrix} e^{P_1^{-1}(\lambda - P_0)(b-a)} \right. \\ \left. + \begin{bmatrix} -(I + \tilde{V})P_1 + (I - \tilde{V}) \end{bmatrix} \right) c = 0. \end{aligned} \quad (25)$$

We know that  $\lambda$  is an eigenvalue of  $A_s$  iff the matrix above is singular. At the beginning of this subsection we showed that the eigenvalues of  $A_s$  satisfy  $\operatorname{Re} \lambda < 0$ , which shows that the matrix above is nonsingular if  $\operatorname{Re} \lambda \geq 0$ , otherwise  $A_s$  would have eigenvalues with  $\operatorname{Re} \lambda \geq 0$ . In summary we have the following result.

*Lemma 4.6:* Consider the system described in Lemma 4.1 with  $\mathcal{J}$  given by (22). Then the eigenvalues of  $A_s$  satisfy  $\operatorname{Re} \lambda < 0$ . Furthermore, if  $\lambda \geq 0$  the matrix on (25) is nonsingular and  $(\lambda - A_s)^{-1}$  is a compact operator.

*Proof:* That the eigenvalues of  $A_s$  satisfy  $\operatorname{Re} \lambda < 0$  and that the matrix is nonsingular was proved above.

Next we study the resolvent of  $A_s$ . First we study the range of  $(\lambda - A_s) : D(A_s) \rightarrow X$  for  $\lambda \geq 0$ . To see this, consider  $(\lambda - A_s)x = y$ , which is equivalent to solve

$$\frac{dx}{dz}(z) = P_1^{-1}(\lambda - P_0)x(z) - P_1^{-1}y(z) \quad (26)$$

for  $x \in D(A_s)$ . The general solution of (26) is given by

$$x(z) = e^{P_1^{-1}(\lambda - P_0)(z-a)} c - \int_a^z e^{P_1^{-1}(\lambda - P_0)(z-\tau)} P_1^{-1}y(\tau) d\tau \quad (27)$$

with  $c$  a constant vector. Since  $x \in D(A_s)$  the boundary conditions (see (17)) are given by (24). Using (27) in (24) yields

$$\begin{aligned} \left( \begin{bmatrix} (I + \tilde{V})P_1 + (I - \tilde{V}) \end{bmatrix} e^{P_1^{-1}(\lambda - P_0)(b-a)} \right. \\ \left. + \begin{bmatrix} -(I + \tilde{V})P_1 + (I - \tilde{V}) \end{bmatrix} \right) c = \\ \begin{bmatrix} (I + \tilde{V})P_1 + (I - \tilde{V}) \end{bmatrix} \int_a^b e^{P_1^{-1}(\lambda - P_0)(b-\tau)} P_1^{-1}y(\tau) d\tau. \end{aligned}$$

We already showed that when  $\operatorname{Re} \lambda \geq 0$  the matrix on the LHS is nonsingular. In that case,  $c$  can be defined uniquely, which implies that (26) has a unique solution in  $D(A_s)$ . This solution is given by (27). Hence  $(\lambda - A_s)^{-1}$  exists when  $\operatorname{Re} \lambda \geq 0$ . In order to prove that (27) defines the resolvent of  $A_s$  we need to show that it is bounded. First observe that the dimension of the domain of  $e^{P_1^{-1}(\lambda - P_0)(z-a)} c$  is finite, thus it is a compact operator (hence bounded), see [6, Th. 2.6-9 b, Th. 8.1-4, Th. 8.1-2 b]. Also, the integral operator on (27) is compact, see Theorem A.3.52 of [4]. Then,  $(\lambda - A_s)^{-1}$  given by (27) is the sum of two compact operators (see see [6, p. 407]), hence  $(\lambda - A_s)^{-1}$  is a compact operator when  $\operatorname{Re} \lambda \geq 0$ . ■

From the Lemma above and Theorem 4.4 we know that this type of closed-loop systems (when  $N = 1$ ) are asymptotically stable. Also, if  $P_0 = 0$  we can see that when  $\lambda$  is a pure imaginary number, the resolvent (27) is clearly a uniformly bounded operator since  $P_1$  has only real eigenvalues and hence the magnitude of  $e^{\lambda P_1^{-1}z}$  is equal to one. Since  $A_s$  generates a contraction semigroup it follows that the conditions on Corollary 3.36 of [5] are satisfied, which proves that, in this case, the closed-loop is exponentially stable.

## V. DYNAMIC FEEDBACK OF AN IMPEDANCE ENERGY-PRESERVING SYSTEM

In this section we generalize the class of static controllers described in the previous section. More precisely, we replace

the static matrix  $\alpha$  with a matrix transfer function  $\alpha(s)$  where  $s \in \mathbb{C}$  is a complex variable.

The state space representation of the controller is given by

$$\begin{aligned} \dot{v}(t) &= A_\alpha v(t) + B_\alpha y(t) \\ y_\alpha(t) &= C_\alpha v(t) + D_\alpha y(t) \end{aligned} \quad (28)$$

where  $v \in \mathbb{R}^m$  is the state of the minimal realization. In this way, equation (15) becomes

$$u = r - y_\alpha = r - C_\alpha v - D_\alpha \mathcal{C} x, \quad (29)$$

where (1c) was used.

In the literature, an  $n \times n$  rational matrix  $H(s)$  is said to be **positive real (PR)** if: i) all elements of  $H(s)$  are analytic in the open right-half plane  $\text{Re}(s) > 0$ , ii) poles of any element of  $H(s)$  on the  $iw$ -axis are distinct, and the associated residue matrix of  $H(s)$  is  $\geq 0$ , iii)  $H(jw) + H^T(-jw) \geq 0 \forall w$  which is not a pole of any element of  $H(jw)$ .

*Definition 5.1 (Tao and Ioannou [7]):* A rational matrix  $H(s)$  is **strictly positive real (SPR)** if  $H(s - \varepsilon)$  is positive real (PR) for some  $\varepsilon > 0$ .

The next lemma is used in the stability analysis (see [7]).

*Lemma 5.2:* Assume that the transfer matrix  $H(s)$  has all its poles in  $\text{Re}(s) < -\gamma$ , where  $\gamma > 0$  and  $(A, B, C, D)$  is a minimal realization of  $H(s)$ . Then  $H(s - \gamma)$  is PR if and only if there exist matrices  $P, Q$  and  $K$  such that  $P = P^T > 0$  and

$$PA + A^T P = -QQ^T - 2\gamma P; \quad PB = C^T - QK; \quad K^T K = D + D^T. \quad (30)$$

Throughout this section the controller is assumed to be SPR. Let  $x \in X$  be the state of the plant,  $v \in \mathbb{R}^m$  the state of the controller, and  $w = \begin{bmatrix} x \\ v \end{bmatrix}$ . Using the feedback control (29) and the fact that  $u(t) = \mathcal{B}x(t)$ , see (1b), we can see that the closed-loop system is now described by

$$\begin{aligned} \dot{w}(t) &= \mathcal{J}_c w(t), \quad w(0) \in \tilde{X} \\ \begin{bmatrix} \mathcal{B} + D_\alpha \mathcal{C} & C_\alpha \end{bmatrix} w(t) &= r(t) \\ \begin{bmatrix} \mathcal{C} & 0 \end{bmatrix} w(t) &= y(t), \end{aligned} \quad (31)$$

where  $\tilde{X} = \begin{bmatrix} X \\ \mathbb{R}^m \end{bmatrix}$  is the state space of the closed-loop system,  $w = \begin{bmatrix} x \\ v \end{bmatrix} \in \tilde{X}$ , and  $\mathcal{J}_c : \tilde{X} \rightarrow \tilde{X}$  is a linear operator defined as

$$\mathcal{J}_c w = \begin{bmatrix} \mathcal{J} & 0 \\ B_\alpha \mathcal{C} & A_\alpha \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \quad (32)$$

with  $D(\mathcal{J}_c) = D(\mathcal{J}) \oplus \mathbb{R}^m$ . The inner product on the space  $\tilde{X}$  is defined as

$$\langle w_1, w_2 \rangle_{\tilde{X}} = \langle x_1, x_2 \rangle_X + \frac{1}{2} v_1^T P v_2 + \frac{1}{2} v_2^T P v_1, \quad (33)$$

where  $P$  is the positive definite matrix found in Lemma 5.2.

*Lemma 5.3:* Let the state of the open-loop system of Figure 1 satisfy  $\frac{1}{2} \frac{d}{dt} \|x(t)\|^2 = u(t)^T y(t)$  and the controller  $\alpha(s)$  be SPR. Then the system (31)-(32) is a boundary control system. Furthermore, the operator  $A_c$  defined by

$$A_c w = \begin{bmatrix} \mathcal{J} & 0 \\ B_\alpha \mathcal{C} & A_\alpha \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \quad (34a)$$

with

$$D(A_c) = \left\{ \begin{bmatrix} x \\ v \end{bmatrix} \in \begin{bmatrix} X \\ \mathbb{R}^m \end{bmatrix} \mid x \in D(\mathcal{J}), \text{ and } \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \in \ker \tilde{W}_D \right\}, \quad (34b)$$

where

$$\tilde{W}_D = \begin{bmatrix} (W_{\text{imp}} + D_\alpha C_{\text{imp}}), & C_\alpha \end{bmatrix}, \quad (34c)$$

generates a contraction semigroup.

*Proof:* First we need to prove that there exists an operator  $B \in \mathcal{L}(U, \tilde{X})$  such that for all  $r \in U$ ,  $Br \in D(\mathcal{J}) \times \mathbb{R}^m$ , and  $\begin{bmatrix} \mathcal{B} + D_\alpha \mathcal{C} & C_\alpha \end{bmatrix} Br = r$ . From the proof of Theorem 4.5 of [1] we know that if the matrix  $\tilde{W}_D$  has full rank, then such operator  $B$  exists. Thus we need to prove that  $\tilde{W}_D$  has full rank. Since the open-loop system is assumed to be impedance energy-preserving, we must have  $VV^T = I$ . From this and Lemma A-2 we can see that the matrix  $(W_{\text{imp}} + D_\alpha C_{\text{imp}})$  has full row-rank. Hence, we can conclude that  $\tilde{W}_D$  in (34c) has also full row-rank.

Equations (34) follow easily from (31) and the proof of Lemma 4.1. Next we need to prove that  $A_c$  generates a semigroup. We will use the Lumer-Phillips theorem (see Theorem 2.27 of [5]). First we prove that  $\langle A_c w, w \rangle \leq 0$ . Let  $w = \begin{bmatrix} x \\ v \end{bmatrix} \in D(A_c)$ , then we have

$$\begin{aligned} \langle A_c w, w \rangle_{\tilde{X}} &= \langle \mathcal{J} x, x \rangle_X + \frac{1}{2} (A_\alpha v + B_\alpha y)^T P v \\ &\quad + \frac{1}{2} v^T P (A_\alpha v + B_\alpha y) \\ &= \langle \mathcal{J} x, x \rangle_X + \frac{1}{2} v^T (A_\alpha^T P + P A_\alpha) v + \frac{1}{2} y^T B_\alpha^T P v + \frac{1}{2} v^T P B_\alpha y. \end{aligned}$$

From Equation (4.8) of [1] and Lemma 5.2 we obtain

$$\begin{aligned} &= \frac{1}{2} \begin{bmatrix} f_\partial^T & e_\partial^T \end{bmatrix} \Sigma \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} + \frac{1}{2} v^T (-QQ^T - 2\gamma P) v \\ &\quad + \frac{1}{2} y^T (C_\alpha - K^T Q^T) v + \frac{1}{2} v^T (C_\alpha^T - QK) y. \end{aligned}$$

Using (A-1) together with (13) and (14) yields

$$\begin{aligned} &= \frac{1}{2} y^T u + \frac{1}{2} u^T y + \frac{1}{2} v^T (-QQ^T - 2\gamma P) v \\ &\quad + \frac{1}{2} y^T (C_\alpha - K^T Q^T) v + \frac{1}{2} v^T (C_\alpha^T - QK) y. \end{aligned}$$

Since  $w = \begin{bmatrix} x \\ v \end{bmatrix} \in D(A_c)$  we have that  $C_\alpha v = -(W_{\text{imp}} + D_\alpha C_{\text{imp}}) \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}$ , see (34), and using again (13) and (14) gives, after simplification

$$\begin{aligned} &= \frac{1}{2} v^T (-QQ^T - 2\gamma P) v - \frac{1}{2} y^T D_\alpha y - \frac{1}{2} y^T D_\alpha^T y \\ &\quad - \frac{1}{2} y^T K^T Q^T v - \frac{1}{2} v^T QK y \end{aligned}$$

and using again Lemma 5.2 yields

$$\langle A_c w, w \rangle_{\tilde{X}} = -\gamma v^T P v - \frac{1}{2} (Ky + Q^T v)^T (Ky + Q^T v). \quad (35)$$

Since  $\gamma > 0$  and  $P$  is positive definite it thus follows from the equation above that  $\langle A_c w, w \rangle_{\tilde{X}} \leq 0$ .

Next we need to prove that the range of  $(I - A_c)$  is equal to  $\tilde{X}$ . In order to do so, we can show that for all  $\begin{bmatrix} f \\ z \end{bmatrix} \in \begin{bmatrix} X \\ \mathbb{R}^m \end{bmatrix}$  there exists  $\begin{bmatrix} x \\ v \end{bmatrix} \in D(A_c)$  such that

$$\begin{bmatrix} f \\ z \end{bmatrix} = \begin{bmatrix} (I - \mathcal{J})x \\ -B_\alpha \mathcal{C}x + (I - A_\alpha)v \end{bmatrix}. \quad (36)$$

Observe that since  $\begin{bmatrix} x \\ v \end{bmatrix} \in D(A_c)$  we must have (see (31))

$$(\mathcal{B} + D_\alpha \mathcal{C})x + C_\alpha v = 0. \quad (37)$$

We need to solve (36) and (37) for  $\begin{bmatrix} f \\ z \end{bmatrix} \in \begin{bmatrix} X \\ \mathbb{R}^m \end{bmatrix}$  given. Recall that  $A_\alpha$  is assumed to have only negative eigenvalues, and hence  $(I - A_\alpha)$  is a nonsingular matrix. Using the lower equation of (36) into (37) yields

$$(\mathcal{B} + \alpha(1)\mathcal{C})x = -C_\alpha(I - A_\alpha)^{-1}z, \quad (38)$$

where  $\alpha(1) = C_\alpha(I - A_\alpha)^{-1}B_\alpha + D_\alpha$ . We now need to find an  $x$  that satisfies (38) and the upper equation in (36). In these two equations let  $\tilde{z} = -C_\alpha(I - A_\alpha)^{-1}z$  and  $x = x_{\text{new}} + \tilde{B}\tilde{z}$  where  $\tilde{B}$  is such that  $(\mathcal{B} + \alpha(1)\mathcal{C})\tilde{B} = I$  (the existence of  $\tilde{B}$  is proved in [1]). This gives

$$(I - \mathcal{J})x_{\text{new}} = f - (I - \mathcal{J})\tilde{B}\tilde{z} \quad (39)$$

$$(\mathcal{B} + \alpha(1)\mathcal{C})x_{\text{new}} = 0. \quad (40)$$

Following the proof of Theorem 4.1 it is not difficult to see that if (40) holds then  $\mathcal{J}$  generates a contraction semigroup. This implies that  $(I - \mathcal{J})$  has an inverse and hence  $x_{\text{new}}$  exists. Thus, for  $\begin{bmatrix} f \\ z \end{bmatrix} \in \begin{bmatrix} X \\ \mathbb{R}^m \end{bmatrix}$  we can find  $\begin{bmatrix} x \\ v \end{bmatrix} \in D(A_c)$  such that (36) and (37) holds. ■

In the rest of this section we denote by  $A_\lambda$  the operator  $A_s$  described in Lemma 4.1 with  $\alpha$  replaced by  $\alpha(\lambda) = C_\alpha(\lambda I - A_\alpha)^{-1}B_\alpha + D_\alpha$  in (18). Next we show that if the resolvent of  $A_\lambda$  is compact for  $\lambda > 0$ , then the associated closed-loop system will also have a compact resolvent.

*Theorem 5.4:* Consider the system described in Theorem 2.2. Let the energy of this system satisfy  $\frac{1}{2} \frac{d}{dt} \|x(t)\|^2 = u(t)^T y(t)$ . Assume that  $\alpha(s)$  is a SPR function and that the resolvent of the operator  $A_\lambda$  described above is compact for  $\lambda > 0$ . Then  $(\lambda I - A_c)^{-1}$  is also compact for  $\lambda > 0$ .

*Proof:* We will use Theorem 8.1-3 of [6], which states that an operator is compact iff it maps every bounded sequence onto a sequence which has a convergent subsequence. First we find the inverse of  $(\lambda I - A_c)$  for  $\lambda > 0$  by following the same procedure used to find (36)–(40). We know that this inverse exists since  $A_c$  generates a contraction semigroup. From (39)–(40) we see, in this case, that  $x_{\text{new}} \in D(A_\lambda)$  and  $x_{\text{new}} = (I - A_\lambda)^{-1}f - (I - A_\lambda)^{-1}(I - \mathcal{J})\tilde{B}\tilde{z}$ , where  $\tilde{z} = -C_\alpha(I - A_\alpha)^{-1}z$ . Since  $x = x_{\text{new}} + \tilde{B}\tilde{z}$  we obtain that

$$x = (I - A_\lambda)^{-1}f - (I - A_\lambda)^{-1}(I - \mathcal{J})\tilde{B}\tilde{z} + \tilde{B}\tilde{z}, \quad (41)$$

and from the lower equation of (36) we get

$$v = (I - A_\alpha)^{-1}B_\alpha \mathcal{C}x + (I - A_\alpha)^{-1}z. \quad (42)$$

Let  $\{k_n\} = \left\{ \begin{bmatrix} f_n \\ z_n \end{bmatrix} \right\} \in \tilde{X} = \begin{bmatrix} X \\ \mathbb{R}^m \end{bmatrix}$  be any bounded sequence in  $\tilde{X}$  and let  $w_n = \begin{bmatrix} x_n \\ v_n \end{bmatrix} \in D(A_c)$  such that  $w_n = (\lambda I - A_c)^{-1}k_n$ . By Theorem 5.3 we know that  $A_c$  generates a contraction semigroup and by the Hille-Yosida theorem it follows that

$\|(\lambda I - A_c)^{-1}\| \leq \frac{1}{\lambda}$  for  $\lambda > 0$ . Hence the sequence  $\{w_n\}$  is bounded too. Since we know that  $(\lambda I - A_\lambda)^{-1}$  is compact and that  $\mathcal{J}\tilde{B}$  is bounded (see Definition 3.3.2 of [4]), we have that  $\{x_n\}$  has a convergent subsequence, see (41). Also, since  $\{v_n\}$  is bounded and belongs to a finite dimensional subspace of  $\tilde{X}$ , it follows that  $\{v_n\}$  has another convergent subsequence. Hence,  $w_n = \begin{bmatrix} x_n \\ v_n \end{bmatrix}$  has a convergent subsequence and therefore  $(\lambda I - A_c)^{-1}$  is compact for  $\lambda > 0$ . ■

Next we give an asymptotic stability result similar to the one in Theorem 4.4.

*Theorem 5.5:* Consider the system given by (31). Let the transfer function  $\alpha(s)$  be a SPR function. Let the resolvent associated with  $A_\lambda$  be compact for  $\lambda > 0$ . Then the system described by (31) with  $r = 0$ , is asymptotically stable.

*Proof:* The proof is similar to that of Theorem 4.4. First we prove this for  $w(0) \in D(A_c)$ . By Lemma 4.1 we know that  $A_c$  generates a contraction semigroup. In this case we have that  $w(t) = T_c(t)x(0) \in D(A_c)$  for all  $t \geq 0$ , see Theorem 2.1.10 of [4]. Define the energy function

$$E_c(t) = \frac{1}{2} \|w(t)\|_{\tilde{X}}^2 = \frac{1}{2} \langle w(t), w(t) \rangle_{\tilde{X}}. \quad (43)$$

Since  $w(0) \in D(A_c)$  we have that  $w(t)$  is differentiable, see [4, §2.1]. By differentiating the equation above and using (31) and (35) we obtain

$$\begin{aligned} \dot{E}_c(t) &= \langle \dot{w}(t), w(t) \rangle_{\tilde{X}} = \langle A_c w(t), w(t) \rangle_{\tilde{X}} \\ &= -\gamma v(t)^T P v(t) - \frac{1}{2} (Ky(t) + Q^T v(t))^T (Ky(t) + Q^T v(t)), \end{aligned} \quad (44)$$

where  $\gamma > 0$  and  $P$  is positive definite. Since  $(\lambda I - A_\lambda)^{-1}$  is compact, it follows from Theorem 5.4 that  $(\lambda I - A_c)^{-1}$  is also compact. Since  $(\lambda I - A_c)^{-1}$  is compact and  $T_c(t)$  is a contraction, it follows from LaSalle's principle that all solutions of (31) asymptotically tend to the maximal invariant set of  $\mathcal{O}_c = \{x \in X \mid \dot{E}_c(t) = 0\}$ . Let  $\mathcal{E}$  be the largest invariant subset of  $\mathcal{O}_c$ . Next we show that  $\mathcal{E} = \{0\}$ . The condition  $\dot{E}_c(t) = 0$  implies, from (44), that  $v(t) = 0$ ; and hence,  $\dot{v}(t) = 0$ . Then by (28) we must have that  $B_\alpha y(t) = 0$ . Since  $\alpha(s)$  is SPR, we have that  $\alpha(jw) + \alpha^T(-jw) > 0$ . This implies that

$$\begin{aligned} y^T(t) [\alpha(jw) + \alpha^T(-jw)] y(t) &> 0 \\ \Rightarrow y^T(t) [D_\alpha + D_\alpha^T] y(t) &> 0 \quad \Rightarrow K^T K > 0. \end{aligned}$$

In the second step the facts  $\alpha(jw) = C_\alpha(jw - A_\alpha)^{-1}B_\alpha + D_\alpha$  and  $B_\alpha y(t) = 0$  were used, and in the third step we used (30). Since  $v(t) = 0$  and  $K^T K > 0$  it follows from (44) that  $y(t) = 0$ , and hence by (28) we also obtain  $y_\alpha(t) = 0$ .

Therefore from (31) and (34) it follows that the invariant solution of (31) in  $\mathcal{O}_c$  reduces to the invariant solution of the associated open-loop system in the domain (17)–(18). The rest of the proof follows from the proof of Theorem 4.4. ■

## VI. SCATTERING ENERGY-PRESERVING SYSTEMS

Here we use the term ‘scattering energy-preserving system’ in the sense of [3]. In that paper the author shows that a scattering energy-preserving system satisfies the relation

$$\frac{1}{2} \frac{d}{dt} \|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2$$

for  $u \in C^2((0, \infty); \mathbb{R}^{2N})$ ,  $x(0) \in D(\mathcal{J})$  and,  $\mathcal{B}x(0) = u(0)$ . In [2] it is shown that, in this case,  $V = 0$ ,  $\tilde{P}_1 = \tilde{P}_2 = I$ ,  $S^{-T}S^{-1} = 4I$ , and  $S_2^{-T}S_2^{-1} = 4I$ . This means that  $W$  has the form  $W = S \begin{bmatrix} I & I \end{bmatrix}$ .

For this type of systems related to the operator  $\mathcal{J}$  we have the following result.

**Theorem 6.1:** Consider the system described in Theorem 2.2 with  $\mathcal{J}e = P_1 \frac{de}{dz}(z)$  and  $\frac{1}{2} \frac{d}{dt} \|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2$ . Then the system is exponentially stable and exactly observable in finite time.

*Proof:* First observe that  $W$  has the form  $W = S \begin{bmatrix} I & I \end{bmatrix}$ . In order to check stability of the system we study the resolvent of  $A_W = \mathcal{J}|_{D(A_W)}$ . Let  $x \in D(A_W)$  and observe that  $y(z) = (\lambda - A_W)x(z) = \lambda x(z) - P_1 \frac{dx}{dz}(z)$ . Thus  $\frac{dx}{dz}(z) = \lambda P_1^{-1}x(z) - P_1^{-1}y(z)$  need to be solved. The general solution is given by

$$x(z) = e^{\lambda P_1^{-1}(z-a)} c - \int_a^z e^{\lambda P_1^{-1}(z-\tau)} P_1^{-1}y(\tau) d\tau \quad (45)$$

where  $c$  is a constant vector. Since  $x \in D(A_W)$  the boundary conditions (see Theorem 2.2) are described by (see (4))

$$WR_{ext} \begin{bmatrix} x(b) \\ x(a) \end{bmatrix} = \begin{bmatrix} P_1 + I & I - P_1 \end{bmatrix} \begin{bmatrix} x(b) \\ x(a) \end{bmatrix} = 0.$$

Using (45) in the equation above yields

$$\begin{aligned} (P_1 + I) \left[ e^{\lambda P_1^{-1}(b-a)} c - \int_a^b e^{\lambda P_1^{-1}(b-\tau)} P_1^{-1}y(\tau) d\tau \right] \\ + (I - P_1)c = 0 \\ \iff \left[ (P_1 + I)e^{\lambda P_1^{-1}(b-a)} + (I - P_1) \right] c = \\ (P_1 + I) \int_a^b e^{\lambda P_1^{-1}(b-\tau)} P_1^{-1}y(\tau) d\tau. \end{aligned}$$

It follows from Lemma 14 of [8] that when  $\lambda \in \mathbb{C} \setminus \mathbb{R}^-$  the matrix on the left hand side is nonsingular. In that case,  $c$  can be defined uniquely, which implies that  $(\lambda - A_W)^{-1}$  exists. Also, it is not difficult to show that  $(\lambda - A_W)^{-1}$  is bounded, hence Equation (45) defines the resolvent operator of  $A_W$ .

It is easy to show that the resolvent (45) is uniformly bounded (see the last paragraph of Subsection IV-A). Since  $A_W$  generates a contraction semigroup it follows from Corollary 3.36 of [5] that the system is exponentially stable. Finally, from Theorem 11.3.8 of [9] one can see that the system is also exactly observable in finite time. ■

**Lemma 6.2:** Consider the system of Theorem 2.2 with  $\mathcal{J}e = P_1 \frac{de}{dz}(z)$  and  $\frac{1}{2} \frac{d}{dt} \|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2$ . Then the resolvent  $(\lambda - A_W)^{-1}$ , with  $\lambda \geq 0$ , is a compact operator, where  $A_W = \mathcal{J}|_{D(A_W)}$ . Furthermore, the operator  $\tilde{A}_W e = P_0 e(z) + P_1 \frac{de}{dz}(z)$  has compact resolvent for  $\lambda > 0$ , i.e.,  $(\lambda - \tilde{A}_W)^{-1}$  is a compact operator.

*Proof:* From Theorem 6.1 we know that the resolvent of  $A_W$  is given by (45) for  $\lambda \geq 0$ . The integral operator on the RHS is clearly compact, see Theorem A.3.52 of [4]. The exponential operator  $e^{\lambda P_1^{-1}(z-a)} c$  is also compact, since it is a finite rank operator, see Theorem 8.1-4 of [6]. Since  $(\lambda - A_W)^{-1}$  in (45) is the sum of two compact operators we have that it is also compact for  $\lambda \geq 0$ , see [6, p. 407].

Using Lemma A-3 we can conclude that  $(\lambda - \tilde{A}_W)^{-1}$  is compact for  $\lambda > 0$  since  $\tilde{A}_W$  is also the generator of a contraction semigroup. ■

**Theorem 6.3:** Consider the system of Theorem 2.2 with  $\mathcal{J}e = P_0 e(z) + P_1 \frac{de}{dz}(z)$  and  $\frac{1}{2} \frac{d}{dt} \|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2$ . Then the system is globally asymptotically stable.

*Proof:* From Lemma 6.2 we know that  $(\lambda - A_W)^{-1}$ , with  $\lambda > 0$ , is a compact operator, where  $A_W = \mathcal{J}|_{D(A_W)}$ . The rest of the proof follows similar to the proof of Theorem 4.4, noting that  $\dot{E}(t) = -\|y(t)\|^2$ . ■

## APPENDIX

**Lemma A-1:** Let  $S_2, V \in \mathbb{R}^{n \times n}$  and  $R_{ext}, \Sigma \in \mathbb{R}^{2n \times 2n}$  where  $\Sigma$  is given by (5). Consider the matrices  $W_{imp}$  and  $C_{imp}$  given by (13) and (14), respectively. If  $V$  satisfies  $VV^T = I = V^T V$  (impedance passive system), then we have

$$C_{imp}^T W_{imp} + W_{imp}^T C_{imp} = \Sigma, \quad (A-1)$$

$$C_{imp} C_{imp}^T = 4S_2 S_2^T, \quad W_{imp} W_{imp}^T = \frac{1}{4} S_2^{-T} S_2^{-1}, \quad (A-2)$$

$$C_{imp} W_{imp}^T = 0, \quad \text{and} \quad W_{imp} C_{imp}^T = 0. \quad (A-3)$$

*Proof:* It follows easily by using algebra and  $VV^T = I$ . ■

**Lemma A-2:** Consider the matrices  $W_{imp}$  and  $C_{imp}$  given by (13) and (14), respectively. Let  $V$  and  $D$  be  $n \times n$  matrices, with  $VV^T = I = V^T V$ . Then, the matrix given by

$$\tilde{W} = W_{imp} + DC_{imp}$$

has full row-rank.

*Proof:* It is easy to show that the matrix  $\tilde{W} \tilde{W}^T$  is nonsingular. ■

The following lemma is not difficult to prove.

**Lemma A-3:** Let  $A = A_u + A_b$  be a closed and densely defined operator on  $X$ , where  $A_b$  is a bounded operator on  $X$  and  $A_u$  is an unbounded operator with  $D(A_u) = D(A)$ . If  $A$  satisfies  $\text{Re} \langle Az, z \rangle \leq 0$  and  $(\lambda - A_u)^{-1}$  is a compact operator for  $\lambda > 0$ . Then the operator  $(\lambda - A)^{-1}$  is a compact operator for  $\lambda > 0$ .

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