

Generic Form of Bayesian Monte Carlo For Models With Partial Monotonicity

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Abstract: This paper presents a generic method for the safety assessments of models with partial monotonicity. For this purpose, a Bayesian interpolation method is developed and implemented in the Monte Carlo process. integrated approach is the generalization of the recently developed techniques used in safety assessment of monotonic models and it substantially increases the efficiency of Monte Carlo method. The formulation of this development is provided in this paper with an example showing its ability to dramatically improve efficiency of simulation. This is achieved by employing prior information obtained from monotonic models and outcomes of the preceding simulations. The theory and numerical algorithms of this method for multi-dimensional problems and their integration with the probabilistic finite element model of a real-world example are presented.

Keywords: Reliability, Bayesian , Dynamic Bounds, Monte Carlo , Gaussian, Beta.

1. INTRODUCTION

In this paper, we introduce a Bayesian Monte Carlo Method for Partially Monotonic (BMCPM) models. Applying the BMCPM significantly reduces simulation time of monotonic models for a desired accuracy levels. Monotonic models are used extensively in practice as shown in [1-3].

There are already two methods introduced for the reliability assessment of monotonic models in [4]. The method of Dynamic Bounds (DB) incorporates the monotonic information of the structures in the reliability assessment. This method is applied to a complex case study in New Orleans [1]. The method of Improved Dynamic Bounds (IDB) is also developed for the reliability assessment of monotonic models given the model response order. The application of this method to flood defence systems is presented in [5]. The Bayesian Monte Carlo method also has been developed to capture model's prior information as presented in [6]. This paper presents a significant improvement by integrating these methods into one generic form. We present a flexible formulation that can be applied to fully monotonic, partially monotonic, and non-monotonic models. This novel approach also integrates the prior information from the neighboring points. These neighboring points represent the information of prior simulations. In other words, we capture the information of prior simulation as well as the partially monotonic models. This premise is based on assumption that the global uncertainty is related to local uncertainties [4, 7]. These outcomes are interesting for the interpolation schemes. We, however, progress further and use the outcomes for the reliability estimation of infrastructures. We use the Monte Carlo method as the basis for our simulations.

With this modeling approach, simulation times can be reduced in predictive tools developed to forecast system reliability estimates. A logical dependence between neighboring points is assumed for each randomly generated point. The uncertainty of this assumption is investigated for all calculated data points and error limits are developed. This approach is flexible as it permits inclusion of additional prior information in the modeling if warranted. Generalize Beta (GB) distribution is used to capture the monotonic prior, and the Gaussian distribution is used for the non-monotonic priors. Regarding the calculation time, the presented approach is recommended for the reliability assessment of complex structures where every realization counts regarding the overall calculation efforts. It is assumed that readers are familiar with the Bayesian techniques [8] and application of Bayes' Theorem in practice [9].

2. MATHEMATICAL FORMULATION

2.1. A NOVEL BAYESIAN INTERPOLATION METHOD

Consider a continuous function U that we wish to estimate at a number of discrete points. We define the u_i set of discrete points by a vector \vec{u} assigned to discrete points (pixels). The elements of observed data points are d_i defined by vector $d = [d_1, \dots, d_n]$, and their locations are stored in a n-dimensional vector. Let $P(u_j | D, I)$ be the univariate probability density function (pdf) for an arbitrary pixel u_j . The data D and informational context I can be found from the simulations and model, respectively. The global uncertainty $\vec{\sigma} = [\sigma^{(1)}, \sigma^{(2)}, \dots, \sigma^{(n)}]$ (e.g., global standard deviation), where each uncertainty is associated with its respective dimension $\vec{x} = [x^{(1)}, x^{(2)}, \dots, x^{(n)}]$ of the limit state equation $G(\vec{x})$ at any point \vec{x}_j is u_j . The global uncertainty was first used in [7, 10] to define a nuisance parameter. With marginalization, the global uncertainty can be written as

$$P(u_j | D, I) = \int \dots \int P(u_j, \vec{\sigma} | D, I) d\vec{\sigma}. \quad (1)$$

By application of Bayes' Theorem, we find

$$P(u_j | D, I) = \int \dots \int P(u_j | \vec{\sigma}) P(\vec{\sigma} | D, I) d\vec{\sigma}. \quad (2)$$

With the global uncertainty ($\vec{\sigma}$) defined, we can now estimate the value of LSE at an arbitrary point x_i from an interpolation function (model) f using information about its neighboring points. Let the estimate be \hat{u}_i . In this model, the value of u_i is estimated by its neighbor points. There are $m+1$ neighboring points for each arbitrary location among these points. The model f_m is defined using Equation (2), where index m is the order of model (function). We can estimate value of LSE at the point x close to the middle of its neighboring points Equation (3) [11-13].

$$f_m(x_i) = \sum_{k=0, k \neq r}^m u_{i-r+k} L_{i-r+k}(x_i), \quad (3)$$

where $r = \text{abs}(\frac{m+1}{2})$, u_i is the LSE responses assigned to point x_i , and L_k is the i -th fundamental polynomial defined as

$$L_k(x_i) = \prod_{k \neq l} \left(\frac{x_i - x_l}{x_k - x_l} \right). \quad (4)$$

2.2. GAUSSIAN ERROR ESTIMATE FOR ONE-DIMENSIONAL PROBLEMS

We first define $P(u_j | \sigma)$ in write hand side (RHS) of Equation (2) for a one-dimensional problem. The value of error with a zero mean could be positive or negative and its unknown variance is σ_j^2 . Assume the standard deviation of error is proportional to the shortest distance from its neighboring data points. We use the Gaussian density function for the error (Equation (5)) of the model as a standard error form [14]. The Gaussian error is defined as

$$P(e_j | \sigma_j) = \frac{1}{\sqrt{2\pi}\sigma_j} \exp \left\{ -\frac{1}{2\sigma_j^2} e_j^2 \right\}. \quad (5)$$

where e_j is the error and σ_j^2 is an unknown variance. By making the change of variable from e_j to u_j , we find the following multivariate pdf for pixels u_j as:

$$P(u_j | \sigma_j) = \frac{1}{\sqrt{2\pi}\sigma_j} \exp \left\{ -\frac{1}{2\sigma_j^2} (u_j - f_m(u_j))^2 \right\}. \quad (6)$$

Following the approach used in [3, 4] to associate the global and local uncertainties, and assuming the logical independence between the errors and making appropriate substitutions, we obtain the posterior for u_j as:

$$P(u_j | D, I) \propto \int \frac{1}{\sigma^{n+1}} \exp \left\{ -\frac{1}{2\sigma^2} \left(\frac{(u_j - f_m(u_j))^2}{\alpha_j^2} - \sum_{i=1}^n \frac{(d_i - f_m(u_i))^2}{\alpha_i^2} \right) \right\} d\sigma. \quad (7)$$

2.3. BETA ERROR ESTIMATE FOR ONE-DIMENSIONAL PROBLEMS

For monotonic model, we first define $P(u_j | \sigma)$ in write hand side (RHS) of Equation (2) for a one-dimensional problem. The value of error with a zero mean could be positive or negative and its unknown variance is σ_j^2 . We use the Generalized Beta (GB) density function for the error to assure monotonic constraint of the model. The GB distribution ensures that $\hat{u}_j = f(x_j)$ is bounded between u_{j-1} and u_{j+1} and its density function for error is a suitable choice. The GB density is defined as

$$P(x | c, d) = \frac{(x-c)^{p-1} (d-x)^{q-1}}{B(p, q)(d-c)^{p+q-1}}, \quad (8)$$

for $c \leq x \leq d$ and $B(p, q)$ is the Beta function. Using the GB distribution at the interval of $[u_{j-1}, u_{j+1}]$ and assuming $u_{j-1} < u_{j+1}$, we have

$$P(x | u_{j-1}, u_{j+1}) = \frac{(x-u_{j-1})^{p-1} (u_{j+1}-x)^{q-1}}{B(p, q)(u_{j+1}-u_{j-1})^{p+q-1}}, \quad (9)$$

where $u_{j-1} \leq x \leq u_{j+1}$. The estimate of the pixel value is $u_j = x$ and the error function is defined as

$$e_j = u_j - u_{j-1}. \quad (10)$$

Substitution of Equation (10) into Equation (9) gives

$$P(u_j | u_{j-1}, u_{j+1}, p_j, q_j) = \frac{(u_j - u_{j-1})^{p_j-1} (u_{j+1} - u_j)^{q_j-1}}{B(p_j, q_j)(u_{j+1} - u_{j-1})^{p_j+q_j-1}}. \quad (11)$$

Now, following the steps indicated in [15, 16] to associate the global and local uncertainties, and assuming the logical independence between the errors and making appropriate substitutions, we obtain the posterior for u_j as:

$$P(u_j | D, I) = \int \prod_{i=1, i \in c}^n \left(\frac{(d_i - u_{i-1})^{p_i-1} (u_{i+1} - d_i)^{q_i-1}}{B(p_i, q_i) (u_{i+1} - u_{i-1})^{p_i+q_i-1}} \right) \times \frac{(u_j - u_{j-1})^{p_j-1} (u_{j+1} - u_j)^{q_j-1}}{B(p_j, q_j) (u_{j+1} - u_{j-1})^{p_j+q_j-1}} d\sigma \quad (12)$$

where $B(.,.)$ is the Beta function as indicted in Equation (8), p_j and q_j are the local Beta parameters obtained by the following equations.

$$p_j = \frac{(u_{j-1} - \hat{u}_j)(-\hat{u}_j u_{j+1} + u_{j+1} u_{j-1} + \sigma_j^2 + \hat{u}_j^2 - \hat{u}_j u_{j-1})}{(u_{j+1} - u_{j-1}) \sigma_j^2}, \quad (13)$$

$$q_j = \frac{-\hat{u}_j u_{j+1}^2 + u_{j-1} u_{j+1}^2 + u_{j+1} \sigma_j^2 + 2\hat{u}_j^2 u_{j+1} - 2\hat{u}_j u_{j-1} u_{j+1} - \sigma_j^2 \hat{u}_j + \hat{u}_j^2 u_{j-1} - \hat{u}_j^3}{(u_{j+1} - u_{j-1}) \sigma_j^2}, \quad (14)$$

2.4. INTEGRATED ERROR ESTIMATE FOR MODELS WITH PARTIAL MONOTONICITY

In this section, we integrate the two distributions into one model to capture prior information of partially monotonic models. This is a necessity for analysis of dependent variables. Assume a limit state equation $G(\vec{x})$, where \vec{x} is composed of vectors \vec{x}_{mon} and \vec{x}_{nonmon} . We use the Generalized Beta and Gaussian distributions to capture the monotonic and non-monotonic part of the model, respectively. These were described in Sections 2.3 and 2.2. In this case, we define a vector of global uncertainties $\vec{\sigma} = [\sigma^{(1)}, \sigma^{(2)}, \dots, \sigma^{(n)}]$, where each global uncertainty $\sigma^{(i)}$ is associated with its corresponding dimension $x^{(i)}$, where $x^{(i)}$ belongs to $\vec{x} = [x^{(1)}, x^{(2)}, \dots, x^{(n)}]$ of the limit state equation $G(\vec{x})$. The exact value of $G(\vec{x})$ at any point \vec{x}_j is u_j . Having the data points at neighborhood of point \vec{x}_j , we can estimate the response model using function f according to Equation (3). This estimate is \hat{u}_j and the error is obtained by the following equation

$$e_j = u_j - \hat{u}_j = u_j - f_q(u_{j_1}, \dots, u_{j_p}). \quad (15)$$

where p is the number of required data points for response estimation and q is the response order. These numbers (p, q) depend on dimensions of the problem and the model order. For example, in a linear estimation of a two dimensional problem, three data points are required, and p is equal to 3 while the model is linear and $q = 1$. In presence of higher number of data points, data points with the shortest distance from u_j are selected since a closer neighbor is assumed to have a greater influence on the estimate than the other neighbors. Equation (2) is used for the multidimensional problems and the first term in its RHS is

$$P(u_j | \vec{\sigma}) = P(u_j | \sigma^{(1)}, \dots, \sigma^{(m)}, \sigma^{(m+1)}, \dots, \sigma^{(n)}), \quad (16)$$

where n is the problem dimension, m is the number of monotonic variables, u_j is the target pixel and $\sigma^{(i)}$ is the global uncertainty associated with i -th dimension. Following the process for monotonic model (Equation (11)) and non-monotonic models (Equation (6)) and assuming independent global uncertainties of different dimensions gives

Results on Figure 2 show that the accuracy of predicted estimates is improving with the progression of MC simulations. Taking advantage of these characteristics saves enormous computational time in the MC simulations.

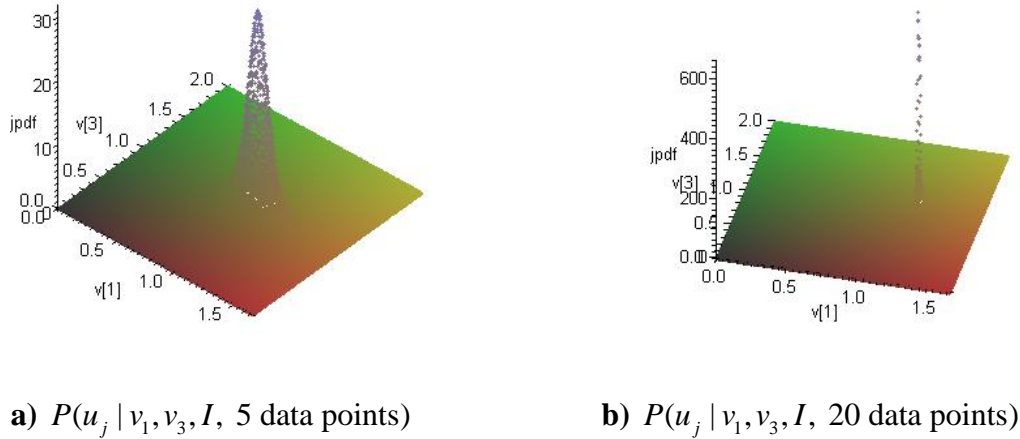


Figure 2. A comparison between the estimated joint pdf of a two-dimensional Gaussian problem (v_1, v_2) for 5 versus 20 data points.

The number of simulations required for the Bayesian Monte Carlo method for the 17th Street I-Wall application for water level +8 ft is shown in

Table 1. Results are provided for the Bayesian Monte Carlo for a monotonic model, Bayesian Monte Carlo for a non-monotonic model, classical Monte Carlo (MC) and Dynamic Bounds (DB) methods, showing that only a fraction of MC simulation is required when the Generalized Beta distribution is used to capture monotonicity. The method of Dynamic Bounds is briefly described here since it has been fully described in previous publications (see References). It is used in the comparisons shown here with a monotonic model. The proposed method in this paper is flexible and can be used for totally monotonic, non-monotonic, and partially monotonic models. This novel approach, therefore, is not subjected to the limitations of use of the DB method. An uncertainty model association is used to relate the local uncertainty σ_i to the global uncertainty σ in the form of $\sigma_i = \alpha_i \sigma$, where α_i corresponds to the cubic root of the distance of x_i with its closest neighbour. The cubic root relation for the third order model response is used [5].

Table 1. The calculated probabilities of failure for the 17th Street I-Wall structure obtained with the BMC for monotonic models, BMC for non-monotonic models, MC and DB methods using product moment correlation [1] for three most influential variables.

| <i>Method</i> | <i>W.L. (ft)</i> | Number of simulations | \hat{p}_f | $\mathcal{G}(\hat{p}_f)$ |
|--|------------------|-----------------------|-------------|--------------------------|
| BMC for monotonic model (G. Beta error estimation) | +8 (2.4 m) | 53 | 51.2 | 0.042 |
| BMC for non-monotonic model (Gaussian error estimation) | +8 (2.4 m) | 185 | 49.2 | 0.048 |
| Dynamic Bounds method | +8 (2.4 m) | 221 | 52 | 0.024 |
| Monte Carlo method | +8 (2.4 m) | 1500 | 52 | 0.024 |

4. CONCLUSIONS

A technique is developed in this paper to significantly increase the efficiency of Monte Carlo method for partially monotonic, totally monotonic and non-monotonic models. This is in great importance for complex finite element applications or other time consuming processes which are employed for modeling linear and nonlinear behavior of structures. This technique integrates advantages of three recently developed reliability methods (DB [19], IDB [4] and BMC [20]) into a generic form for the Monte Carlo family.

The theoretical formulation and numerical implementation details of a novel Bayesian interpolation method developed for this purpose are provided. The proposed Bayesian model integrates the Gaussian and Generalized Beta density functions. Our proposed method enjoys inclusion of a new concept in Bayesian formulation that relates the global and local uncertainties [10]. As a result, an unbiased estimate for the Monte Carlo method has been obtained. In other words, this novel technique preserves fundamental properties of the classical MC method, and greatly improves the computational efficiency by using different types of prior information.

This newly developed method is applied to investigate load-response characteristics of the 17th Street Flood Wall. Results of previous numerical realizations (simulations), termed here as the prior information, are used in the current simulation with the Bayes theory and the Bayesian Monte Carlo method. The prior information is obtained from previously completed Monte Carlo simulations. A partial or total monotonicity of a model is considered as another source of prior information integrated to the simulation by use of the Generalized Beta distribution. As a result, separation of data points in this manner avoids unnecessary simulations in the Monte Carlo method and substantially reduces computational burdens.

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