A Hamiltonian Formulation for Nonlinear Wave-Body Interactions*

E.F.G. van Daalen **E. van Groesen** **P.J. Zandbergen**†

Abstract

Here we show that gravity driven water waves — on a fluid of arbitrary variable depth — interacting with rigid bodies floating freely in or below the free surface can be described as an infinite-dimensional Hamiltonian system. It will appear that, with appropriate choices for the canonical variables and the Hamiltonian, the complete set of equations of motion — i.e. the nonlinear free surface conditions and the hydrodynamic equations of motion for the rigid bodies — can be written in a canonical form.

1. Introduction

For all we know, the first description of the evolution of water waves in terms of a Hamiltonian density and generalized variables was made by Zakharov (1968), who presented the canonical equations of motion for an infinite-depth free surface potential flow in a homogeneous gravity field. Few years later, Broer (1974) independently showed that — for a fluid of *arbitrary* depth — the nonlinear kinematic and dynamic free surface conditions constitute an infinite-dimensional Hamiltonian system, with the elevation and the free surface potential as the canonical variables and the total energy as the Hamiltonian density. In a subsequent paper Broer, van Groesen, and Timmers (1976) presented a corrected proof of the equivalence of the canonical equations with the free surface conditions, and paid attention to conservation laws and the stability of approximate (Boussinesq-type) models. Other contributions to these special wave descriptions are due to Miles (1977, 1981). For an overview of Hamiltonian formulations in fluid mechanics (including free surface flows) we refer to Salmon (1988).

The first optimal variation principle for nonlinear water waves is due to Luke (1967), who showed that the *full* set of governing equations for the classical water-wave problem can be obtained from a *single* variation principle. In Luke's formulation, which is related to those given by Clebsch (1859), Bateman (1932) and Friedrichs (1933), the *integrated* Bernoulli pressure plays the role of the Lagrangian density, and the velocity potential and the free surface elevation are the independent variables. Miloh (1984) presented an extension of Luke's principle for water waves interacting with multiple floating bodies oscillating at a common frequency.

For a large number of dynamical systems the transition from a Lagrangian principle to a Hamiltonian formulation can easily be made by means of the so-called *Legendre transformation*. The essence of this transformation is the definition of 'canonical' momenta which are conjugate to the chosen 'canonical' coordinates. For a certain class of these systems, these canonical variables allow the transition from a *single second-order* Euler-Lagrange equation to a *pair of first-order* canonical equations, usually written in the well-known skew-symmetric matrix form, see e.g. Goldstein (1980). The possibility of this transition with respect to the water-wave problem was first recognized by van Groesen (1978); here it will be applied to the nonlinear wave-body problem.

The main purpose of this paper is to extend the 'Zakharov/Broer/Miles' Hamiltonian formulation to water waves in hydrodynamic interaction with *freely* floating bodies. The outline is as follows: in section 2 a variation principle is presented for water waves interacting with an unrestrained body floating in or below the free surface. With the integrated pressure as the Lagrangian for the fluid, and with kinetic minus potential energy as the Lagrangian for the body, the proposed variation principle is shown to yield the field equation, the equations of motion for the fluid and the body, and the Neumann boundary conditions on the bottom and the wetted body surface. In section 3 canonical variables for both the fluid and the body are found by means of direct Legendre transformations. With the total energy as the Hamiltonian, the nonlinear free surface conditions and the hydrodynamic equations of motion for the body are written in a canonical form. We close our discussion with concluding remarks in section 4.

2. Variation Principle

In this section Luke's (1967) variation principle for the classical water-wave problem is extended to the hydrodynamic interaction with a freely floating body.

The system under consideration consists of a fluid, bounded by the impermeable bottom B (which is not necessarily even), the free surface F, and the wetted surface S of a rigid body, see Figure 1. In the horizontal directions x and y, the fluid domain is cut off by a cylindrical vertical surface Σ of infinite radius; Σ extends from the bottom to the free surface.

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†All authors: University of Twente, Faculty of Applied Mathematics, P.O. Box 217, 7500 AE Enschede, The Netherlands.

The body mass and moments about its principal axes of inertia are denoted by M and $\vec{I} = (I_1, I_2, I_3)^T$ respectively. The position of the body is specified by the centre of gravity $\vec{x}_G = (x_1, x_2, x_3)^T$, corresponding to the surge, sway and heave motions respectively, and the body orientation by the 'roll-pitch-yaw vector' $\vec{\theta}_G = (\theta_1, \theta_2, \theta_3)^T$. As usual, gravity is acting in negative z-direction.

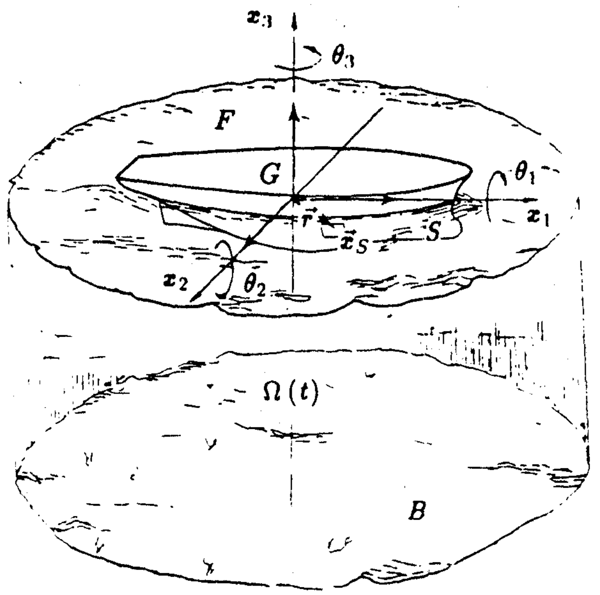

Figure 1: Definition of fluid domain and floating body.

Following Luke, the Lagrangian for the fluid is given by

$$\mathcal{L}_f \equiv \iiint_{\Omega(t)} p \, d\Omega = -\iiint_{\Omega(t)} \left(\phi_t + \frac{1}{2}(\nabla\phi \cdot \nabla\phi) + gz \right) d\Omega \quad (1)$$

i.e. the Bernoulli pressure p integrated over the transient fluid domain $\Omega(t)$; the fluid density is taken as unity.

The Lagrangian for the body is taken as kinetic minus potential energy, that is

$$\mathcal{L}_b \equiv \mathcal{K}_b - \mathcal{P}_b \quad (2)$$

where

$$\mathcal{K}_b = \frac{1}{2} M \dot{\vec{x}}_G \cdot \dot{\vec{x}}_G + \frac{1}{2}\left(\vec{I} \otimes \dot{\vec{\theta}}_G\right) \cdot \dot{\vec{\theta}}_G \quad (3)$$

$$\mathcal{P}_b = M g \vec{e}_3 \cdot \vec{x}_G \quad (4)$$

In (3) the symbol $\otimes$ is used to define the component-wise product of two vectors: $\vec{a} \otimes \vec{b} \equiv (a_1 b_1, a_2 b_2, a_3 b_3)^T$.

The Lagrangian for the *total* system, consisting of the fluid and the body, is defined as the sum of the separate Lagrangians, that is

$$\mathcal{L}_s \equiv \mathcal{L}_f + \mathcal{L}_b \quad (5)$$

With the above definitions the proposed variation principle reads

$$\delta J = 0 \quad \text{with} \quad J \equiv \int_{t_1}^{t_2} \mathcal{L}_s \, dt \quad (6)$$

for all variations in the free surface elevation η, the velocity potential ϕ, and the body position $\vec{x}_G$ and orientation $\vec{\theta}_G$. These variations are subject to the restrictions that they vanish at the end points of the time interval, i.e. at times $t = t_1$ and $t = t_2$; moreover, the variations in η and ϕ equal zero on the vertical boundary at infinity, i.e. on Σ. Then, following the standard procedure in the calculus of variations, (1-6) yields

$$\begin{aligned} \delta J = & - \int_{t_1}^{t_2} \left\{ \iiint_{\Omega(t)} (\delta\phi_t + \nabla\phi \cdot \nabla\delta\phi) \, d\Omega \right\} dt \\ & + \int_{t_1}^{t_2} \left\{ \iint_F p \, \delta\eta \, R^{-1} dS \right\} dt \\ & + \int_{t_1}^{t_2} \left\{ \iint_S p \, (\delta\vec{x}_S \cdot \vec{n}) \, dS \right\} dt \\ & + \int_{t_1}^{t_2} \left\{ M \ddot{\vec{x}}_G \cdot \delta\vec{x}_G - M g \vec{e}_3 \cdot \delta\vec{x}_G \right\} dt \\ & + \int_{t_1}^{t_2} \left\{ \left(\vec{I} \otimes \ddot{\vec{\theta}}_G\right) \cdot \delta\vec{\theta}_G \right\} dt = 0 \end{aligned} \quad (7)$$

where $\vec{n}$ is the unit normal vector along $\partial\Omega \supset S$, and

$$R(\eta) \equiv \left(1 + \eta_x^2 + \eta_y^2\right)^{1/2} \quad (8)$$

giving $dS = R \, dx \, dy$.

In (7) $\vec{x}_S$ denotes the position of a point on the wetted body surface S; the change in $\vec{x}_S$ due to the variations in $\vec{x}_G$ and $\vec{\theta}_G$ is given by

$$\delta\vec{x}_S = \delta\vec{x}_G + \delta\vec{\theta}_G \times \vec{r} \quad (9)$$

where $\vec{r} \equiv \vec{x}_S - \vec{x}_G$, see Figure 1.

Taking into account the motion of $\Omega(t)$, we may write

$$\iiint_{\Omega(t)} \delta\phi_t d\Omega = \frac{\partial}{\partial t} \iiint_{\Omega(t)} \delta\phi \, d\Omega - \iint_F \eta_t \delta\phi \, R^{-1} dS - \iint_S \left(\dot{\vec{x}}_S \cdot \vec{n}\right) \delta\phi \, dS \quad (10)$$

The first term on the right-hand side of (10) vanishes due to the restriction $\delta\phi = 0$ at times $t = t_1$ and $t = t_2$.

With Green's first identity we obtain:

$$\iiint_{\Omega(t)} \nabla\phi \cdot \nabla\delta\phi \, d\Omega = \iint_{\partial\Omega} \phi_n \delta\phi \, dS - \iiint_{\Omega(t)} \nabla^2\phi \, \delta\phi \, d\Omega \quad (11)$$

Due to the restriction $\delta\phi = 0$ on $\Sigma \subset \partial\Omega$, the corresponding contribution on the right-hand side of (11) vanishes.

Integration by parts, and using the restrictions $\delta\vec{x}_G = \vec{0}$ and $\delta\vec{\theta}_G = \vec{0}$ at times $t = t_1$ and $t = t_2$, gives

$$\int_{t_1}^{t_2} M\dot{\vec{x}}_G \cdot \delta\dot{\vec{x}}_G \, dt = -\int_{t_1}^{t_2} M\ddot{\vec{x}}_G \cdot \delta\vec{x}_G \, dt \quad (12)$$

$$\int_{t_1}^{t_2} \left(\vec{I} \otimes \dot{\vec{\theta}}_G\right) \cdot \delta\dot{\vec{\theta}}_G \, dt = -\int_{t_1}^{t_2} \left(\vec{I} \otimes \ddot{\vec{\theta}}_G\right) \cdot \delta\vec{\theta}_G \, dt \quad (13)$$

With (7-13) the proposed variation principle reads

$$\begin{aligned} \delta J = & \int_{t_1}^{t_2} \left\{ \iint_F p\,\delta\eta\, R^{-1} dS + \iiint_{\Omega(t)} \nabla^2\phi\,\delta\phi\, d\Omega \right\} dt \\ - & \int_{t_1}^{t_2} \left\{ \iint_F (R\,\phi_n - \eta_t)\, R^{-1} \delta\phi \, dS \right\} dt \\ - & \int_{t_1}^{t_2} \left\{ \iint_S \left(\phi_n - \dot{\vec{x}}_S \cdot \vec{n}\right) \delta\phi \, dS \right\} dt \\ - & \int_{t_1}^{t_2} \left\{ \iint_B \phi_n \delta\phi \, dS \right\} dt \\ + & \int_{t_1}^{t_2} \left\{ \left(\iint_S p\,\vec{n}\, dS - Mg\vec{e}_3 - M\ddot{\vec{x}}_G \right) \cdot \delta\vec{x}_G \right\} dt \\ + & \int_{t_1}^{t_2} \left\{ \left(\iint_S p\,(\vec{r} \times \vec{n})\, dS - \vec{I} \otimes \ddot{\vec{\theta}}_G \right) \cdot \delta\vec{\theta}_G \right\} dt \\ = & \; 0 \end{aligned} \quad (14)$$

From this it is clear that invariance of J with respect to a variation in the free surface elevation η yields the dynamic free surface condition:

$$p = -\left(\phi_t + \frac{1}{2}(\nabla\phi \cdot \nabla\phi) + gz\right) = 0 \quad \text{on} \quad F \quad (15)$$

Similarly, invariance of J with respect to a variation in the velocity potential ϕ yields the field equation:

$$\nabla^2\phi = 0 \quad \text{in} \quad \Omega(t) \quad (16)$$

the kinematic free surface condition:

$$\eta_t + \eta_x\phi_x + \eta_y\phi_y = \phi_z \quad \text{on} \quad F \quad (17)$$

the 'contact' condition on the wetted body surface:

$$\phi_n = \dot{\vec{x}}_S \cdot \vec{n} \quad \text{on} \quad S \quad (18)$$

and the impermeability condition on the bottom:

$$\phi_n = 0 \quad \text{on} \quad B \quad (19)$$

Finally, invariance of J with respect to variations in the body position $\vec{x}_G$ and orientation $\vec{\theta}_G$ yields the equations of motion for the body:

$$\iint_S p\,\vec{n}\, dS - Mg\vec{e}_3 = M\ddot{\vec{x}}_G \quad (20)$$

$$\iint_S p\,(\vec{r} \times \vec{n})\, dS = \vec{I} \otimes \ddot{\vec{\theta}}_G \quad (21)$$

Thus it has been proven that (1-6) is a proper variation principle for nonlinear gravity waves interacting with a body floating freely in or below the free surface. The extension to multiple bodies is straightforward.

3. Hamiltonian Formulation

In this section the Hamiltonian theory for the classical water-wave problem is extended to the interaction with an unrestrained body floating in or below the free surface. All definitions from the previous section with respect to the fluid domain and the body are adopted here.

With regard to the body, the 'coordinate' $\vec{\xi}$ is defined as the combination of the position vector $\vec{x}_G$ and the orientation vector $\vec{\theta}_G$:

$$\vec{\xi} \equiv \begin{bmatrix} \vec{x}_G \\ \vec{\theta}_G \end{bmatrix} \quad (22)$$

Similarly, the 'normal' $\vec{\nu}$ along the wetted body surface S is defined as

$$\vec{\nu} \equiv \begin{bmatrix} \vec{n} \\ \vec{r} \times \vec{n} \end{bmatrix} \quad (23)$$

and the diagonal 'mass' matrix $\mathcal{M}$ is defined as

$$\text{diag}(\mathcal{M}) \equiv [M, M, M, I_1, I_2, I_3]^T \quad (24)$$

With the above definitions the Lagrangian for the system can be written as

$$\begin{aligned} \mathcal{L}_s = & - \iiint_{\Omega(t)} \left(\phi_t + \frac{1}{2}(\nabla\phi \cdot \nabla\phi) + gz\right) d\Omega \\ & + \frac{1}{2}\mathcal{M}\dot{\vec{\xi}} \cdot \dot{\vec{\xi}} - \mathcal{M}g\vec{e}_3 \cdot \vec{\xi} \end{aligned} \quad (25)$$

Taking into account the evolution of the fluid domain and integrating by parts, we may write

$$\begin{aligned} \mathcal{L}_s = & - \frac{\partial}{\partial t} \iiint_{\Omega(t)} \phi \, d\Omega + \iint_F \phi\,\eta_t \, dx\, dy \\ & + \mathcal{M}\dot{\vec{\xi}} \cdot \dot{\vec{\xi}} + \iint_S \phi \left(\dot{\vec{\xi}} \cdot \vec{\nu}\right) dS \\ & - \iiint_{\Omega(t)} \left(\frac{1}{2}(\nabla\phi \cdot \nabla\phi) + gz\right) d\Omega \\ & - \frac{1}{2}\mathcal{M}\dot{\vec{\xi}} \cdot \dot{\vec{\xi}} - \mathcal{M}g\vec{e}_3 \cdot \vec{\xi} \end{aligned} \quad (26)$$

where $\dot{\vec{\xi}} \cdot \vec{\nu}$ is the normal velocity of a point on S. The first term on the right-hand side of (26) integrates out to the end points of the time interval considered.

Considering (26), we may write formally

$$\mathcal{L}_s(\eta_s, \dot{\eta}_s) = \pi_s \dot{\eta}_s - \mathcal{H}_s(\eta_s, \pi_s) \quad (27)$$

with the following definitions: the canonical coordinates are given by

$$\eta_s = (\eta_f, \vec{\eta}_b) \equiv \left(\eta, \vec{\xi}\right) \quad (28)$$

i.e. the free surface elevation and the body position and orientation.

The canonical conjugate momenta are given by

$$\pi_s = (\pi_f, \vec{\pi}_b) \equiv \left(\Phi, \mathcal{M}\dot{\vec{\xi}} + \iint_S \phi\, \vec{\nu}\, dS\right) \quad (29)$$

i.e. the free surface potential $\Phi = [\phi]_{z=\eta}$ and the rigid body impulse plus a 'Kelvin impulse' contribution of the velocity potential over the wetted body surface.

The Hamiltonian is given by

$$\begin{aligned}\mathcal{H}_s &= \iiint_{\Omega(t)} \left(\frac{1}{2}(\nabla\phi \cdot \nabla\phi) + gz\right) d\Omega \\ &+ \frac{1}{2}\mathcal{M}\dot{\vec{\xi}} \cdot \dot{\vec{\xi}} + \mathcal{M}g\vec{e}_3 \cdot \vec{\xi} \end{aligned} \quad (30)$$

i.e. the total energy of the system.

Then, with ϕ satisfying the following boundary value problem (BVP):

$$\nabla^2\phi = 0 \quad \text{in} \quad \Omega(t) \quad (31)$$

$$\phi_n = \dot{\vec{\xi}} \cdot \vec{\nu} \quad \text{on} \quad S \quad (32)$$

$$\phi_n = 0 \quad \text{on} \quad B \quad (33)$$

$$\phi(x, y, z) = \Phi(x, y) \quad \text{on} \quad F \quad (34)$$

we can formulate the following theorem:

Theorem: *The equations of motion for gravity driven water waves interacting with a body floating freely in or below the free surface describe an infinite-dimensional Hamiltonian system in the canonically conjugate variables π_s and η_s and with the total energy $\mathcal{H}_s$ as Hamiltonian: the canonical equations*

$$\partial_t \begin{pmatrix} \pi_s \\ \eta_s \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta\mathcal{H}_s/\delta\pi_s \\ \delta\mathcal{H}_s/\delta\eta_s \end{pmatrix} \quad (35)$$

are equivalent with the nonlinear free surface conditions and the hydrodynamic equations of motion for the body.

The proof of this theorem follows directly from the next lemma:

Lemma: ***The variational derivatives of the kinetic fluid energy***

$$\mathcal{K}_f(\Phi, \eta) = \iiint_{\Omega(t)} \frac{1}{2}(\nabla\phi \cdot \nabla\phi)\, d\Omega \quad (36)$$

are given by:

$$\delta_\Phi \mathcal{K}_f = R(\eta) \left(\frac{\partial\phi}{\partial n}\right)_{z=\eta} \quad (37)$$

$$\delta_\eta \mathcal{K}_f = \frac{1}{2}(\nabla\phi \cdot \nabla\phi)_{z=\eta} - \delta_\Phi \mathcal{K}_f \left(\frac{\partial\phi}{\partial z}\right)_{z=\eta} \quad (38)$$

Proof of (37): Keep η and $\vec{\xi}$ fixed and let ϕ vary such that its variation $\delta\phi$ corresponds to a change $\delta\Phi$. Then the first variation in $\mathcal{K}_f(\Phi, \eta)$ reads

$$\begin{aligned}\delta\mathcal{K}_f(\delta\Phi) &= \iiint_{\Omega(t)} \nabla\phi \cdot \nabla\delta\phi\, d\Omega \\ &= \iiint_{\Omega(t)} \left[\nabla \cdot (\nabla\phi\, \delta\phi) - \nabla^2\phi\, \delta\phi\right] d\Omega \end{aligned} \quad (39)$$

With Gauss' divergence theorem and (31-33) we obtain

$$\delta\mathcal{K}_f(\delta\Phi) = [(\nabla\phi \cdot \vec{n})\, dS]_{z=\eta}\, \delta\Phi \quad (40)$$

Using $dS = R\, dx\, dy$, the proof is completed.

Proof of (38): Now, vary η and assume that the solution ϕ of the BVP is correspondingly modified for the varied fluid domain. At the modified free surface we have to lowest order:

$$\begin{aligned}\phi(\eta + \delta\eta) &= \phi(\eta) + \delta\eta \left(\frac{\partial\phi}{\partial z}\right)_{z=\eta} \\ &\equiv \Phi + \delta_\eta\Phi \end{aligned} \quad (41)$$

With the above result the total effect of a variation $\delta\eta$ in $\mathcal{K}_f$ is found to be

$$\delta\mathcal{K}_f(\delta\eta) + \delta\mathcal{K}_f(\delta_\eta\Phi) = \delta\eta \left\{\frac{1}{2}(\nabla\phi \cdot \nabla\phi)_{z=\eta}\right\} \quad (42)$$

and hence, to lowest order

$$\delta_\eta\mathcal{K}_f + \delta_\Phi\mathcal{K}_f \left(\frac{\partial\phi}{\partial z}\right)_{z=\eta} = \frac{1}{2}(\nabla\phi \cdot \nabla\phi)_{z=\eta} \quad (43)$$

from which the second part of the lemma follows.

4. Concluding Remarks

The full set of equations of motion for gravity driven water waves interacting with unrestrained bodies floating in or below the free surface has been presented in a canonical form. The transition from the variational (Lagrangian) principle to the Hamiltonian formulation was made along the formal line of a Legendre transformation, yielding the canonical momenta conjugate to the chosen canonical coordinates for the system and the total energy as the Hamiltonian.

The fact that a system of gravity driven water waves interacting with freely floating bodies can be described as a Hamiltonian system has several important implications. In the first place, this Hamiltonian structure allows a systematic account of conservation laws by considering the symmetries of the system; see Benjamin and Olver's (1982) analysis of constants of the motion for the water-wave problem and van Daalen's (1993) generalization of these invariants to the nonlinear wave-body problem. For instance, note that from (35) it follows directly that

$$\dot{\mathcal{H}}_s = \frac{\delta\mathcal{H}_s}{\delta\eta_s}\dot{\eta}_s + \frac{\delta\mathcal{H}_s}{\delta\pi_s}\dot{\pi}_s = 0 \quad (44)$$

ter the total energy is a constant of the motion. Figure 2 shows the exchange of energy in a simple wave-body system consisting of an extincting cylinder on the free surface of a wave tank. The computations have been carried out with a two-dimensional panel method based on a Green's formulation for the velocity potential.

Secondly, well-chosen approximations to the canonical equations (35) may provide numerical schemes which conserve integrated densities (for instance, the total energy). The above and other implications motivated these investigations.

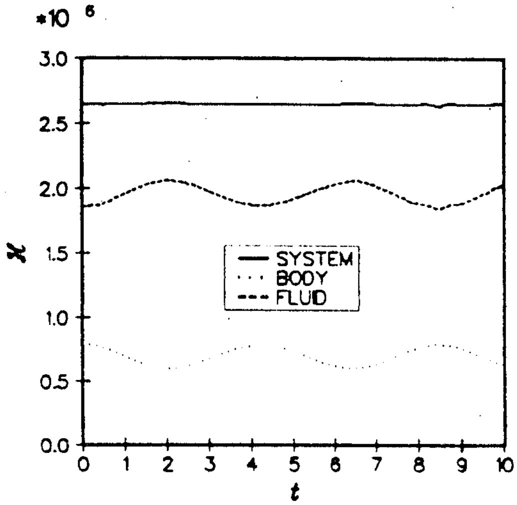

Figure 2: Exchange of energy in wave-body system.

References

BATEMAN, H. 1933. *Partial Differential Equations of Mathematical Physics.* Cambridge University Press.

BENJAMIN, T.B., AND OLVER, P.J. 1982. *Hamiltonian structure, symmetries and conservation laws for water waves.* J. Fluid Mech. **125**:137-185.

BROER, L.J.F. 1974. *On the Hamiltonian theory of surface waves.* Appl. Sci. Res. 29:430-446.

BROER, L.J.F., GROESEN, E.W.C. VAN, AND TIMMERS, J.M.W. 1976. *Stable model equations for long water waves.* Appl. Sci. Res. **32**:619-636.

CLEBSCH, A. 1859. *Über die Integration der Hydrodynamischen Gleichungen.* Journal für die Reine und Angewandte Mathematik **56**:1-10.

DAALEN, E.F.G. VAN 1993. *Numerical and Theoretical Studies of Water Waves and Floating Bodies.* Ph.D. thesis, University of Twente. Enschede, The Netherlands.

FRIEDRICHS, K. 1933. Mathematische Annalen **109**:60.

GOLDSTEIN, H. 1980. *Classical Mechanics.* Addison-Wesley. Reading, Massachusetts.

GROESEN, E.W.C. VAN 1978. *Variational Methods in Mathematical Physics.* Ph.D. thesis, Eindhoven University. Eindhoven, The Netherlands.

LUKE, J.C. 1967. *A variational principle for a fluid with a free surface.* J. Fluid Mech. **27**(2):395-397.

MILES, J.W. 1977. *On Hamilton's principle for surface waves.* J. Fluid Mech. **83**(1):153-158.

MILES, J.W. 1981. *Hamiltonian formulations for surface waves.* Appl. Sci. Res. **37**:103-110.

MILOH, T. 1984. *Hamilton's principle, Lagrange's method and ship motion theory.* J. Ship Res. **28**(4):229-237.

SALMON, R. 1988. *Hamiltonian fluid mechanics.* Ann. Rev. Fluid Mech. **20**:225-256.

ZAKHAROV, V.E. 1968. *Stability of periodic waves of finite amplitude on the surface of a deep fluid.* J. Appl. Mech. Techn. Phys. **2**:190-194.