

SLIDING MODE CONTROL OF SPATIAL MECHANICAL SYSTEMS DECOUPLING TRANSLATION AND ROTATION

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ABSTRACT

This paper looks at the robust trajectory control of spatial mechanical systems using sliding mode techniques. Two distinctions of the proposed method from reported methods are: (1) The measure of attitudinal error used is intrinsically defined, Euclidean-geometric, and intuitive. From Euler's theorem it follows that given a desired and actual attitude of a rigid body there exists an axis and angle of rotation relating the two attitudes. This defines a relative rotation vector, which is used as an intrinsically defined, intuitive measure of error. Reported methods use algebraic differences of entities such as generalized coordinates representing attitude. While functionally correlated to attitudinal error, these measures are not intrinsically defined. (2) A novel, dynamically nonlinear sliding function is used that results in a simple control law. The parameters of this function are dynamically and geometrically intuitive. Simulation results are given for a spacecraft tracking a complex desired trajectory.

I. INTRODUCTION

Sliding mode control is a well known method of robust trajectory control (Hung et al., 1993; Slotine and Sastry, 1983; Slotine and Li, 1991; Utkin, 1977; Utkin, 1993). Rather than controlling the states of the system directly, the desired error behavior of the system is specified in terms of a sliding surface in the state space. A discontinuous, so-called variable structure control law can be used to drive the state to the sliding surface. The state then "chatters" along the surface with a desired Filippov velocity so that the desired error dynamics are achieved. This method is robust to model uncertainty because of the discontinuous feedback. In practice this can result in unacceptably high control activity and can excite unmodelled dynamics. For this reason continuous feedback is often used so that the state remains within a so-called boundary layer near the sliding surface, with a velocity approximating the desired Filippov velocity.

This paper looks at the robust trajectory control of spatial mechanical systems using sliding mode techniques. This topic has been addressed in the literature in the context of spacecraft attitude control. A number of excellent papers have been written on this topic, for example (Dwyer and Ramirez, 1988; Iyer and Singh, 1988; Robinett and Parker, 1996; Tsiotras, 1996; Vadali, 1986). Distinctions of the proposed method from reported methods are (1) the attitudinal error measure is intrinsically defined, intuitive and Euclidean-geometric, that is physical geometric not abstract differential geometric, (2) a novel, dynamically nonlinear sliding function is used that results in a simple control law. The parameters of this function are dynamically and geometrically intuitive. Simulation results are given for a spacecraft tracking a complex desired trajectory. Methods reported in the literature have typically been illustrated using trivial, even stationary, desired trajectories.

The attitude of a rigid body such as an idealized vehicle or robot end-effector can be represented in many ways (Shuster, 1993). Representations previously used in sliding mode attitude control include Euler angles (Iyer and Singh, 1988), Euler parameters (complete or reduced) (Robinett and Parker, 1996; Vadali, 1986) and (modified) Cayley-Rodrigues parameters (Dwyer and Ramirez, 1988; Tsiotras, 1996). For example, Vadali (1986) uses a reduced set of Euler parameters (i.e., just the vector part) to regulate the attitude of a spacecraft. The sliding surface is defined as a linear combination of the body-relative angular velocities and the reduced Euler parameters. The desired reduced Euler parameters are assumed to be zero. Robinett and Parker (1996) use a complete set of Euler parameters to make a spacecraft track a desired attitude. They define attitude error as the algebraic difference between the reference Euler parameters and the actual parameters. The sliding surface is defined to be a linear combination of the attitude error and its temporal derivative. Iyer and Singh (1988) use Euler angles to make a spacecraft track a desired attitude. They define attitude error as the difference between the desired Euler angles and the actual Euler angles. The sliding

surface is defined to be a linear combination of this error and its temporal derivative. Dwyer and Sira-Ramirez (1988) use Cayley-Rodrigues parameters to regulate the attitude of a spacecraft. They define a sliding surface to be an abstract nonlinear function of the body-relative angular velocity and the desired and actual attitude parameters. The actual sliding surface is to be specified by the control system designer.

A major difference between these methods and the one proposed in this paper is that the proposed attitude error is defined in an intrinsic, Euclidean-geometric, intuitive way. The reported error measures have been algebraic differences between actual and desired attitude parameters. Although the attitude parameters themselves may be geometric, their algebraic difference is generally not. For example, Euler parameters (unitary quaternions) are geometric entities. The difference of two Euler parameters is not in general an Euler parameter. This difference can be used as an index of error, but it is not a Euclidean geometric entity.

From Euler's theorem it follows that given a desired and actual attitude of a rigid body then there exists an axis and angle of rotation relating the two attitudes. This defines a relative rotation vector, which can be used as an intuitive, intrinsically defined, Euclidean-geometric measure of attitudinal error. The desired error dynamics (sliding function) are defined in terms of this vector. A nonobvious sliding function is defined that results in a simple control law. The parameters of this function have a geometric interpretation, which further distinguishes this method from previously reported methods.

The outline of the rest of the paper is as follows: The control problem is defined in Sec. II. Tracking error is defined in Sec. III. Section IV defines novel sliding variables and surfaces and gives a geometric interpretation of the control parameters. The control law is presented in Sec. V. Section VI shows the design and simulation of a robust tracking controller for a spacecraft.

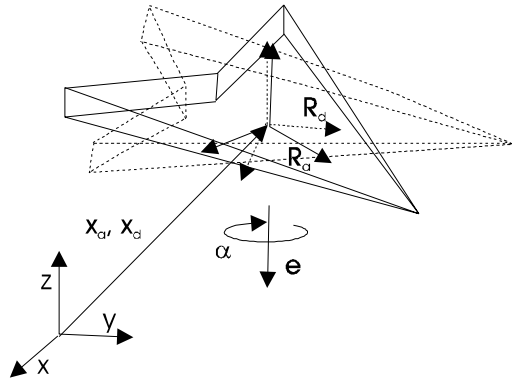


Figure 1: Desired and actual attitudes of a spacecraft

II. CONTROL PROBLEM

Consider the problem of making a spatial rigid body follow a desired trajectory. The configuration of the body can be represented by (1) a vector \mathbf{x} representing the displacement of some point on the body from a reference point, and (2) a direction cosine matrix $\mathbf{R} = [\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3]$ representing the attitude of the body with respect to an inertial reference frame. Unit vectors \mathbf{e}_1 , \mathbf{e}_2 and \mathbf{e}_3 define a body frame. Figure

1 depicts two configurations of an idealized spacecraft. Dotted lines indicate the desired configuration; solid lines indicate the actual configuration. The actual configuration is represented by \mathbf{x}_a and \mathbf{R}_a ; the desired configuration is represented by \mathbf{x}_d and \mathbf{R}_d . The kinematics of the body are simply

$$\frac{d\mathbf{x}_a}{dt} = \dot{\mathbf{x}}_a \quad \text{and} \quad \frac{d\mathbf{R}_a}{dt} = \mathbf{R}_a \tilde{\mathbf{w}}_a \quad (1)$$

where $\dot{\mathbf{x}}_a$ is the translational velocity in the inertial frame in inertial coordinates, \mathbf{w}_a is the angular velocity with respect to the inertial frame in body coordinates (or simply *body-relative angular velocity*), and in general tilde denotes the skew-symmetric matrix associated with a vector:

$$\tilde{\mathbf{a}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \quad (2)$$

The dynamics of this system are assumed to be second order:

$$\ddot{\mathbf{h}}_a = \mathbf{f} + \mathbf{B}\mathbf{u}, \quad \text{where} \quad \mathbf{h}_a = \begin{bmatrix} \dot{\mathbf{x}}_a^t & \mathbf{w}_a^t \end{bmatrix}^t \quad (3)$$

In this equation \mathbf{f} is a vector (tuple) of functions, \mathbf{B} is the input matrix, and \mathbf{u} is the control vector. In general \mathbf{f} and \mathbf{B} can be functions of any measurable variable including states. Let $\hat{\mathbf{f}}$ and $\hat{\mathbf{B}}$ be estimates of \mathbf{f} and \mathbf{B} . The extent of the model uncertainty is assumed to be bounded in the following way:

$$\|\hat{\mathbf{f}} - \mathbf{f}\| \leq \mathbf{F} \quad (4)$$

$$\mathbf{B} = (\mathbf{I}_6 + \mathbf{D})\hat{\mathbf{B}}, \quad \text{where} \quad \|\mathbf{D}\| \leq \mathbf{D}_B \quad (5)$$

where all inequality constraints and the absolute value function are applied element-wise. Vector \mathbf{F} bounds the uncertainty of \mathbf{f} , matrix \mathbf{D}_B bounds the uncertainty of \mathbf{B} .

Assume that the estimated input matrix, $\hat{\mathbf{B}}$, and the input matrix, \mathbf{B} , are always nonsingular and that the estimated input matrix equals the input matrix in the absence of parametric uncertainty ($\mathbf{D} = \mathbf{0}$). The control problem is make the actual configuration $\mathbf{x}_a(t)$ and $\mathbf{R}_a(t)$ track the desired configuration $\mathbf{x}_d(t)$ and $\mathbf{R}_d(t)$ as closely as possible in the presence of model uncertainty.

III. TRACKING ERROR

The tracking error can be separated into translational and attitudinal error. The translational error, \mathbf{x}_e , is defined conventionally:

$$\mathbf{x}_e = \mathbf{x}_a - \mathbf{x}_d \quad (6)$$

The attitudinal error is defined unconventionally. From Euler's theorem it follows that given a desired and actual attitude of a rigid body, \mathbf{R}_d and \mathbf{R}_a , then there exists an axis \mathbf{e} and angle α of rotation relating the two attitudes. This defines a relative rotation vector, which can be used as a measure of error:

$$\mathbf{q}_e = \alpha \mathbf{e} \quad (7)$$

where the angle of rotation α is restricted to the interval $[0, \pi]$ so that \mathbf{q}_e is uniquely defined. Note that this measure of error is not an algebraic difference of desired and actual attitude parameters such as used in the reported methods.

The rotation vector is not a true vector because an addition operator is not defined. In particular, addition of rotation vectors does not correspond to composition of rotations. Attitudinal error can also be expressed as an error rotation matrix, \mathbf{R}_e , which is related to $\tilde{\mathbf{q}}_e$ by the matrix exponential:

$$\mathbf{R}_e = \mathbf{R}_d^t \mathbf{R}_a = e^{\tilde{\mathbf{q}}_e} \quad (8)$$

Euler's theorem in general and computation of the matrix exponential are discussed in, for example, (Murray et al. 1994; Shuster 1993). Euler's theorem and more generally screw theory are used extensively in kinematic analysis. They are not widely used in defining measures of error or in deriving control laws.

Given \mathbf{R}_e , let $\ln(\mathbf{R}_e)$ denote the \mathbf{q}_e such that \mathbf{R}_e is the matrix exponential of $\tilde{\mathbf{q}}_e$. Although \mathbf{q}_e is defined intrinsically, we have computed it using body coordinates. The actual attitude in (actual) body coordinates is always the identity matrix, $\mathbf{I} = \mathbf{R}_a^t \mathbf{R}_a$. The desired attitude in body coordinates is $\mathbf{R}_a^t \mathbf{R}_d$. Matrix $\mathbf{R}_e = \mathbf{R}_d^t \mathbf{R}_a$ is the rotation operator that transports the desired attitude to the actual attitude, expressed in body coordinates, because

$$\mathbf{I} = (\mathbf{R}_d^t \mathbf{R}_a) \mathbf{R}_a^t \mathbf{R}_d \quad (9)$$

This is consistent with Fig. 1, where the direction \mathbf{e} of \mathbf{q}_e is depicted downwards. This is the axis of rotation for which a positive angle α of rotation transports the body with the desired attitude to the body with the actual attitude.

The derivative of \mathbf{R}_e can be expressed as an angular velocity, \mathbf{w}_e :

$$\dot{\mathbf{R}}_e = \mathbf{R}_e \tilde{\mathbf{w}}_e \quad (10)$$

From (8) it follows that

$$\tilde{\mathbf{w}}_e = \mathbf{R}_e^t \dot{\mathbf{R}}_e = \tilde{\mathbf{w}}_a - \mathbf{R}_a^t \mathbf{R}_d \tilde{\mathbf{w}}_d \mathbf{R}_d^t \mathbf{R}_a \quad (11)$$

where \mathbf{w}_a is the actual body-relative angular velocity and \mathbf{w}_d is the desired body-relative angular velocity. This is equivalent to the vector equation

$$\mathbf{w}_e = \mathbf{w}_a - \mathbf{R}_a^t \mathbf{R}_d \mathbf{w}_d \quad (12)$$

which is in (actual) body coordinates.

Combining the orientation and translation error into one vector, we define the tracking error as

$$\mathbf{h}_e = \begin{bmatrix} \mathbf{x}_e \\ \mathbf{q}_e \end{bmatrix} = \begin{bmatrix} \mathbf{x}_a - \mathbf{x}_d \\ \ln(\mathbf{R}_d^t \mathbf{R}_a) \end{bmatrix} \quad (13)$$

Computation of \mathbf{h}_e requires computation of the \mathbf{q}_e corresponding to $\mathbf{R}_e = \mathbf{R}_d^t \mathbf{R}_a$. There are several ways to perform this computation. Probably the simplest is to use on- and off-diagonal elements of the

matrix product to determine the rotation vector (Goeree, 1995), this process is very similar to finding the Euler parameters of an orthonormal matrix. Other methods can be found in Murray et al. (1994) and Shuster (1993).

IV. SLIDING SURFACES

The first step of the sliding control methodology is to define a sliding surface that achieves a set of desired error dynamics. This effectively replaces a tracking problem by an equivalent stabilizing problem.

Sliding variable and surface

Define the sliding variable \mathbf{s} as:

$$\mathbf{s} = \begin{bmatrix} \mathbf{s}_t \\ \mathbf{s}_o \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{x}}_e \\ \mathbf{w}_e \end{bmatrix} + \mathbf{L} \begin{bmatrix} \mathbf{x}_e \\ \mathbf{q}_e \end{bmatrix} \quad (14)$$

where

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_o \end{bmatrix} \quad (15)$$

and \mathbf{L}_t and \mathbf{L}_o are positive definite 3×3 matrices. The rotational sliding variable, \mathbf{s}_o , is an algebraic linear combination of \mathbf{w}_e and \mathbf{q}_e . N.B., angular velocity difference \mathbf{w}_e is *not* the rate of change of error \mathbf{q}_e ($\mathbf{w}_e \neq \dot{\mathbf{q}}_e$). The desired error dynamics corresponding to $\mathbf{s}_o = \mathbf{0}$ are therefore dynamically nonlinear. Thus the sliding variable defined in (14) is *not* equivalent to the alternative basic sliding mode definition

$$\mathbf{s}' = \dot{\mathbf{h}}_e + \mathbf{L} \mathbf{h}_e \quad (16)$$

We choose the different sliding variable of eq. (14) because it results in a simple control law and because the resulting dynamic behavior is intuitive, despite the dynamic nonlinearity. Because the dynamics are nonlinear we must analyze the stability of the error dynamics corresponding to the sliding surface. The following analysis of the translational error dynamics is conventional. The analysis of the rotational error dynamics is new.

Define the sliding surface to be the set where the sliding variable is zero:

$$S = \left\{ (\mathbf{x}_a, \mathbf{R}_a, \dot{\mathbf{x}}_a, \mathbf{w}_a) \mid \mathbf{s}(\mathbf{x}_e, \mathbf{q}_e, \dot{\mathbf{x}}_e, \mathbf{w}_e) = \mathbf{0} \right\} \quad (17)$$

The tracking error goes to zero if the sliding variable is kept on the sliding surface. First consider the translational part:

$$\mathbf{s}_t = \dot{\mathbf{x}}_e + \mathbf{L}_t \mathbf{x}_e \quad (18)$$

Define function V_t to be the square of the Euclidean norm of the translational error:

$$V_t = \mathbf{x}_e^t \mathbf{x}_e \quad (19)$$

The rate of change of V_t is

$$\dot{V}_t = 2\mathbf{x}_e^t \dot{\mathbf{x}}_e = 2\mathbf{x}_e^t (\mathbf{s}_t - \mathbf{L}_t \mathbf{x}_e) \quad (20)$$

If the sliding variable is kept at zero then the rate of change is strictly negative:

$$\dot{V}_t = -2\mathbf{x}_e^t \mathbf{L}_t \mathbf{x}_e \leq -2\mathbf{x}_e^t \mathbf{x}_e \mathbf{g}_{\min} < 0 \quad (21)$$

where $\mathbf{g}_{\min} > 0$ is the minimum eigenvalue of \mathbf{L}_t . Obviously V_t is a Lyapunov function if the sliding variable is kept at the sliding surface. This implies that the translational error goes to zero if the sliding variable is kept at zero.

Consider now the attitudinal part:

$$\mathbf{s}_o = \mathbf{w}_e + \mathbf{L}_o \mathbf{q}_e \quad (22)$$

Define function V_o to be the square of the rotation angle:

$$V_o = \mathbf{a}^2 = \mathbf{q}_e^t \mathbf{q}_e. \quad (23)$$

In Appendix A it is shown that the rate of change of the rotation angle equals the projection of the angular error velocity on the rotation vector:

$$\dot{\mathbf{a}} = \mathbf{e}^t \mathbf{w}_e = \frac{1}{\mathbf{a}} \mathbf{q}_e^t \mathbf{w}_e \quad (24)$$

Note that this implies that the orientational error decreases as long as the angle between \mathbf{q}_e and \mathbf{w}_e is greater than 90° . The rate of change of V_o is

$$\dot{V}_o = 2\mathbf{a}\dot{\mathbf{a}} = 2\mathbf{q}_e^t \mathbf{w}_e = 2\mathbf{q}_e^t (\mathbf{s}_o - \mathbf{L}_o \mathbf{q}_e) \quad (25)$$

If the sliding variable is kept at zero then the rate of change is strictly negative:

$$\dot{V}_o = -2\mathbf{q}_e^t \mathbf{L}_o \mathbf{q}_e \leq -2\mathbf{q}_e^t \mathbf{q}_e \mathbf{g}_{\min} < 0 \quad (26)$$

where $\mathbf{g}_{\min} > 0$ is the minimum eigenvalue of \mathbf{L}_o . Obviously V_o is a Lyapunov function if the sliding variable is kept at zero. This implies that the attitudinal error goes to zero if the sliding variable is kept at the sliding surface.

Geometrical interpretation of the control parameters

The rest of this paper will assume for the most part that \mathbf{L} is diagonal. If matrix $\mathbf{L} = \text{diag}(\mathbf{I}_{t1}, \mathbf{I}_{t2}, \mathbf{I}_{t3})$ has strictly positive elements, then the three differential equations defining the translational part of the sliding variable vector are decoupled:

$$s_{ti} = \dot{x}_{ei} + \mathbf{I}_{ti} x_{ei} \quad (27)$$

Assuming that each s_i is identically zero, the solution is

$$x_{ei}(t) = x_{ei}(0)e^{-\mathbf{I}_{ti}t} \quad (28)$$

The reciprocal of diagonal element $1/\mathbf{I}_{ti}$ is the time constant at which the translational error x_{ei} asymptotically approaches zero. In other words, \mathbf{I}_{ti} specifies the desired translational error dynamics for the i -axis. A similar interpretation holds for any symmetric, positive-definite \mathbf{L} . The translational error dynamics are decoupled along the principal axes of \mathbf{L} .

If matrix $\mathbf{L}_o = \text{diag}(\mathbf{I}_{o1}, \mathbf{I}_{o2}, \mathbf{I}_{o3})$ has strictly positive elements, then the three differential equations defining the attitudinal part of the sliding variable vector are decoupled

$$s_{oi} = \mathbf{w}_{ei} + \mathbf{I}_{oi} \mathbf{q}_{ei} \quad (29)$$

In Appendix B it is shown that for small attitudinal errors the rate of change of the rotation vector can be approximated to first order by the angular error velocity:

$$\dot{\mathbf{q}}_e \approx \mathbf{w}_{oe} \quad (30)$$

Hence, for small errors the sliding variable can be approximated by

$$s_{oi} \approx \dot{\mathbf{q}}_{ei} + \mathbf{I}_{oi} \mathbf{q}_{ei} \quad (31)$$

which yields the solution:

$$\mathbf{q}_{ei}(t) \approx \mathbf{q}_{ei}(0)e^{-\mathbf{I}_{oi}t} \quad (32)$$

The reciprocal of diagonal element $1/\mathbf{I}_{oi}$ is the time constants at which the attitudinal error \mathbf{q}_{ei} about axis \mathbf{e}_i asymptotically approaches zero if the sliding variable is kept at zero. In other words, \mathbf{I}_{oi} specifies the desired attitudinal error dynamics for the \mathbf{e}_i axis. A similar interpretation holds for any symmetric, positive-definite \mathbf{L}_o . For small rotations the attitudinal error dynamics are decoupled along the principal axes of \mathbf{L}_o .

The geometrical interpretation of matrices \mathbf{L} and \mathbf{L}_o make their selection more intuitive.

Bounds on the sliding variable translate into bounds on the tracking error

Consider the first-order differential equation associated with each translational sliding variable (27). This can be interpreted as a first-order, low-pass filter with cutoff frequency λ_{ti} . Each error $x_{ei}(t)$ is a filtered version of $s_{ti}(t)$. Hence if the magnitude of $s_{ti}(t)$ is bounded by \mathbf{f}_{ti} for all time, then the magnitude of $x_{ei}(t)$ is bounded as well (Slotine and Li, 1991):

$$|x_{ei}| \leq \mathbf{e}_{ti} = \frac{\mathbf{f}_{ti}}{\mathbf{I}_{ti}} \quad (33)$$

For small attitudinal errors approximation (30) can be used. By a similar argument, if the magnitude of $s_{oi}(t)$ is bounded by \mathbf{f}_{oi} for all time, then the magnitude of $\mathbf{q}_{ei}(t)$ is bounded:

$$|\mathbf{q}_{ei}| \leq \mathbf{e}_{oi} = \frac{\mathbf{f}_{oi}}{\mathbf{I}_{oi}} \quad (34)$$

In fact, the error can be shown to be bounded for the completely nonlinear case, allowing for possibly large attitudinal errors. From (25) the rate of change of \mathbf{a} satisfies

$$\dot{\mathbf{a}} + \mathbf{a} \mathbf{e}^t \mathbf{L}_o \mathbf{e} = \mathbf{e}^t \mathbf{s}_o \quad (35)$$

where again \mathbf{e} is the unit vector in the direction of \mathbf{q}_e . It follows that

$$\dot{\mathbf{a}} + \mathbf{g}_{\min} \mathbf{a} \leq \|\mathbf{s}_o\|_2 \quad (36)$$

where \mathbf{g}_{\min} is the smallest eigenvalue of \mathbf{L}_o . This differential equation can be interpreted as a first order low pass filter with bounded input. Assuming that the magnitude of $\mathbf{s}_o(t)$ is bounded by \mathbf{f}_o for all time, then the attitudinal error \mathbf{a} is bounded as well:

$$\mathbf{a} \leq \mathbf{e}_o = \frac{\mathbf{f}_o}{\mathbf{g}_{\min}} \quad (37)$$

Sliding conditions

To guarantee that the sliding variable \mathbf{s} indeed approaches the sliding surface $\mathbf{s} = \mathbf{0}$ we choose the control law such that each s_i^2 is a Lyapunov-like function. So that the rate of change of these functions is negative definite we require for each index i that

$$\frac{1}{2} \frac{d}{dt} s_i^2 \leq -\mathbf{h}_i |s_i| \quad (38)$$

These conditions are referred to as the *sliding conditions*. If the sliding conditions are met, the distance in state space from the sliding variable to the sliding surface decreases monotonically. Once on the sliding surface, the sliding variable stays zero so that the sliding surface is an invariant set.

Furthermore, one can show that if the sliding variable is initially off the sliding surface, the sliding surface will be reached in finite time. This finite *reaching time* is bounded (Asada and Slotine, 1985; Slotine and Sastry, 1983; Slotine and Li, 1991):

$$t_{ri} \leq \frac{|s_i(t=0)|}{\mathbf{h}_i} \quad (39)$$

Control parameter \mathbf{h} determines the maximum reaching time.

V. CONTROL LAW

The selected control law that keeps the sliding variable on the sliding surface and that satisfies the sliding conditions consist of two parts: an *equivalent* control term and a *robust* control term. The first term uses the dynamic model of the system to keep the sliding variable on the sliding surface. The robust term ensures that the sliding conditions are met in the presence of model uncertainty.

Equivalent control

The equivalent control ensures that the sliding variable, once at the sliding surface, remains on the surface assuming a perfect model. To do this the rate of change of the sliding variable must be identically zero:

$$\dot{\mathbf{s}} \equiv \mathbf{0} = \begin{bmatrix} \ddot{\mathbf{x}}_e \\ \dot{\mathbf{w}}_e \end{bmatrix} + \mathbf{L}\dot{\mathbf{h}}_e \quad (40)$$

where the definition of the sliding variable (14) is used. Differentiating the expression for the angular error velocity (12) and using the definition of the translational tracking error (6) we obtain:

$$\mathbf{0} = \begin{bmatrix} \ddot{\mathbf{x}}_a \\ \dot{\mathbf{w}}_a \end{bmatrix} - \begin{bmatrix} \mathbf{R}_a^t \mathbf{R}_d \ddot{\mathbf{w}}_d - \tilde{\mathbf{w}}_a \mathbf{R}_a^t \mathbf{R}_d \dot{\mathbf{w}}_d \\ \mathbf{R}_a^t \mathbf{R}_d \dot{\mathbf{w}}_d - \tilde{\mathbf{w}}_a \mathbf{R}_a^t \mathbf{R}_d \mathbf{w}_d \end{bmatrix} + \mathbf{L}\dot{\mathbf{h}}_e \quad (41)$$

Using the estimate of the system dynamics (3) and solving for the control $\mathbf{u} = \mathbf{u}_{eqv}$:

$$\mathbf{u}_{eqv} = \hat{\mathbf{B}}^{-1} \left(-\hat{\mathbf{f}} + \begin{bmatrix} \ddot{\mathbf{x}}_d \\ \mathbf{R}_a^t \mathbf{R}_d \dot{\mathbf{w}}_d - \tilde{\mathbf{w}}_a \mathbf{R}_a^t \mathbf{R}_d \mathbf{w}_d \end{bmatrix} - \mathbf{L}\dot{\mathbf{h}}_e \right) \quad (42)$$

Robust control term

A robust control term is subtracted from the equivalent control to ensure that the sliding conditions are satisfied:

$$\mathbf{u} = \mathbf{u}_{eqv} - \mathbf{u}_r \quad (43)$$

Let \mathbf{k} be a six-dimensional vector with strictly positive elements. The robust term is assumed to be of the form

$$\mathbf{u}_r = \hat{\mathbf{B}}^{-1} \tilde{\mathbf{k}} \text{sign}(\mathbf{s}) \quad (44)$$

where the sign function operates element-wise on \mathbf{s} and $\tilde{\mathbf{k}}$ denotes the diagonal matrix with elements of vector \mathbf{k} on the diagonal. In order to satisfy the sliding conditions (38) \mathbf{k} must be chosen such that

$$\frac{1}{2} \frac{d}{dt} s_i^2 = s_i \dot{s}_i \leq -\mathbf{h}_i |s_i| \quad (45)$$

To this end, we first compute the rate of change of the sliding variable, which expression is substituted into the sliding conditions (45). Corresponding conditions for \mathbf{k} are then derived. Rewriting (41), the rate of change of \mathbf{s} is

$$\dot{\mathbf{s}} = \begin{bmatrix} \ddot{\mathbf{x}} \\ \dot{\mathbf{w}} \end{bmatrix} - \mathbf{z} + \mathbf{L}\dot{\mathbf{h}}_e \quad (46)$$

where

$$\mathbf{z} = \begin{bmatrix} \ddot{\mathbf{x}}_d \\ \mathbf{R}_a^t \mathbf{R}_d \dot{\mathbf{w}}_d - \tilde{\mathbf{w}}_a \mathbf{R}_a^t \mathbf{R}_d \mathbf{w}_d \end{bmatrix} \quad (47)$$

Substitution of the dynamic model (3) and the control law (43), (42), (44) yields

$$\dot{\mathbf{s}} = \mathbf{f} + \hat{\mathbf{B}}\hat{\mathbf{B}}^{-1} \left[-\hat{\mathbf{f}} + \mathbf{z} - \mathbf{L}\dot{\mathbf{h}}_e - \tilde{\mathbf{k}} \text{sign}(\mathbf{s}) \right] - \mathbf{z} + \mathbf{L}\dot{\mathbf{h}}_e \quad (48)$$

Using the relation between the estimated and the actual input matrix (5):

$$\dot{\mathbf{s}} = (\mathbf{f} - \hat{\mathbf{f}}) - \tilde{\mathbf{k}} \text{sign}(\mathbf{s}) - \mathbf{D}\tilde{\mathbf{k}} \text{sign}(\mathbf{s}) + \mathbf{D}\mathbf{a} \quad (49)$$

where

$$\mathbf{a} = -\hat{\mathbf{f}} + \mathbf{z} - \mathbf{L}\dot{\mathbf{h}}_e \quad (50)$$

Substituting (49) back into the sliding conditions (45), for each component it is true that

$$s_i \dot{s}_i = s_i \left(f_i - \hat{f}_i - \sum_{j=1}^6 (\Delta_{ij} k_j \text{sign}(s_j) + \Delta_{ij} a_j) \right) - k_i \text{sign}(s_i) s_i \quad (51)$$

Choosing \mathbf{k} such that $s_i \dot{s}_i$ is smaller than $-\mathbf{h}|s_i|$ satisfies the sliding conditions. An upper bound is obtained by taking the maximum values of the elements and by using the bounds (4),(5) on the model uncertainty:

$$s_i \dot{s}_i \leq \left(F_i + \sum_{j=1}^6 (D_{ij} k_j + D_{ij} |a_j|) \right) |s_i| - k_i |s_i| \leq -\mathbf{h}|s_i| \quad (52)$$

Dividing both sides by $|s_i|$ and rearranging terms yields

$$k_i - \sum_{j=1}^6 D_{ij} k_j \geq F_i + \sum_{j=1}^6 D_{ij} |a_j| + \mathbf{h}_i \quad (53)$$

Eliminating abbreviations (47), (50) and writing the inequality in vector form we have

$$(\mathbf{I}_6 - \mathbf{D})\mathbf{k} \geq \mathbf{F} + \mathbf{D} \left[-\hat{\mathbf{f}} + \begin{bmatrix} \ddot{\mathbf{x}}_d \\ \mathbf{R}'_d \mathbf{R}_d \dot{\mathbf{w}}_d - \tilde{\mathbf{w}}_d \mathbf{R}'_d \mathbf{R}_d \mathbf{w}_d \end{bmatrix} - \mathbf{L} \mathbf{h}_e \right] + \mathbf{h} \quad (54)$$

where the absolute value function and the inequality constraint apply element-wise. Any \mathbf{k} satisfying this constraint can be chosen. In particular we can choose

$$\mathbf{k} = (\mathbf{I}_6 - \mathbf{D})^{-1} \left[\mathbf{F} + \mathbf{D} \left[-\hat{\mathbf{f}} + \begin{bmatrix} \ddot{\mathbf{x}}_d \\ \mathbf{R}'_d \mathbf{R}_d \dot{\mathbf{w}}_d - \tilde{\mathbf{w}}_d \mathbf{R}'_d \mathbf{R}_d \mathbf{w}_d \end{bmatrix} - \mathbf{L} \mathbf{h}_e \right] + \mathbf{h} \right] \quad (55)$$

This is how \mathbf{k} was chosen in the simulations. The complete control law is now specified by equations (14), (42), (43), (44) and (55).

Use of a boundary layer

The discontinuous control law that satisfies the sliding conditions achieves in theory perfect tracking in the presence of model uncertainty. In practice switching is not instantaneous, leading to control chattering and extremely high control activity, which may excite unmodelled high-frequency dynamics.

To eliminate chattering, the discontinuous control law is approximated by a continuous control law in a thin *boundary layer*, B , neighboring the switching surface:

$$B = \left\{ (\mathbf{x}_a, \mathbf{R}_a, \dot{\mathbf{x}}_a, \mathbf{w}_a) \mid |s_i(\mathbf{x}_e, \mathbf{q}_e, \dot{\mathbf{x}}_e, \mathbf{w}_e)| \leq \mathbf{f}_i \right\} \quad (56)$$

where $\mathbf{f}_i > 0$ is the boundary layer thickness. Inside the boundary layer, the discontinuous function $\text{sign}(s_i)$ in the robustness term (44) is replaced by the continuous saturation function:

$$\text{sat}(s_i / \mathbf{f}_i) = \begin{cases} s_i / \mathbf{f}_i & \text{if } |s_i| \leq \mathbf{f}_i \\ \text{sign}(s_i) & \text{if } |s_i| > \mathbf{f}_i \end{cases} \quad (57)$$

Outside the boundary layer the control law is chosen as before. Hence, outside the boundary layer the sliding conditions (38) are satisfied. All system trajectories approach the boundary layer. Once inside the boundary layer the sliding conditions guarantee that the system trajectories remain there, so that the boundary layer is an invariant set. Given error bounds (33), (34), (37), interpolation of the control law in the boundary layer leads to tracking within a guaranteed precision ϵ rather than “perfect” tracking.

One can show (Slotine and Sastry, 1983; Slotine and Li, 1991) that smoothing the control discontinuity inside the boundary layer in effect assigns a low pass filter structure to the local dynamics of the sliding variables, s_i . The associated cutoff frequencies are k_i / \mathbf{f}_i . Chattering can be eliminated by choosing the boundary layer thickness such that unmodelled high-frequency dynamics are not excited.

VI. SPACECRAFT CONTROL

This section presents the design and simulation of a spatial sliding controller for a spacecraft. This illustrates the design procedure and shows a potential application of the method. The spacecraft is assumed to be a rigid body acted upon by six actuators. Three actuators exert control thrusts along the distinguished body axes. Three actuators exert control torques about the body axes. This could be achieved in practice using, e.g., gimballed, proportional thrusters and momentum wheels.

Three kinds of uncertainties are considered: (1) the mass of the spacecraft is not exactly known, (2) the principal moments of inertias are not exactly known, and (3) the actuator axes are not exactly aligned with the principal axes of inertia.

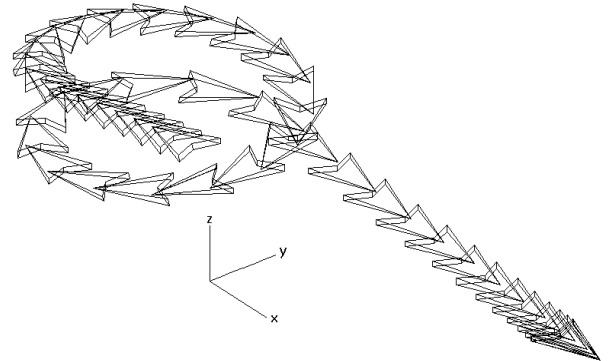


Figure 2: Desired trajectory for the spacecraft

Consider the control problem of making the spacecraft track the desired trajectory shown in Fig. 2. The desired trajectory consists of three segments. First the spacecraft moves in a straight line with a constant translational velocity. It then accelerates downwards in a helical motion. Finally the spacecraft “lands” on the x-y plane according to a parabolic curve. During the helical motion the desired attitude is aligned with the tangent, normal and bi-normal vectors of the path of the center of mass. The angular velocities are discontinuous at the transitions between the segments. These infeasible parts of the trajectory can be regarded as perturbations.

Control law

The translational equation of motion of the spacecraft is

$$m\ddot{\mathbf{x}}_a = \mathbf{R}_a \mathbf{G} \quad (58)$$

where m is the total mass, \mathbf{x}_a is the position of the centroid with respect to an inertial frame, \mathbf{R}_a is the spacecraft attitude with respect to the inertial frame, and \mathbf{G} is the body-relative actuator force. The rotational equations of motion are:

$$\dot{\mathbf{w}}_a = -\mathbf{J}^{-1} \tilde{\mathbf{w}}_a \mathbf{J} \mathbf{w}_a + \mathbf{J}^{-1} \mathbf{t} \quad (59)$$

where \mathbf{w}_a is the body-relative angular velocity, \mathbf{J} is the moment of inertia matrix, and \mathbf{t} is the body-relative actuator torque. These two equations can be put in the form of the system model (3) used in the derivation of the general control law. The functions \mathbf{f} and \mathbf{B} are

$$\mathbf{f} = \begin{bmatrix} \mathbf{0} \\ -\mathbf{J}^{-1} \tilde{\mathbf{w}}_a \mathbf{J} \mathbf{w}_a \end{bmatrix} \quad (60)$$

$$\mathbf{B} = \begin{bmatrix} \frac{1}{m} \mathbf{R}_a & \mathbf{0} \\ \mathbf{0} & \mathbf{J}^{-1} \end{bmatrix} \quad (61)$$

assuming $\mathbf{u} = [\mathbf{G}^t \quad \mathbf{t}^t]^t$. Let the uncertainty of the moment of inertia matrix be bounded as follows:

$$\mathbf{J} = (\mathbf{I}_3 + \mathbf{D}_J) \hat{\mathbf{J}}, \quad |\mathbf{D}_J| \leq \mathbf{D}_J \quad (62)$$

where again the inequality constraint is applied element-wise. Let the uncertainty on the mass be bounded by

$$m_{\min} \leq m \leq m_{\max} \quad (63)$$

Given the bounds on the uncertainty of the mass and the inertia matrix, the next step is to compute bounds on the uncertainty of \mathbf{f} and \mathbf{B} . Consider the difference between the actual and estimated dynamic function \mathbf{f} :

$$\mathbf{f} - \hat{\mathbf{f}} = \begin{bmatrix} \mathbf{0} \\ -\mathbf{J}^{-1} \tilde{\mathbf{w}}_a \mathbf{J} \mathbf{w}_a + \hat{\mathbf{J}}^{-1} \tilde{\mathbf{w}}_a \hat{\mathbf{J}} \mathbf{w}_a \end{bmatrix} \quad (64)$$

Substitution of the bounds on the uncertainty on the moment of inertia matrix (62) gives:

$$\mathbf{f} - \hat{\mathbf{f}} = \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{J}}^{-1} \left[-(\mathbf{I}_3 + \mathbf{D}_J)^{-1} \tilde{\mathbf{w}}_a (\mathbf{I}_3 + \mathbf{D}_J) + \tilde{\mathbf{w}}_a \right] \hat{\mathbf{J}} \mathbf{w}_a \end{bmatrix} \quad (65)$$

If the magnitudes of the eigenvalues of \mathbf{D}_J are much smaller than one, then the following first-order approximation can be used (Atkinson, 1987):

$$(\mathbf{I}_3 + \mathbf{D}_J)^{-1} \approx \mathbf{I}_3 - \mathbf{D}_J \quad (66)$$

In practice this means the uncertainty must not be too large. Using this approximation we obtain

$$\mathbf{f} - \hat{\mathbf{f}} = \begin{bmatrix} \mathbf{0} \\ -\hat{\mathbf{J}}^{-1} \left[-\tilde{\mathbf{w}}_a \mathbf{D}_J + \mathbf{D}_J \tilde{\mathbf{w}}_a \right] \hat{\mathbf{J}} \mathbf{w}_a \end{bmatrix} \quad (67)$$

The magnitude of uncertainty can next be bounded by using the Euclidean matrix norm and the bounds on the uncertainty on the moment of inertia matrix (62):

$$\|\mathbf{f} - \hat{\mathbf{f}}\| \leq \left[\frac{\mathbf{0}}{2 \|\hat{\mathbf{J}}^{-1}\|_2 \|\tilde{\mathbf{w}}_a\|_2^2 \|\mathbf{D}_J\|_2 \|\hat{\mathbf{J}}\|_2 \mathbf{I}_3} \right] = \mathbf{F} \quad (68)$$

Again the uncertainty of \mathbf{B} is bounded by a to-be-computed matrix \mathbf{D}_B according to (5). These bounds will be expressed in terms of bounds on the uncertainty on the mass and the inertia matrix. From (5) and the equations of motion (60) we have

$$\mathbf{I}_6 + \mathbf{D}_B = \mathbf{B} \hat{\mathbf{B}}^{-1} = \begin{bmatrix} \frac{1}{m} \mathbf{R}_a & \mathbf{0} \\ \mathbf{0} & \mathbf{J}^{-1} \end{bmatrix} \begin{bmatrix} \hat{m} \mathbf{R}'_a & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{J}} \end{bmatrix} = \begin{bmatrix} \frac{\hat{m}}{m} \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}^{-1} \hat{\mathbf{J}} \end{bmatrix} \quad (69)$$

The moment of inertia matrix, \mathbf{J} , and its estimate, $\hat{\mathbf{J}}$, are symmetric matrices. Using this property and the approximation (66) it follows that

$$\mathbf{J}^{-1} \hat{\mathbf{J}} = (\hat{\mathbf{J}} \mathbf{J}^{-1})^t = \left((\mathbf{I}_3 + \mathbf{D}_J)^{-1} \right)^t \approx (\mathbf{I}_3 - \mathbf{D}_J)^t = \mathbf{I}_3 - \mathbf{D}'_J \quad (70)$$

The uncertainty on the input matrix can then be written as:

$$\mathbf{D}_B = \begin{bmatrix} \left(\frac{\hat{m}}{m} - 1 \right) \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & -\mathbf{D}'_J \end{bmatrix} \quad (71)$$

An upper bound of the extent of uncertainty on the input matrix is given by

$$\mathbf{D}_B = \begin{bmatrix} \left(\frac{m_{\max}}{m_{\min}} - 1 \right) \mathbf{I}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}'_J \end{bmatrix} \quad (72)$$

With these bounds on the uncertainty the control law presented in Sec. V can be used. Summarizing, the sliding variable is defined by (14). The control input, \mathbf{u} , is given by (43). It consists of an equivalent control term, \mathbf{u}_{eqv} , (42) and a robust control term, \mathbf{u}_r , (44). Control parameter \mathbf{k} is chosen according to (55). Bounds \mathbf{F} and \mathbf{D}_B are given by (68) and (72). The saturation function (57) is used inside the boundary layer.

Model and controller parameters

The model parameters were chosen arbitrarily. The following mass and moments of inertia were used for simulation of the dynamics of the spacecraft:

$$m = 10 \text{ kg}, \quad \mathbf{J} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix} \text{ kg m}^2 \quad (73)$$

The estimated parameters are different to show that the controller is robust. The estimated mass and the extent of the uncertainty on the mass are

$$m_{\min} = 9.5 \text{ kg} \leq m \leq m_{\max} = 12 \text{ kg} \quad (74)$$

$$\hat{m} = \frac{m_{\max} + m_{\min}}{2} = 10.75 \text{ kg} \quad (75)$$

Qualitatively, the mass estimation error was assumed to be less than 26%. The estimated moments of inertia were chosen such that there was an error in the estimation of the direction of the principal axes of inertia and on the values of the principal moments of inertia:

$$\hat{\mathbf{J}} = \begin{bmatrix} 0.1030 & -0.0009 & -0.0021 \\ -0.0009 & 0.1920 & -0.0012 \\ -0.0021 & -0.0012 & 0.3120 \end{bmatrix} \text{ kg m}^2 \quad (76)$$

Matrix \mathbf{D}_J is a bound on the estimation error

$$\mathbf{D}_J = \begin{bmatrix} 0.0322 & 0.0049 & 0.0072 \\ 0.0099 & 0.0459 & 0.0046 \\ 0.0217 & 0.0068 & 0.0420 \end{bmatrix} \quad (77)$$

The spectrum of this matrix is

$$\mathbf{s}(\mathbf{D}_J) = 0.059 \ll 1 \quad (78)$$

Qualitatively, the moment of inertia matrix estimation error was assumed to be less than 6%. The control parameters were also chosen arbitrarily:

$$\mathbf{L}_t = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 30 \end{bmatrix} \frac{1}{s}, \quad \mathbf{L}_v = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 30 \end{bmatrix} \frac{1}{s} \quad (79)$$

$$\mathbf{h} = [10 \ 15 \ 20 \ 10 \ 15 \ 20]^t \quad (80)$$

$$\mathbf{f} = [0.1 \ 0.1 \ 0.1 \ 0.1 \ 0.1 \ 0.1]^t \quad (81)$$

Tuple \mathbf{h} has units of m/s^2 and rad/s^2 , as appropriate. Tuple \mathbf{f} has units of m/s and rad/s , as appropriate. For perfect tracking the initial conditions must equal the initial desired state. In order to simulate the reaching phase (the behavior of the system away from the boundary layer), a mismatch in the initial conditions was used. The initial deviation from the desired position was $[0.5 \ -0.5 \ 0.5]^t$ m. Initially, the spacecraft was rotated 10° about $1/\sqrt{14}[1 \ 2 \ 3]^t$ with respect to the desired attitude.

Simulation results

To reduce the number of plots, only results for attitude tracking are shown. The translational tracking method is conventional, with results similar to those for attitude. The actual trajectory is barely distinguishable from the desired trajectory shown in Fig. 2, and thus not shown.

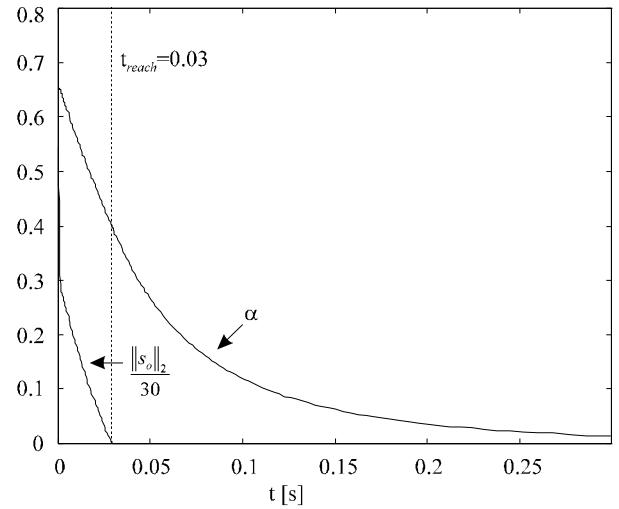


Figure 3: Norm of \mathbf{s}_o and α as functions of time

The Euclidean norm of the sliding variable \mathbf{s}_o and the attitudinal error α are shown in Fig. 3 as functions of time. Consider the behavior of the sliding variable. Initially the sliding variable is nonzero due to a mismatch of initial conditions; the system is off the sliding surface. The deviation from the sliding surface decreases until it reaches zero at 0.03s. The sliding variable then stays in the boundary layer; in the simulation it was essentially on the sliding surface. This behavior is characteristic for sliding mode control. The sliding conditions (38) guarantee that if the system is initially off the surface, the surface (or boundary layer) will be reached in finite time. The phase in which the sliding variable decreases to zero is the *reaching phase*. Using (39), an upper bound for the reaching time is

$$t_{reach} \leq \frac{|s_i(t=0)|_{\max}}{h_{i\min}} = \frac{9}{10} = 0.9 \text{ s} \quad (82)$$

The plot shows that the reaching time is 0.03s and thus well within the upper bound.

After the sliding surface is reached the sliding variable stays on the surface. This behavior is characteristic for sliding control. Satisfying the sliding conditions (38) guarantees that if the sliding variable is on the sliding surface, it stays on the surface. The controller is in the *sliding regime* if the sliding variable is zero. The error then decays to zero.

Figure 4 shows the error vector, $\mathbf{q}_e = \mathbf{h}_e[4..6]$, as a function of time. The figure shows also the lines tangent to the curves at time $t=0.03$ s. For small attitudinal errors the error decays approximately exponentially to zero with time constants $1/\lambda_i$. Note that Fig. 3 shows that the error magnitude α at time $t=0.03$ s is approximately 0.4 rad (20°). The time constants can be computed from the reaching time and the crossings of the tangent lines with the time axis:

$$\begin{aligned} h_{e4}: t_4 &= 0.130 - 0.03 = 0.100\text{s} \\ h_{e5}: t_5 &= 0.080 - 0.03 = 0.050\text{s} \\ h_{e6}: t_6 &= 0.063 - 0.03 = 0.033\text{s} \end{aligned} \quad (83)$$

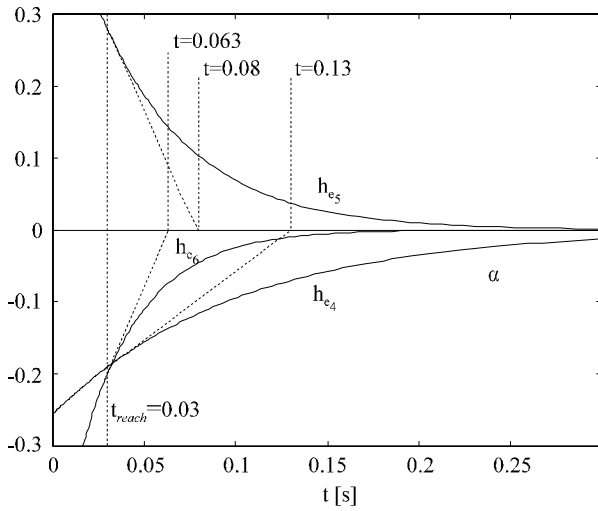


Figure 4: Attitudinal error as a function of time.

These time constants are consistent with those determined by matrix \mathbf{L}_0 :

$$\begin{aligned} \frac{1}{\mathbf{I}_{o1}} &= \frac{1}{10} = 0.100s \\ \frac{1}{\mathbf{I}_{o2}} &= \frac{1}{20} = 0.050s \\ \frac{1}{\mathbf{I}_{o3}} &= \frac{1}{30} = 0.033s \end{aligned} \quad (84)$$

The absolute value of $s_4 = s_{o1}$ and $h_{e4} = \mathbf{q}_{e1}$ are plotted as functions of time in Fig. 5. The switching instants between the rectilinear and helical segments, and between the helical and parabolic segments are at 1s and 4.755s, respectively. The plot shows that the discontinuity of the desired angular velocity at the switching instants acts as a perturbation. Focus now on the intervals between the switching instants, especially between 1.5s and 4.7s. In this interval the bounds on the sliding variable can be translated directly into bounds on the tracking error. Sliding variable s_4 is bounded by $4 \cdot 10^{-4}$. The tracking error, h_{e4} , is bounded by $4 \cdot 10^{-5}$, which is a factor $\mathbf{I}_4 = 10$ smaller.

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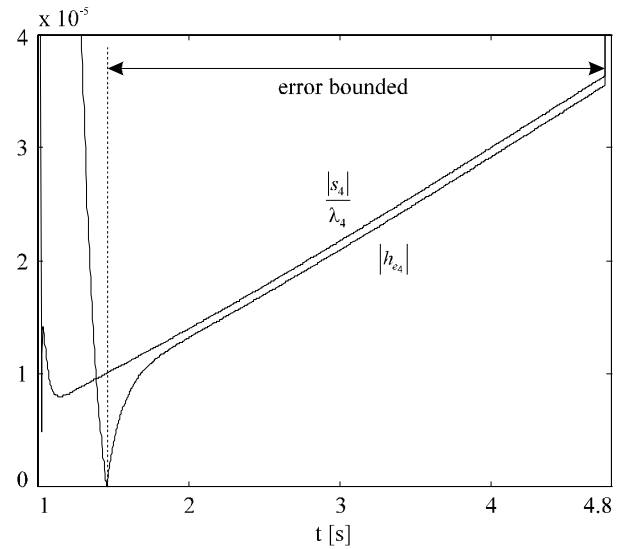


Figure 5: Absolute value s_4 and h_{e4} as functions of time.

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A. RATE OF CHANGE OF ROTATION ANGLE

This appendix derives the relation between the rate of change of the rotation angle, $\dot{\mathbf{a}}$, and the angular error velocity, \mathbf{w}_e . The basic equation relating these two variables is

$$\frac{d}{dt}e^{\tilde{\mathbf{q}}_e} = \dot{\mathbf{R}}_e = \mathbf{R}_e \mathbf{w}_e \quad (85)$$

First consider $\mathbf{a} \in (0, \mathbf{p})$, so that \mathbf{a} is neither zero nor \mathbf{p} . The trace of both sides of (85) can be computed. The trace of the left-hand side is (Strang, 1988)

$$\begin{aligned} \operatorname{tr}\left(\frac{d}{dt}e^{\tilde{\mathbf{q}}_e}\right) &= \frac{d}{dt}\operatorname{tr}\left(e^{\tilde{\mathbf{q}}_e}\right) = \frac{d}{dt}(1 + 2\cos(\mathbf{a})) \\ &= -2\sin(\mathbf{a})\dot{\mathbf{a}} \end{aligned} \quad (86)$$

In general it is true that (Murray et al, 1994):

$$\mathbf{R}_e = e^{\tilde{\mathbf{q}}_e} = \mathbf{I} + \frac{1 - \cos(\mathbf{a})}{\mathbf{a}^2} \tilde{\mathbf{q}}_e^2 + \frac{\sin(\mathbf{a})}{\mathbf{a}} \tilde{\mathbf{q}}_e \quad (87)$$

Substituting this into (85) and taking the trace of the right-hand side yields

$$\operatorname{tr}(\mathbf{R}_e \tilde{\mathbf{w}}_e) = \frac{\sin(\mathbf{a})}{\mathbf{a}} \operatorname{tr}(\tilde{\mathbf{q}}_e \tilde{\mathbf{w}}_e) = -\frac{2\sin(\mathbf{a})}{\mathbf{a}} \mathbf{q}'_e \mathbf{w}_e \quad (88)$$

From (85), (86) and (88) it follows that

$$-2\sin(\mathbf{a})\dot{\mathbf{a}} = -\frac{2\sin(\mathbf{a})}{\mathbf{a}} \mathbf{q}'_e \mathbf{w}_e \quad (89)$$

Because this must hold for arbitrary \mathbf{a} except zero and \mathbf{p} we conclude that

$$\dot{\mathbf{a}} = \frac{1}{\mathbf{a}} \mathbf{q}'_e \mathbf{w}_e = \mathbf{e}^t \mathbf{w}_e \quad (90)$$

where \mathbf{e} is the unit vector in the direction of \mathbf{q}_e . Consider now the case that $\mathbf{a} = 0$. The Taylor expansion of the matrix exponential in the neighborhood of $\mathbf{a} = 0$ is

$$\mathbf{R}_e = e^{\tilde{\mathbf{q}}_e} = \mathbf{I} + \tilde{\mathbf{q}}_e + \frac{1}{2!} \tilde{\mathbf{q}}_e^2 + \dots \quad (91)$$

Because the matrix exponential is an analytic function, so that Taylor expansion of the derivative is the derivative of the Taylor expansion, we have to first order:

$$\frac{d}{dt}e^{\tilde{\mathbf{q}}_e} \approx \dot{\tilde{\mathbf{q}}_e} \quad (92)$$

Again using the Taylor expansion we have to first order:

$$\mathbf{R}_e \tilde{\mathbf{w}}_e \approx \tilde{\mathbf{w}}_e \quad (93)$$

From (85), (92) and (93) it follows that near $\mathbf{a} = 0$

$$\dot{\mathbf{q}}_e \approx \mathbf{w}_e \quad (94)$$

with $\dot{\mathbf{q}}_e = \mathbf{w}_e$ at $\alpha = 0$. The rate of change of \mathbf{a} is then

$$\dot{\mathbf{a}} = \|\mathbf{w}_e\|_2 = \mathbf{e}^t \mathbf{w}_e \quad (95)$$

where \mathbf{e} is the unit vector in the direction of \mathbf{w}_e . Thus for the interval $\mathbf{a} \in [0, \mathbf{p})$

$$\dot{\mathbf{a}} = \mathbf{e}^t \mathbf{w}_e \quad (96)$$

where \mathbf{e} is the unit vector in the direction of \mathbf{q}_e if $\mathbf{a} \in (0, \mathbf{p})$, \mathbf{e} is the unit vector in the direction of \mathbf{w}_e if $\mathbf{a} = 0$.

B. RATE OF CHANGE OF THE ROTATION VECTOR

In this appendix the rate of change of the rotation vector \mathbf{q}_e is expressed as a function of the error angular velocity and the rotation vector. Consider:

$$\frac{d}{dt}(\mathbf{R}_e - \mathbf{R}_e^t) = \dot{\mathbf{R}}_e - \dot{\mathbf{R}}_e^t \quad (97)$$

From (87) it follows that

$$\mathbf{R}_e - \mathbf{R}_e^t = \frac{2\sin(\mathbf{a})}{\mathbf{a}} \tilde{\mathbf{q}}_e \quad (98)$$

From (10) it follows that

$$\frac{d}{dt}\left(\frac{2\sin(\mathbf{a})}{\mathbf{a}} \tilde{\mathbf{q}}_e\right) = \mathbf{R}_e \mathbf{w}_e + \mathbf{w}_e \mathbf{R}_e^t \quad (99)$$

Carrying out the differentiation and rearranging:

$$\frac{2\sin(\mathbf{a})}{\mathbf{a}} \dot{\tilde{\mathbf{q}}_e} = \mathbf{R}_e \mathbf{w}_e + \mathbf{w}_e \mathbf{R}_e^t + \frac{2\sin(\mathbf{a})}{\mathbf{a}^2} \dot{\mathbf{a}} \tilde{\mathbf{q}}_e - \frac{2\cos(\mathbf{a})}{\mathbf{a}} \dot{\mathbf{a}} \tilde{\mathbf{q}}_e \quad (100)$$

In general for vectors \mathbf{v} and \mathbf{w} and square matrix \mathbf{A} the following identity is true:

$$\tilde{\mathbf{v}} = \mathbf{A} \tilde{\mathbf{w}} + \tilde{\mathbf{w}} \mathbf{A}^t \text{ if and only if } \mathbf{v} = (\operatorname{tr}(\mathbf{A})\mathbf{I} - \mathbf{A}^t) \mathbf{w} \quad (101)$$

Applying this identity to (100) yields:

$$\frac{2\sin(\mathbf{a})}{\mathbf{a}} \dot{\tilde{\mathbf{q}}_e} = (\operatorname{tr}(\mathbf{R}_e)\mathbf{I} - \mathbf{R}_e^t) \mathbf{w}_e + 2 \frac{\sin(\mathbf{a}) - \mathbf{a} \cos(\mathbf{a})}{\mathbf{a}^2} \dot{\mathbf{a}} \tilde{\mathbf{q}}_e \quad (102)$$

As shown in Appendix A (94), for $\alpha = 0$

$$\dot{\tilde{\mathbf{q}}_e} = \mathbf{w}_e \quad (103)$$

Equation (102) is valid for $\mathbf{a} \in (0, \mathbf{p})$, whence

$$\dot{\mathbf{q}}_e = \frac{\mathbf{a}}{2\sin(\mathbf{a})} (\operatorname{tr}(\mathbf{R}_e)\mathbf{I} - \mathbf{R}_e^t) \mathbf{w}_e + \frac{1}{\mathbf{a}} \dot{\mathbf{a}} \tilde{\mathbf{q}}_e - \frac{\cos(\mathbf{a})}{\sin(\mathbf{a})} \dot{\mathbf{a}} \tilde{\mathbf{q}}_e \quad (104)$$

Using (96) it follows that

$$\dot{\mathbf{q}}_e = \left[\frac{\mathbf{a}}{2\sin(\mathbf{a})} (\operatorname{tr}(\mathbf{R}_e)\mathbf{I} - \mathbf{R}_e^t) + \left(1 - \mathbf{a} \frac{\cos(\mathbf{a})}{\sin(\mathbf{a})}\right) \mathbf{e} \mathbf{e}^t \right] \mathbf{w}_e \quad (105)$$

where \mathbf{e} is the unit vector in the direction of \mathbf{q}_e .