SENSIBLE MATHEMATICS:

SEARCHING FOR CHARACTERISTICS USING LESSON STUDY

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This paper describes the search of six Dutch teachers for characteristics of sensible mathematics. The mathematics teachers investigated their theoretically founded classroom practices using lesson study. Two successive research lessons about the introduction of the derivative were jointly planned, implemented, and live observed. The teachers revised and re-teached the research lessons based on collaborative discussions at school and reflections at the university. The results of the study show that making sense of the derivative starts with encouraging students to communicate intuitively using own words. This is followed by the development of visualizations (icons) and finally results in the use of numbers (symbols): operations with numbers and reasoning about operations with numbers.

INTRODUCTION

This study focusses on teachers' collaborative investigation to determine characteristics of sensible mathematics. In 2008 the Dutch government recognized a stagnated progress in numeracy at scientific studies as a consequence of a lack of students' mastering of mathematical skills. This resulted in an increased attention for algorithms and correct calculations at secondary schools. The balance moved from a focus on Skemp's (1976) relational understanding to instrumental understanding of mathematical concepts.

Research at the University of Twente focusses on the effects of teacher design teams. In this context a number of mathematics teachers collaborated with the intention to improve mathematics education. The researcher (first author) invited six mathematics teachers to start a lesson study team. Teacher selection was based on good experiences the researcher had with the teachers during teacher trainee supervision. Lesson study is a professional development strategy in which teachers collaboratively investigate teaching and learning practices by means of live classroom observations and post-lesson discussions (Stepanek, Appel, Leong, Mangan, & Mitchell, 2007). We used lesson study to collaboratively investigate the characteristics of sensible mathematics.

Making sense of mathematics arises initially through coherent perception and action, and develops through coherent use of operations in arithmetic and algebra (Tall, 2012). The lesson study team's investigation of making sense of mathematics builds on Tall's (2008) philosophy of long-term mathematical thinking in relation with Bruner's (1966) framework of representations.

PROBLEM DEFINITION AND RESEARCH QUESTION

Lesson study is in Japan widely used and deeply rooted for over a century. Lesson study makes teaching approaches more practical and understandable to teachers through a deeper understanding of content and student thinking (Murata, 2011). In 2009 a four-year lesson study project was initiated at the University of Twente. The first project year focussed on the effects of lesson study on teachers' professional development. The results showed complexities with regard to culture differences with Japan (Verhoef & Tall, 2011). The second project year showed a positive effect of the use of GeoGebra in the context of the introduction of the derivative. This paper reports the third project year in which the search for characteristics of sensible mathematics was central. Our research question is:

What are the characteristics of sensible mathematics in the context of the introduction of the derivative?

THEORETICAL FRAMEWORK

(a) A sensible approach to mathematics

A sensible approach to mathematics takes account of the structures of mathematics and of the increasing levels of sophistication as learning progresses from sense through perception, then through the relationships of operation and a developing sense of reason (Chin &Tall, 2012). This approach relates to Bruner's (1966) modes of representation. Bruner distinguished: (a) action based enactive representation, (b) image based iconic representation, and (c) symbolic representation including not only written and spoken language but also the symbolism of arithmetic and the language of logic. In his 'three worlds' Tall (2008) combined enactive and iconic representations into a long-term development of conceptual embodiment. The enactive and iconic modes of human perception and action develop into the mental world of perceptual and mental thought experiment. Operational symbolism develops from embodied actions, such as counting and measuring, and encapsulates as symbols in arithmetic. The higher level of logic specified by Bruner is seen as a distinct level of axiomatic formalism based on set-theoretic definitions and formal proof. This distinction is essential in a sensible approach to calculus which is based on visual properties of graphs and symbolic operations with functions prior to the major change to the formal theory of mathematical analysis.

We suggest that, to make sense of mathematical thinking, the teacher should be aware of the changing needs of the student in new situations, to build on previous success and to realize that what worked before will need a new approach to make sense of the new situation. To do this we consider how the learner makes sense through perception based on fundamental conceptual embodiment and thought experiment, then through the coherent relationships in operational symbolism, and later in terms of reasoning based on definition and deduction. In school mathematics, reasoning develops in various forms, like refining ways of thinking by formulating observed regularities as principles such as the transition from the practical slope of a graph to the theoretical symbolic calculation of the ratio $\Delta y/\Delta x$ and dy/dx. In this paper we interpret sense making of mathematics in terms of *perception*, *operation* and *reasoning*. We distinguish the practical enactive and iconic representations, and the theoretical symbolic representation of the derivative. That implies sense making of the derivative as the visual changing slope of a graph y=f(x), which is linked to the symbolic calculation of the practical slope (f(x+h)-f(x))/h. The ratio can be visualized and imagined to stabilize on the theoretical slope f'(x).

(b) Lesson study as a strategy for professional development

Lesson study can be typified as a *live* research lesson. The live research lesson creates a unique learning opportunity for teaching. Lewis, Perry and Murata (2006) describe three specific areas that develop through the lesson study process: (1) teachers' knowledge, (2) teachers' commitment, and (3) community and learning resources. While teaching is considered an independent and often isolated practice in many countries, lesson study brings teachers together to share goals, discuss ideas, and work collaboratively.

Murata (2011) reports the following five attention points. Firstly, lesson study is centred around teachers' interests. Teachers should perceive lesson study goals to be important and relevant for their own classroom practice. Secondly, lesson study is student focussed. The lesson study activities should direct teachers' attention to student learning and the relation between learning and teaching. Thirdly, lesson study has a research potential. Teachers share physical observation experiences and these provide research opportunities. Fourthly, lesson study is a reflective process. Teachers have to reflect on their teaching practice and subsequent student learning in an educational community. Fifthly, in lesson study teachers work interdependently and collaboratively. Isoda (2010) characterized the lesson study cycle process as consisting of the following collaborative elements: planning (preparation), doing (observation), and seeing (discussion and reflection). He advocated the use of scientific literature as a basis for deepening teaching strategies.

RESEARCH METHOD

Participants

Six mathematics teachers from different secondary schools participated in the lesson study team during the school year 2011-2012, see Table 1. The first three teachers participated in previous years. School management facilitated the teachers by giving them half a day weekly for participating in the lesson study project.

Table 1: Description of participants

	Work experience in 2010	Education and teaching experience
A	17 years	BSc math + MSc math education; lower level to upper level high school students
B	14 years	BSc math + BSc math education; mostly upper level high school students
C	one year	BSc engineering + MSc math education; mostly upper level high school students
D	26 years	MSc math + MSc math education; mathematics teacher team leader

E	19 years	BSc math + MSc math education; lower level to upper level high school students
F	one year	MSc math + BSc math education; a mathematics PhD

Besides the teachers, the lesson study team consisted of four staff members of the University of Twente: a mathematician, a mathematics teacher trainer, a PhD-candidate and the researcher (first author). The staff members had specific roles in the lesson study team.

Research instruments

The research instruments consisted of three lesson plans, field notes of student observations and written reports of the discussions at the teachers' school, and the plenary reflections at the university. The observers were participants of the lesson study team plus interested school colleagues.

Context of the study

The teachers revised the textbook with regard to the introduction of the derivative with a focus on sensible mathematics. They intended to pick up the textbook approach, with a focus on mastering differentiation rules, after the introduction.

Based on last year's experiences with lesson study, the teachers decided to use GeoGebra for sense making of the derivative. The teachers started a process of zooming in at a fixed point on the graph (Bruner's enactive representation). They continued with a process of awareness using the visualization of rate of change. They wanted to introduce an icon (Bruner's iconic representation) to develop a link to the use of numbers (Bruner's symbolic representation).

The teachers worked in three pairs (P1, P2 and P3). In each pair, one teacher did not have any previous experience with lesson study. The pairs started successively teaching two lessons. Firstly, P1 started with the first research lesson. P2 continued the same day at another location. Secondly, P1 continued with the second research lesson next day while P2 continued later. The lessons were planned collaboratively, observed and discussed at the teacher's school. The first four lessons (from P1 and P2) were evaluated in a plenary meeting at the university. This resulted in a revision of the research lessons, and this was used in class by P3. This lesson was collaboratively discussed at the teacher's school and plenary evaluated at a university meeting.

Data collection, processing and analysis

P1's lesson plan was summarized. The other lesson plans were described in relation with P1's lesson plan. The field notes of the student observations were classified with regard to Chin and Tall's (2012) categorizations: perception, operation and reasoning. Remarkable (discussion and reflection) report statements were coded as practical (enactive, iconic) or theoretical (symbolic) based on Bruner's (1966) framework of representations. The classifications, codes and analysis were member checked with the teachers afterwards.

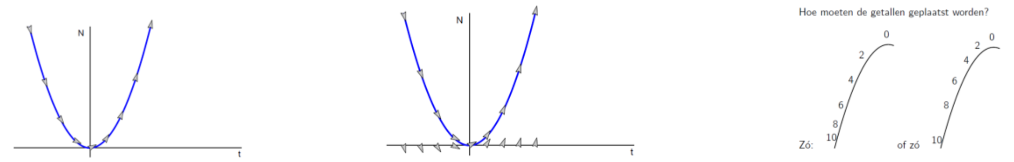
RESULTS

Lesson plans

Below, first the results of the lesson plans will be reported, followed by field notes of student observations, and finally elements from the discussions and plenary reflections. P1's lesson plan emphasized student interaction. The teachers tried to make sense by activating students' communication explicitly in their lesson plan (Figure 1).

The teacher introduces the increasing and decreasing graph in comparison with a jumping frog in the first lesson. The Power Point sheet shows the words: increasing/decreasing; monotonic; tangent; slope. The teacher gives each student pair one assignment. One student of each pair, sitting with backs against each other, receives an arbitrary graph on paper. The student describes the given graph in own words, the other student tries to draw the graph on his empty paper.

The teacher continues plenary with the graph of a parabola. He has drawn arrows on the graph (first two figures below). The teacher reminds the students of the computer game Angry Birds, making sense to the graph's change in one point. The teacher shows the third figure below and asks 'Do you know the right place of these numbers'?



The teacher continues the second lesson using numbers (slopes) illustrating a change each. He uses squares on his board and puts line segments in here with the comparable slopes. The students get an arbitrary graph on paper each. The teacher asks to put numbers – illustrating a change each - at some fixed points on the delivered graph.
The teacher ends plenary with the calculation of the slope of a straight line through two closed points on the graph, suggesting this gives one answer exactly: the change in one point on the graph.

Figure 1: Lesson plan of the first pair

P2's lesson plan emphasizes operations with symbols. The teachers replace student interaction with worksheets. They want to reveal students thinking on paper as much as possible. The worksheet focusses on reasoning about numbers as rates of change. Figure 2 lists the core problem.

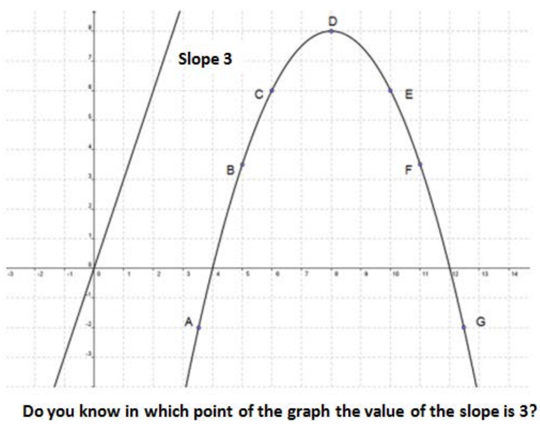


Figure 2: The slope in a point on the graph

The teachers use the problem in which students describe a graph in own words at the end of the lesson instead of in the beginning. The teachers don't use arrows. They emphasize local straightness in a point nearby the top (more curved). They introduce the zooming in process as an analogy with a view at the earth from space. The teachers continue the procedure of zooming in according to two closed points using GeoGebra suggesting that the process of zooming in gives the same result (one slope). After that they calculate the derivative in different points on the graph and establish that the numbers are elements of one straight line. They end with the ratios $\Delta y/\Delta x$ and $(f(x+\Delta x) - f(x)/\Delta x)$.

P3's first lesson starts again with a student describing a graph in own words, and continues with the graph of the parabola with arrows, see the first graph in Figure 1.The students solve problems on a worksheet in pairs. The teachers ask the students to add numbers to the arrows on the graph. They continue with asking numbers as rates of change, see Figure 2. They end with asking the right numbers on the right places, see Figure 3.

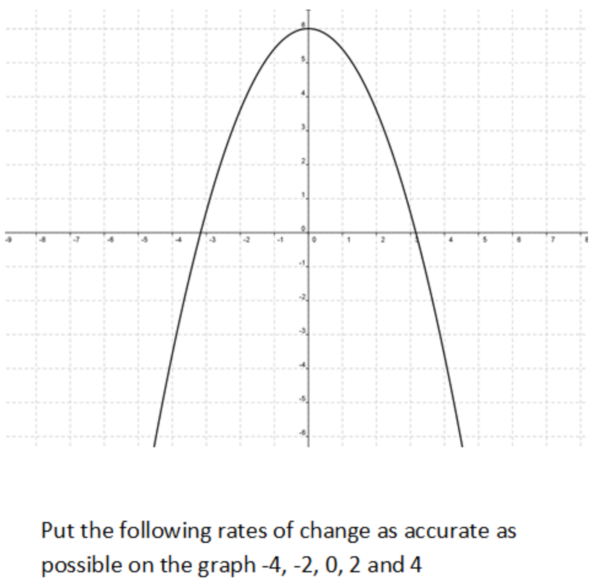


Figure 3: Right number on the right place

Field notes of student observations

Table 2 lists characteristic field notes of the student observations. The first column lists the successive pairs and the successive two lessons. The rest of the columns shows the classifications *perception*, *operation* and *reasoning*. The cells contain characteristic field notes of student observations per pair. The dotted line marks the reflective meeting at the university after P2's lessons.

Table 2: Characteristic field notes of student observations

	Perception Students see:	Operation Students:	Reasoning Students wonder whether:
P1-1	- the words 'increasing/ decreasing; monotonic; tangent; slope' stand on the board	- draw arrows (dove tails) on the graph	- does it involve the ratio on the coordinate system or the rate of increasing?
P1-2	- line segments in a square on the board indicating the change from picture to number	- fold paper and slide with it - calculate the difference quotient	- why numbers are important, if you are able to fold the change?
P2-1	nothing at all	- draw chords, not the tangent line in a point on the graph - try by calculating - calculate x- and y-coordinates	nothing at all
P2-2	- zooming in with GeoGebra - local straightness - a chord on a small interval - zooming out gives the tangent	- slide the lines, estimate and calculate - calculate the difference quotient	nothing at all
P3-1	- the words 'increasing/ decreasing; monotonic; tangent; slope' are given in advance	- sketch on the worksheet - slide a line with a ruler - draw chords instead of a tangent line - don't mention the word tangent	- do arrows on a straight line be the same ? - do the closer the arrows on the x-axis be, the flatter the graph? - does an arrow on the x-axis show the height of the graph?
P3-2	-zooming in with GeoGebra local straightness see that Δx becomes smaller	- make a curve straight - slide the lines, estimate and calculate - formulate a straight line	- what happens if A coincides B? - what happens if Δx becomes zero? - does dy/dx not be defined, dx=0?

Perception: the perception moves to the visualization of symbolic representations during the lesson study process. The words 'increasing/ decreasing; monotonic; tangent; slope' in P1's first lesson stimulates students' communication. P1's students start to fold paper when the teacher asks to put numbers – illustrating a change each - at some fixed points on the delivered graph (see Figure 1, the last sentence of the last but one paragraph). P2 does not focus on perception in the first lesson.

Operation: the number of student activities increases during the lesson study process.

Reasoning: the intuitive reasoning becomes more important. P2 does not focus on reasoning at all. P1's students are trying to refine the icon 'arrow' (as a dove tail) to a line segment in a square. P2's students are impeded by the incorrect use of the 'tangent line method', learned from the physics teacher. They start with chords on a large interval (to prevent measurement errors as in physics). P3's students start to reason about the icon 'arrow'. They moved to reasoning about Δx and dx without any transition. These students have to use symbols to be able to differentiate functions.

Reports from discussions and reflections

Table 3 shows the characteristic teacher comments from the discussions and the reflections based on the student observations. The first column lists the successive pairs and the successive two lessons. The rest of the columns shows the components in types of mathematical thinking: practical (enactive and iconic) and theoretical (symbolic subdivided in local or global). The cells contain characteristic comments from the reports. The dotted line marks the reflective meeting after P2's lessons.

Table 3: Teachers' reflections

	Practical		Theoretical	
	Enactive Reflection on action	Iconic Reflection on visualization	Symbolic (local) Reflection on symbols	Symbolic (global) Reflection on symbols
P1-1	- *no* words increasing/ decreasing; monotonic; tangent; slope in advance	- an arrow with direction refers to a move, preference an 'arrow' without any direction	- students not simply work with numbers after working with arrows	- differentiation globally should develops gradually
P1-2	- maximum and minimum don't be the problem	- preference of an arrow without direction	- awareness of coaching to numbers	- support with calculations /difference quotient
P2-1	conflict with physics: derivative means chord	- the use of an icon is necessary	- change the context to a coordinate system	- calculations with a difference quotient
P2-2	- misconception that drawing a chord on a small interval when zooming in, gives a tangent line	- the use of an icon is necessary without giving a direction	- give equations of parabola and line and support students to reason about the slope	- calculations with a difference quotient
P3-1	- cut with a scissors, sew like a sewing machine does and emphasize two sides	- line segment with a dot halfway works best	- avoid stagnation in a chord, continue in coordinate system	- local discontinuous; connect increasing and monotonic increasing
P3-2	- zooming in on paper is not possible - calculation with small Δx takes time!	nothing at all	- students think that a Δx of 0,001 is exactly!	- there is a lack of numeracy

The teachers prefer to stimulate students' intuitive communication in own words, *not* giving the words 'increasing/ decreasing; monotonic; tangent; slope' in advance (enactive representation). They introduce new ideas like cutting with a scissors, and sewing like a sewing machine does and emphasize two sides. The icon develops during the lesson study from a dove tail shaped arrow (suggesting a movement), via an arrow without direction, to a line segment with a dot halfway. For the students the transition to the use of numbers is impeded by a too large gap between perception (practical) and calculation (theoretical) by themselves. For the teachers the idea grows to introduce a coordinate system to insert both a line segment related to the slope value and a graph in relation with its equation. The teachers agree to stimulate communication with the students regarding three possibilities to calculate the slope at an interval: the interval [A-x, A+x], the interval [A, A+x] and the interval [A-x, x]. The discussions focus on a too large or a too small number. Another possibility is the use of counter examples like the relation with the graph of y=abs(x).

CONCLUSIONS AND DISCUSSION

The study shows that the well-thought-off choice of an icon influences operational symbolism positively when the icon was simple chosen (without any extra information). The choice of an icon, a dove tail at the graph, seems to hide a line segment inside from the top to the bottom of the arrow, which may give rise to the idea that the concept of the derivative is inseparable from a difference quotient. Subsequently, the difference quotient gives rise to the differential quotient with which dividing by zero appears as an obstacle. The arrow as a line segment and a v-sign on top may give rise to the assumption that there is a continuous move, because the direction is given and it resembles a vector used in physics. The line segment with halfway a dot is the first step to the relational understanding of the concept of a vector field as a basis for understanding differential equations in a later phase (Figure 4).

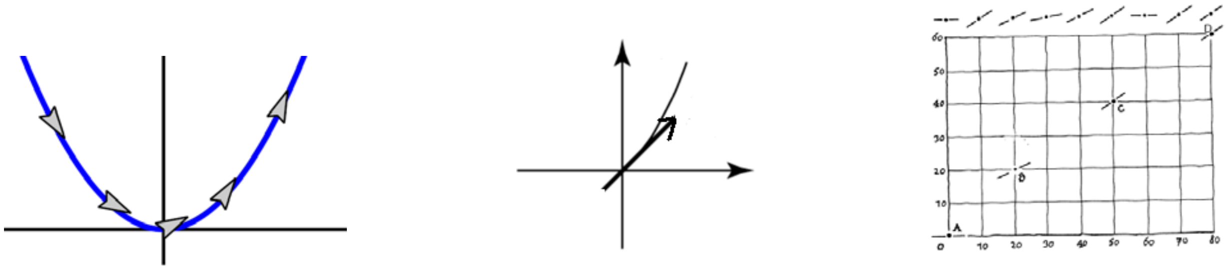


Figure 4: Different representations of arrows

The experiences of the development of an icon as being characteristic for sensible mathematics were based on the student observations in the context of lesson study. Essentially, lesson study focusses on classroom practices. The discussions after class and the reflections contribute to relational understanding by the teachers. The consulted theories encourage teachers to revise their lessons outside their textbook approach. This stimulates the lesson study process (Oshimaa, Horinoa, Oshimab, Yamamotoc, Inagakid, Takenakae, Yamaguchif, Murayamaa, & Nakayamaf, 2006).

Three characteristics typify sensible mathematics. Firstly perception: the teachers introduce conceptual embodiment by the zooming in process as an analogy with a view at the earth from space, underlining sense making of the derivative. Secondly operation: the teachers continue by cutting with a scissors enabling the students to operate their perceptions, followed by searching a relation between the scissors' cut and a straight line. Thirdly reasoning: the teachers introduce operational symbolism by a coordinate system in which the students are able to calculate. They inserted both an iconic line segment related to the symbolic slope value and a graph in relation with its equation. The teachers' collaborative problem was to switch from numbers on a graph locally (see Figure 3) to calculations with ratios $\Delta y/\Delta x$ and $(f(x+\Delta x) - f(x)/\Delta x)$ globally. They expected students' recognition of the zooming in process. The teachers were strongly impeded by their textbook approach - with the focus on the limit process. The study shows that the teachers are inclined to fall back to their textbook, in spite of the implementation of another teaching approach. This explains that P3 tries to teach the differentiation rules.

This study contributes to the realization of Tall's (2012) theory of long-term mathematical development integrating by Bruner's (1966) iconic representations in practice. Freudenthal (1984) argues that Bruner's theory of cognitive growth (that results in mathematical concept development) follows the ordering of real phenomena in order to constitute mental objects, the principle of re-invention. In this context of the derivative the students order the movements at a curve. The students are challenged to name the rates of change on an arbitrary graph in own words. The teachers focus on making sense of the derivative starting by ordering phenomena, and continue with the use of enactive and iconic representations.

REFERENCES

Bruner, J. S. (1966). *Towards a Theory of Instruction*. New York: Norton.

Chin, K.E., & Tall, D. (2012). Making Sense of Mathematics through Perception, Operation & Reason: The case of Trigonometric Functions. *Proceedings of PME 36*. Taipei, Taiwan.

Freudenthal, H. (1984). *Didactische fenomenologie van wiskundige structuren.* Utrecht: OW&OC.

Isoda, M. (2010). Lesson Study: problem solving approaches in mathematics education as a Japanese Experience. *Procedia Social and Behavioral Sciences, 8,*17–27.

Lewis, C. C., Perry, R., & Murata, A. (2006). How should research contribute to instructional improvement? The case of Lesson Study. *Educational Researcher, 35*(3), 3-14.

Murata , A. (2011). Introduction: Conceptual Overview of Lesson Study. In L.C. Hart, A. Alston & A. Murata (Eds.). *Lesson Study Research and Practice in Mathematics Education* (pp. 269-279). New York: Springer.

Oshimaa, J., Horinoa, R., Oshimab, R., Yamamotoc, T., Inagakid, S., Takenakae, M., Yamaguchif, E., Murayamaa, I., & Nakayamaf, H. (2006). Changing Teachers' Epistemological Perspectives: A case study of teacher–researcher collaborative lesson studies in Japan. *Teaching Education*, *17*(1), 43–57.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, *77*, 20–26.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.

Stepanek, J., Appel, G., Leong, M., Mangan, M. T., & Mitchell, M. (2007). *Leading lesson study: A practical guide for teachers and facilitators*. Thousand Oaks, CA: Corwin Press.

Tall, D. O. (2008). The Transition to Formal Thinking in Mathematics. *Mathematics Education Research Journal*, *20*(2), 5-24.

Tall, D. O. (2012). Making Sense of Mathematical Reasoning and Proof. Plenary to be presented at *Mathematics & Mathematics Education: Searching for Common Ground: A Symposium in Honor of Ted Eisenberg, April 29-May 3, 2012, Ben- Gurion University of the Negev, Beer Sheva, Israel.*

Verhoef, N.C., & Tall, D.O. (2011). Teacher's professional development through lesson study: effects on mathematical knowledge for teaching. *Proceedings of PME 35*. Ankara, Turkye.