32 *E. van Groesen*

THE SELF-ORGANIZATION HYPOTHESIS FOR 2D NAVIER-STOKES EQUATIONS

32.1 Introduction

In this short contribution we will sketch a rigorous proof of the "self-organization hyopthesis" for the 2D Navier-Stokes equations (see [1] for a detailed analysis).

In the physical literature this hypothesis is roughly described by saying that the long-time behaviour of the system must be related to the solution of a certain minimization problem. If the system admits two functionals that are conserved when dissipation is neglected, the minimization problem is that of minimizing the functional that decays fastest in the presence of dissipation on level sets of the slower decaying one.

For the problem under consideration, the energy E and enstrophy W serve as such functionals and W is minimized at prescribed E. Arguments that are usually used to support this hypothesis arise from considering non-linear mode interactions (which show an inverse and a normal cascade for the spectral energy and the enstrophy density respectively), together with the selective dissipation due to the viscosity (see [2,3,4] and the references therein).

In this paper we describe how a simple analysis of the deterministic equations (instead of the spectral densities) will show that for a large class of initial data this minimization problem indeed determines the spatial structure of the asymptotic motion.

After the completion of this work it appeared that FOIAS and SAUT [5] have considered this asymptotic behaviour in great detail and also for more general situations. The presentation below is simpler and follows closely the idea behind the self-organization hypothesis. In fact, this line of reasoning has proved to be useful in other cases too, and a description for more general classes of dissipative Poisson systems will be published elsewhere (see also [8]).

32.2 Preliminaries

The Navier-Stokes equations for a flow in a plane with uniform density $\rho = 1$ can be described with the Eulerian velocity $\underline{v}$ and the component of the vorticity $\underline{\omega}$ perpendicular to the plane, $\underline{\omega}$ =

rot $\underline{v} = (0,0,\omega)$, like

$$\nabla.\underline{v} = 0 \text{ and } \omega_t + \underline{v}.\nabla\omega = \nu\Delta\omega,$$

where $\nu > 0$ is the viscosity coefficient. In order to study 2D flow structures on a period grid, let $R = [0,\pi] \times [0,\pi]$ and take as boundary conditions $\underline{v}.\underline{n} = 0$ on ∂R. An appropriate continuation of a solution on R will then provide a smooth solution on a periodic grid with period 2π in both directions. With the boundary condition $\omega = 0$ on ∂R, there is a one-to-one correspondence between $\underline{v}$ and ω when related as above, and we can consider all the functionals as they appear below as functionals of ω.

For the Euler equations, i.e. when $\nu = 0$, the energy E and the enstrophy W are constants of the motion and given by

$$E = \int \frac{1}{2} \underline{v}^2 \text{ and } W = \int \frac{1}{2} \omega^2$$

(integrations over R). Let us first consider the minimization problem referred to in the self-organization hypothesis. The solutions are, apart from a scaling factor, independent of the prescribed value of E, and they are the same as those of the minimization problem for the functional Q which is the Rayleigh quotient of W and E:

$$Q = W/E.$$

For further appreciation it is important to note that Q depends only on the so-called **spatial structure** of a function ω, and not on its amplitude:

$$Q(\omega) = Q(\hat{\omega}), \text{ where } \hat{\omega} = \frac{\omega}{||\omega||}$$

(with $||\ ||$ the L_2-norm on R).

The critical points of Q (on functions ω satisfying $\omega = 0$ on ∂R) are precisely the eigenfunctions of the Laplace operator on R.

For $\underline{k} \in \mathbf{N} \times \mathbf{N}$ these eigenvalues and eigenfunctions are

$$\lambda_{\underline{k}} = |\underline{k}|^2 = k_1^{\ 2} + k_2^{\ 2},\ \Omega_{\underline{k}} = \sin k_1\ x.\sin k_2 y.$$

Denote the increasing set of eigenvalues by μ_k, $\mathbf{k} \in \mathbf{N}$ and let $\hat{\Omega}$ be the normalized eigenfunction $\Omega_{(1,1)}$.

Then the variational characterization of the lowest eigenvalue shows that

$$\mu_1 = Q(\hat{\Omega}) = \min \{Q)\omega) | \omega = 0 \text{ on } \partial R\}.$$

Now it is to be observed that with every $\Omega_{\underline{k}}$, which is an exact, time-independent solution of the Euler equations, the functions

$$\Omega_{\underline{k}} \exp(-\nu\lambda_{\underline{k}} t)$$

are exact solutions of the Navier-Stokes equations, which will be called planar Taylor vortices (after Taylor [6]).

32.3 Viscous Evolution of the Functionals E and W

In the presence of dissipation due to viscosity, $\nu > 0$, the functionals E and W are no longer constant on a solution; their time derivatives are given by

$$\dot{E} = -\nu \int \omega^2 \text{ and } \dot{W} = -\nu \int (\nabla\omega)^2.$$

If we introduce the **dissipation rate quotient** as the functional

$$\Lambda = \dot{W}/\dot{E},$$

the time derivative of Q can be expressed like

$$\dot{Q} = -2\nu Q[\Lambda - Q].$$

Using the Cauchy-Schwartz inequality it is easily shown that $\Lambda(\omega) - Q(\omega)$ is sign definite (positive) and zero only if $\hat{\omega} = \hat{\Omega}_{\underline{k}}$ for some $\underline{k}$. Since $Q(\omega) \geq \mu_1 > 0$ for all ω, we obtain the following fundamental result:

Proposition 1: For any solution $\omega(t)$ it holds that $\dot{Q}(\omega(t)) \leq 0$, and $\dot{Q}(\omega(t)) = 0$ iff $\omega(t)$ is some planar Taylor vortex.

32.4 Self-organization

As a consequence of this result we find that for any initial value ω_o the value $Q(\omega(t))$ at the corresponding solution decreases monotonically to some definite value, to be denoted by $q(\omega_o)$, which must necessarily be μ_k for some $k \in \mathbf{N}$. These quantisized limit values determine manifolds $I_k := \{\omega_o | q(\omega_o) = \mu_k\}$ that are invariant for the flow. In particular, if $Q(\omega_o) < \mu_2$ then $\omega_o \in I_1$, and for $\omega_o \in I_1$ we have $Q(\omega(t)) \to \mu_1$, from which it follows that $\hat{\omega}(t) \to \hat{\Omega}$. Using a more refined analysis of the difference $\Lambda(\omega) - Q(\omega)$ near $\hat{\Omega}$, we can prove:

Proposition 2: For the solution $\omega(t)$ with initial data $\omega_o \in I_1$, the convergence of the spatial structure to $\hat{\Omega}$:

$$\hat{\omega}(t) = \frac{\omega(t)}{||\omega(t)||} \to \hat{\Omega} \text{ as } t \to \infty,$$

can be more precisely described as follows:

$$\omega(t) = \alpha(t)\,\hat{\Omega} + \xi(t),$$

where $\alpha(t)$ is a scalar function satisfying $c_1 \exp(-\nu\mu_1 t) \le |\alpha(t)| \le c_2 \exp(-\nu\mu_1 t)$ for some constants c_1 and c_2, and ξ is orthogonal to $\hat{\Omega}$ (in L_2-sense) and decreases exponentially faster than $\alpha(t)$:

$$||\xi(t)\}|| = |\alpha(t)|.O(e^{-\delta t}), \text{ with } \delta = \nu(1-\mu_1/\mu_2)(\mu_2-\mu_1).$$

For initial data $\omega_o \in I_k$, $k \neq 1$, the limiting behaviour can also be studied. However, these limiting structures will be unstable since, as a consequence of proposition 1, any perturbation ξ with $Q(\xi) < \mu_k$ at a time T sufficiently large will imply $Q(\omega(T) + \xi) < \mu_k$, and thus $q(\omega(T) + \xi) < \mu_k = q(\omega(T))$.

Hence, practically speaking, in the presence of random perturubations all solutions will eventually have the asymptotic behaviour as described in proposition 2. This result specifies and justifies the self-organization hypothesis mentioned before.

32.5 Remarks

(1) Any solution $\omega(t)$ defines a curve $t \to (E(\omega(t)), W(\omega(t))$ in a 2-dimensional E-W diagram. The functional Q determines the angle of a point with the E-axis, and Λ is the tangent to the curve. The self-organization phenomenon is reflected in the fact that this solution curve is tangent to the $Q = \mu_1$-line at the origin.

(2) In a spectral setting, Q and Λ turn out to be the mean squared wave number with weight function the spectral energy and enstrophy density respectively. The results described here show that both these means tend to the smallest wavenumber $\underline{k} = (1,1)$.

(3) The asymptotic results above, and in [5], do not explain the appearance of small scaled vortices as has been observed in recent computer simulations by McWilliams [7].

References

[1] E. van Groesen (1988). *Physica* **148A**, 312.

[2] A. Hasagawa (1985). *Adv. Physics* **34**, 1.

[3] R.H. Kraichnan and D. Montgomery (1980). *Rep. Prog. Phys.* **43**, 547.

[4] Z. Yoshida, T. Uchida and N. Inoue (1984). *Phys. Fluids* **227**, 1785.

[5] C. Foias and J.C. Saut (1984). *Ind. Univ. Math. J.* **33**, 459.

[6] G.T. Taylor (1923). *Philos. Mag.* **46**, 671.

[7] J.C. McWilliams (1984). *J.F.M.* **146**, 21.

[8] E. van Groesen (1988). A deterministic approach towards self-organisation in continuous media. *Cont. Math.*, to appear.