GENUS-TWO SOLUTIONS TO THE KADOMTSEV - PETVIASHVILI EQUATION

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Abstract. In this paper we consider dynamical aspects of multi-directional waves described by the Kadomtsev-Petviashvili (KP) equation. We investigate some analytically known solutions: the two-soliton interacting waves and their periodic equivalents. It is shown that the behaviour of the interaction of two-solitons can be classified by a parameter $A \geq 0$ (depending on the amplitudes of pure one-solitons and the angles of interactions). In the limiting case when $A = 0$, it is found that the two-soliton reduces to a three-branch soliton.

1. Introduction

The Kadomtsev-Petviashvili (KP) equation

$$(f_t + 6ff_x + f_{xxx})_x + 3f_{yy} = 0, \tag{1}$$

which was introduced in 1970 (see [1]), is of both physical and mathematical interest. It is the generalization of the following well known Korteweg-de Vries (KdV) equation

$$f_t + 6ff_x + f_{xxx} = 0. \tag{2}$$

The KP equation has been known as a model for dispersive, weakly nonlinear waves which are essentially uni-directional with weak transverse effects. In this paper we discuss exact solutions of KP equation. Analytic solutions of KP equation are available in the form of theta function (see for example in [4], [6]), but it is difficult to understand. Genus-two solutions are discussed briefly in section 2, mainly to show that in limiting cases some genus-two solutions can be seen as a superposition of two corresponding cnoidal waves. Some investigations on the effect of the interaction are also included. The effect of the interaction is discussed analytically in section 3 for the simpler case; i.e. the interaction of two solitons.

2. Genus-Two Solutions

KP-equation is a nonlinear equation that has very special properties: it is completely integrable. We will use only one facet of this integrability, namely the fact that special solutions can be written down explicitly. We will show these formulae below. Usually with nonlinear equations, it is not possible to write down solutions in an analytic way (not even a few, let alone whole classes of solutions as we will see below). On the other hand, the interpretation of these analytic expressions is not easy, and we will investigate some aspects below.

In the first subsection we consider periodic wave trains, in particular the interaction of two colliding trains. The explicit expression with Riemann theta function is shown in a limiting case to consist of a linear superposition of two wave trains. In the next subsection we focus on the interaction process that is in fact highly nonlinear. Then it is no restriction to consider the two-soliton waves that exist as an interaction of two solitons (instead of wave trains).

For the following (both the periodic and the soliton case), the basic observation is that the exact solutions are easiest obtained after the transformation:

$$f = 2\partial_x^2 \, ln \, \theta. \tag{3}$$

2.1. GENUS-TWO, LIMITING CONDITION AND SUPERPOSITION OF CNOIDALS

When looking for periodic waves, the simplest exact solution is the one parameter family of KdV cnoidal wave. For KP this amounts to the plane wave

$$f(x,y,t) = cn(\phi; m). \tag{4}$$

Here cn is the cnoidal function, with elliptic modulus m, and the phase variable is given by $\phi = \mu x + \nu y - \omega t + \delta$. The wave vector (μ, ν) determines the propagation direction (perpendicular to the plane wave). The propagation speed of the wave is a rather complicated expression depending on the wave vector and the modulus m (see [4]). An alternative formulation of the solution (4) in terms of a series, with the transformations (3), is given by the Riemann theta function of genus-one:

$$\theta = \sum_{m=-\infty}^{\infty} e^{\frac{b}{2}m^2 + im\phi}. \tag{5}$$

Here the parameter $b < 0$ determines the amplitude and the speed of the wave. In a good approximation, the amplitude a as a function of b is obtained from(5) when $\phi = 0$ as follows

$$\sum_{m=-\infty}^{\infty} e^{\frac{1}{2}bm^2}. \tag{6}$$

The larger $-b$, the lower the amplitude, and for $b \mapsto -\infty$ the solution f approaches the sinusoidal waves of infinitesimal amplitude (the solutions of the linearized equation).

More interesting solutions, the so called genus-2 solutions, are given as follows. For two phase factors ϕ_1, ϕ_2 that determine two directions of propagation :

$$\phi_j = \mu_j x + \nu_j y - \omega_j t + \delta_j, j = 1, 2 \tag{7}$$

the Riemann-theta function of genus two

$$\theta(\phi_1, \phi_2) = \Sigma_{m_1\ m_2}\ e^{\frac{1}{2}bm_1^2}\ e^{im_1\phi_1}\ e^{\frac{1}{2}(d+b\lambda^2)m_2^2}\ e^{im_2\phi_2}\ e^{b\lambda m_1 m_2} \tag{8}$$

defines the class of solutions

$$f(x,y,t) = 2\ \partial_x^2\ ln\ \theta(\phi_1, \phi_2). \tag{9}$$

The parameters are $b < 0, d < 0$ and λ, which should satisfy $0 < \lambda^2 \leq \frac{1}{2}$, and $d + b\lambda^2 \leq b$. The interpretation of the formula (8) becomes somewhat clearer by taking the limit $b\lambda \mapsto 0$. Then, formally, the double series (8) reduces to a product

$$\theta = \theta_1 \times \theta_2, \tag{10}$$

and the solution f becomes a superposition of two non interacting waves of genus-one, determined by the parameters b and d

$$f = f_1 + f_2, \tag{11}$$

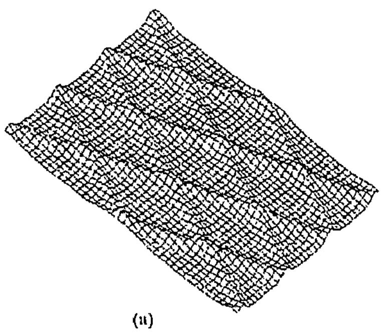
(a) (b)

Figure 1. The difference between a genus 2 solution where $\mu_1 = \mu_2 = 0.5$, $\nu_1 = -\nu_2 = 0.2$ $b = d = -2$ (+++), with the superposition of two corresponding cnoidal waves (—) (a) $\lambda = 0.7$, (b) $\lambda = 0.001$

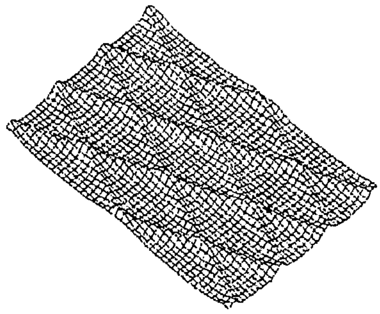
with $f_k = 2\partial_x^2 \, ln \, \theta_k$. For $b\lambda$ not too small there is a clear nonlinear interaction which is difficult to analyze from the formula. An investigation is presented in the next subsections.

2.2. AN INVESTIGATION ON THE EFFECT OF THE INTERACTION

Since $\lambda = 0$ is not allowed and $b = 0$ is impossible, we are interested in observing how the genus-2 solutions behave for small λ. For certain value of b with λ varies the result is that the effect of interaction is small (large) for λ small (large). At a fixed point y, this can be seen in fig. 1. On 2-D surface the difference between linear superposition of two wave trains and genus-2 solution for λ large is recognized easily in [5], but for small λ it is hardly distinguishable, see fig. 2.



(a)



(b)

Figure 2. (a) Genus 2 solution with $\mu_1 = \mu_2 = 0.5$, $\nu_1 = -\nu_2 = 0.2$, $b = d = -2$, $\lambda = 0.001$ (b) Superposition of two corresponding cnoidal waves at (a).

3. Interaction of two-solitons

In this section we will study the nonlinear interaction process in somewhat more detail. As is quite common in the literature (see [3] for obliquely interacting wave, and [2] for uni directional waves), we will simplify the analysis by replacing the periodic interaction by the interaction of two solitary waves. In the first subsection 3.1 we will present the solitary equivalent of the periodic genus-2 solution. Restricting to an investigation of the spatial pattern, we will analyze the pattern at a fixed time by treating the y-variable as a timelike quantity.

3.1. TWO-SOLITONS, SUPERPOSITION REPRESENTATION

A solitary wave in the direction of a wave vector $\mathbf{k} = (\mu, \nu)^T$ is a single hump wave form that decays rapidly to zero in the propagation direction. For KP equation, the governing formula of the solitary wave reads

$$f = 2\partial_x^2 \; ln \; \theta \tag{12}$$

with the phase function $\phi = \mathbf{k}.\mathbf{x} - ct$, where $k = \mid \mathbf{k} \mid$ and

$$\theta = 1 + e^{2\phi}. \tag{13}$$

A simple calculation produces the well known result

$$f = \mu\partial_x \; tanh \; \phi = \mu^2 \; sech^2\phi. \tag{14}$$

Multi-direction waves, known as genus-two solutions, can be written down in the non-periodic case in an easier way than in the periodic case. We look for solutions depending on two phases in the form

$$f = u(x + \rho_1 y - c_1 t, \quad x + \rho_2 y - c_2 t). \tag{15}$$

This solution is a steady state in a moving coordinate frame of reference:

$$x \longmapsto x - \nu_1 t, \quad y \longmapsto y - \nu_2 t$$

where $\nu_1 = \frac{\rho_1 c_2 - \rho_2 c_1}{\rho_1 - \rho_2}$, $\nu_2 = \frac{c_1 - c_2}{\rho_1 - \rho_2}$. Then we have $f = u(x + \rho_1 y,\ x + \rho_2 y)$. Furthermore, the directions with respect to the y-axis, determined by $\rho_{1,2}$ can be made symmetrical ($\rho_1 = -\rho_2 = r$) by another transformation

$$x \longmapsto x + py, \quad y \longmapsto y + qy$$

where

$$p = \frac{r(\rho_1 + \rho_2 + 2\rho_1\rho_2)}{\rho_2 - \rho_1}, \quad q = \frac{r(2 + \rho_1 + \rho_2)}{\rho_2 - \rho_1}. \tag{16}$$

From now on the variable t is dropped from the representation of genus-two solutions. Then a genus-2 solution (two-soliton solution) can be written down in two different equivalent ways.
A first way, based on the transformation (3), is written as (see [7] and [4])

$$\theta = 1 + e^{2\phi_1} + e^{2\phi_2} + Ae^{2\phi_1 + 2\phi_2}, \tag{17}$$

where A is given as

$$A = \frac{4(\mu_1 - \mu_2)^2 - (\rho_1 - \rho_2)^2}{4(\mu_1 + \mu_2)^2 - (\rho_1 - \rho_2)^2}. \tag{18}$$

In the absence of ρ, A reduces to the corresponding A for KdV. Note that, unlike in KdV where $0 < A < 1$, here A can take any value.
Another way, following the approach in [2] describing the two-soliton solutions for KdV, is to write the solution directly as a linear superposition:

$$f = u_1 + u_2 \tag{19}$$

where (for symmetrical interaction)

$$\begin{aligned}
u_j &= 2\mu_j \partial_x\, tanh\ g_j \\
g_1 &= \phi_1 + \tfrac{1}{2}\ ln\{\tfrac{1 + Ae^{2\phi_2}}{1 + e^{2\phi_2}}\}, \quad g_2 = \phi_2 + \tfrac{1}{2}\ ln\{\tfrac{1 + Ae^{2\phi_1}}{1 + e^{2\phi_1}}\} \\
\phi_1 &= \mu_1(x + ry) + \delta_1, \quad \phi_2 = \mu_2(x - ry) + \delta_2 \\
A &= \frac{(\mu_1 - \mu_2)^2 - r^2}{(\mu_1 + \mu_2)^2 - r^2}.
\end{aligned}$$

We assume from now on that $\mu_1 > \mu_2$, and consider y as time-like variable. As in KdV, conservation of mass for each individual soliton u_1 and u_2 is immediately shown as follows (assuming that $A > 0$, which implies that the range of g_j is $(-\infty, \infty)$)

$$\int_{-\infty}^{\infty} u_j dx = \int_{-\infty}^{\infty} 2\mu_j \; sech^2(v)d(v) = 4\mu_j, \quad j = 1, 2. \tag{20}$$

Define the center of masses $x_1(y)$ and $x_2(y)$ such that $\int_{-\infty}^{x_j(y)} u_j dx = \int_{x_j(y)}^{\infty} u_j dx = 2\mu_j, \; j = 1, 2$. Then we have

$$\int_{-\infty}^{g_j} sech^2(v)d(v) = \int_{g_j}^{-\infty} sech^2(v)d(v) = 1 \tag{21}$$

where g_j are evaluated at $x = x_j(y), \; j = 1, 2$ respectively. We have from (21) that the path of each soliton is given by

$$g_1 \equiv \phi_1 + \tfrac{1}{2} \, ln\{\tfrac{1+Ae^{2\phi_2}}{1+e^{2\phi_2}}\} = 0, \quad g_2 \equiv \phi_2 + \tfrac{1}{2} \, ln\{\tfrac{1+Ae^{2\phi_1}}{1+e^{2\phi_1}}\} = 0, \tag{22}$$

respectively. The phase shift of each soliton is $-\frac{1}{2} \, lnA$. This follows from (22) by fixing one of the phase variable and letting the second tend to $\pm\infty$. The intersection point which can be interpreted as the peak of the interaction, is obtained by intersecting (22)

$$\phi_1 + \tfrac{1}{2} \, ln\{\tfrac{1+Ae^{2\phi_2}}{1+e^{2\phi_2}}\} = 0 = \phi_2 + \tfrac{1}{2} \, ln\{\tfrac{1+Ae^{2\phi_1}}{1+e^{2\phi_1}}\}, \tag{23}$$

which gives

$$\phi_1 = \phi_2 = -\tfrac{1}{4} \, lnA. \tag{24}$$

When the constants δ_i that appear in the definitions of ϕ_i are assigned the values $\delta_i = -\frac{1}{4}\ln A, i = 1, 2$, the point of interaction of the paths as given by (24) becomes the origin $(0, 0)$ in the (x, y) plane. Interacting paths which differ from KdV are shown in figure 3.

3.2. LENGTH OF THE INTERACTION

As in [2], we define the beginning of the interaction by y_0 and the end of interaction by y_1. Let Δ be a small positive number which is used to indicate the portion of the overlapping solitons. Then y_0 and y_1, are defined implicitly as follows

$$\int_{A_0}^{x_0} u_1 dx = 2\mu_1(1 - \Delta), \quad \int_{x_0}^{B_0} u_2 dx = 2\mu_2(1 - \Delta) \tag{25}$$

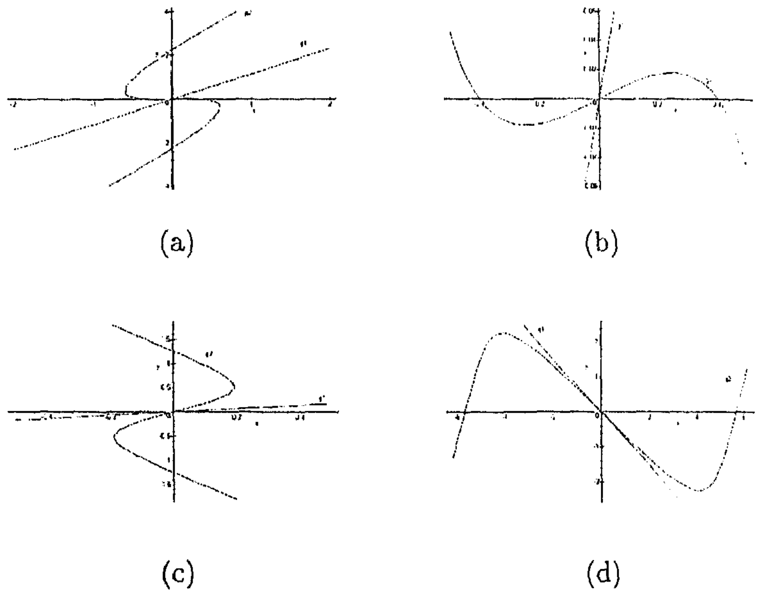


Figure 3. (a) The path of solitons for $\mu_1 = 0.9,\ \mu_2 = 0.35,\ \rho_1 = -0.81,\ \rho_2 = -0.5$, (b) blowup of (a), (c) The path of solitons for $\mu_1 = 0.9,\ \mu_2 = 0.1,\ \rho_1 = -3,\ \rho_2 = 0.36$, (d) blowup of (c)

$$\int_{B_1}^{x_1} u_2 dx = 2\mu_2(1-\Delta), \quad \int_{x_1}^{A_1} u_1 dx = 2\mu_1(1-\Delta). \tag{26}$$

Here A_0, A_1 and B_0, B_1 are the center of masses of u_1 and u_2 at the point y_0 and y_1 respectively and x_0, x_1 represent the points between the two solitons where the "Δ overlap" occurs corresponding to before and after interaction respectively. Since the path of each soliton is symmetrical with respect to $(0,0)$, we have $x_1 = -x_1$ and $y_1 = -y_0$ and the length of interaction $L = 2y_1$. Then direct calculation of integrals in (25) and (26) gives

$$L = \frac{\mu_1 + \mu_2}{-2\mu_1\mu_2 r}\, ln\{\frac{(1-\Delta)A^{\frac{1}{2}} + \sqrt{(1-\Delta)^2 A + (2-\Delta)\Delta}}{\Delta}\}. \tag{27}$$

We will now summarize in the following theorem.

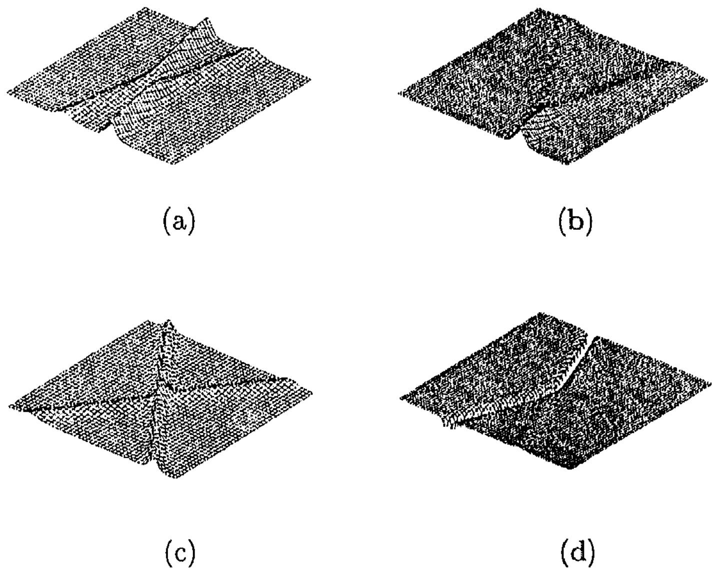


Figure 4. Interacting solitons (a) for $0 < A < 1$, (b) for $A = 0$, (c) for $A > 1$ and (d) for $A < 0$

Proposition 1

(i) *If $r < \mu_1 - \mu_2$, then $0 < A < 1$, and each soliton undergoes negative shift ($= \frac{1}{2}\, ln(A)$), mass conservation, exact interaction and splitting. The length of the interaction approaches ∞ as $r \longrightarrow 0$.*

(ii) *If $r > \mu_1 + \mu_2$, then $A > 1$, and each soliton undergoes positive shift ($= \frac{1}{2}\, ln(A)$), mass conservation, exact interaction and splitting. The length of interaction approaches 0 as $r \longrightarrow \infty$.*

(iii) *If $\mu_1 - \mu_2 < r \leq \mu_1 + \mu_2$, then singularity (at finite time) occurs in the form of unbounded solutions.*

(iv) *If $r = \mu_1 - \mu_2$, then the solution u forms a three-branch soliton along $\phi_1 =$ constant on the upperhalf plane, along $\phi_2 =$ constant on the lowerhalf plane, and along $\phi_1 - \phi_2 =$ constant on the righthalf plane.*

Proof:
The mass conservation in (20) is valid when g ranges from $-\infty$ to ∞, i.e. when $A > 0$. The phase shifts $(\frac{1}{2}ln(A))$ is positive when $A > 1$ and negative when $0 < A < 1$. The conclusion about the length of interaction follows immediately from (27). This completes the proof of parts (i) and (ii).
For the case (iii), we have $A < 0$ and therefore g_1, g_2 are no longer defined

in the whole R^2 and singularities occur when $1 + Ae^{2\phi_i} = 0$, $i = 1$ or 2. When $A = 0$, then using (12) and (17) u reduces to $u = 2\partial_x^2 ln(\theta)$, where $\theta = 1 + e^{2\phi_1} + e^{2\phi_2}$. The first and the second solitons reduce to
$u_1 = 2\mu_1 \frac{\mu_1 + (\mu_1 - \mu_2)e^{2\phi_2}}{1+e^{2\phi_2}}\ sech^2(\phi_1 - \frac{1}{2}\ ln(1 + e^{2\phi_2}))$, and
$u_2 = 2\mu_2 \frac{\mu_2 + (\mu_1 - \mu_2)e^{2\phi_1}}{1+e^{2\phi_1}}\ sech^2(\phi_1 - \frac{1}{2}\ ln(1 + e^{2\phi_1}))$ respectively.
We will show that there are exactly three nonvanishing phase directions $\phi_1 = constant, \phi_2 = constant$ and $\phi_1 - \phi_2 = constant$.
Along $\phi_1 = constant$, as $y \to +\infty$, we have $\phi_2 = \frac{\mu_2}{\mu_1}\phi_1 - 2\mu_2 ry \to -\infty$, and therefore $\theta \to 1 + e^{2\phi_1}$. Hence along $\phi_1 = constant$, as $y \to +\infty$, $u \to 2\partial_x^2 ln(1 + e^{2\phi_1})$, i.e. the pure first soliton.
Along $\phi_2 = constant$, as $y \to -\infty$, we have $\phi_1 = \frac{\mu_1}{\mu_2}\phi_2 + 2\mu_1 ry \to -\infty$, and therefore $\theta \to 1 + e^{2\phi_2}$. Hence along $\phi_2 = constant$, as $y \to -\infty$, $u \to 2\partial_x^2 ln(1 + e^{2\phi_2})$, i.e. the pure second soliton.
Along $\Phi = \phi_1 - \phi_2 = constant$, as $y \to -\infty$, we have
$\phi_2 = \frac{\mu_2}{\mu_1 - \mu_2}\Phi - \frac{2\mu_1\mu_2 r}{\mu_1\mu_2} y \to +\infty$. Rewrite θ as $\theta = e^{2\phi_2}[1 + e^{2(\phi_1 - \phi_2)} + e^{-2\phi_2}]$. The first exponential factor vanishes under the operator $2\partial_x^2 ln$ and the second factor $\to 1 + e^{2(\phi_1 - \phi_2)}$ as $y \to -\infty$. Hence, as $y \to -\infty$ we have $u \to 2\partial_x^2 ln(1 + e^{2(\phi_1 - \phi_2)})$, i.e. the pure one-soliton in the phase direction $\phi_1 - \phi_2$. Finally it will be shown that there is no other nonvanishing phase direction. Suppose there is a nonvanishing phase direction $\xi = \phi_1 + \alpha\phi_2 = constant$ on the upperhalf plane. If ξ for large y lies above $\phi = 0$ ($\alpha > 0$ or $\alpha < -\frac{\mu_1}{\mu_2}$), then along $\xi = constant$, $\phi_1 \to \infty$ as $y \to \infty$. By rewriting $\theta = e^{2\phi_2}[1 + e^{-2\phi_1} + e^{\frac{2}{\alpha}\xi - \frac{2(1+\alpha)}{\alpha}\phi_1}]$, then the second factor $\to 1$ as $y \to \infty$. If ξ for large y lies below $\phi_1 = 0$, then along $\xi = constant$, as $y \to \infty$ we have $\phi_1, \phi_2 \to -\infty$. This implies that $\theta \to 1$ as $y \to \infty$ (along $\xi = constant$). This contradicts the fact that ξ is a nonvanishing phase direction. Similar argument applies in the lower half plane.

4. CONCLUSION

A classification of genus-two solutions of KP equation has been obtained in which for limiting cases some genus-2 solutions can be seen as a superposition of two genus-1 solutions. In getting indepth view we considered the simpler case, i.e. interaction of two-solitons. During the interaction we recognize different types of behavior (parameterized by A): positive phase shift with the peak larger than their amplitudes, negative phase shift with the peak smaller than the largest amplitude, mass conservation, period of interaction and singularities. In the limiting case when $A = 0$, the two-soliton reduces to a three-branch soliton of amplitudes $2\mu_1^2, 2\mu_2^2$ and $2(\mu_1 - \mu_2)^2$. This can be viewed as an interaction of two solitons which results in one other soliton of different amplitude.

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