Voidage Waves in Bubbly Flows

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1. INTRODUCTION

Gas/liquid flows can support waves of various kinds. One type is formed by acoustic waves which display some striking properties, such as a sound speed ranging from 30 m/s, say, at low frequencies to amply above the sound speed in pure liquid near the natural frequency of the bubbles. Pressure waves of this kind will be discussed in other lectures during this Seminar. Whenever bubbles are so small that they move along with the liquid, due to strong viscous forces, these are the only waves which may occur. This ceases to be the case when relative motion between the two phases is possible. Pressure waves may alter their appearance because of this but also a new type of wave becomes possible, viz. concentration or voidage waves. Along these waves perturbations in the local void fraction or concentration propagate. Propagation speed and frequency of these waves are low with respect to those of pressure waves and density changes in both phases can therefore be neglected.

Voidage waves in bubbly liquids are an example of so-called kinematic waves. These waves are described by a continuity equation giving a relation between the time rate of change of the concentration of some quantity and the gradient of the flux of this quantity together with a functional relation between this flux and the concentration. As a specific example consider the settling of small particles under the action of gravitational forces. When the particle concentration is denoted by α and the local particle velocity by U_p (i.e. the particle velocity averaged over a region large with respect to the particle size but small with respect to macroscopic length scales, like a wavelength) the conservation of particles requires:

$$\frac{\partial \alpha}{\partial t} + \frac{\partial}{\partial x} \alpha U_p = 0 \ ,$$

which can also be written as the wave equation

$$\frac{\partial\alpha}{\partial t} + (U_p + \alpha \frac{dU_p}{d\alpha}) \frac{\partial\alpha}{\partial x} = 0 \ , \tag{1.1}$$

when a functional relation $U_p = U_p(\alpha)$ exists.
Relation (1.1) means that wavelets carrying perturbations in α travel along characteristics $x(t)$ given by

$$\frac{dx}{dt} = U_p + \alpha \frac{dU_p}{d\alpha} \ . \tag{1.2}$$

For dilute suspensions Batchelor (1972) has shown that

$$U_p = U_o \ (1-6.55\alpha + O(\alpha^2)) \ ,$$

U_o being the settling velocity of an isolated particle, which means, upon insertion in (1.2), that the speed of propagation of a concentration wave in this case is given by

$$\frac{dx}{dt} = U_o \ (1-13.1\alpha).$$

Bubbly flow is an example of a case in which the relationship between the flux and the concentration is not a functional one but follows from the pertinent equation expressing conservation of momentum. Then also dynamic waves may occur leading to interaction and competition between these waves and the kinematic waves.
Classic papers on the subject are by Kynch (1952) and by Lighthill & Whitham (1955 a,b). General discussions and additional references are given in Whitham (1974) and by Kluwick (1977).
Early applications of the theory to two-phase flows can be found in Zuber (1964) and Wallis (1969) and more recent and elaborate discussions are by Schneider (1982) and Kluwick (1983) on sedimentation waves and Liu (1982) on waves in fluidized beds.
Our interest in the subject stems from the well-known "problem of complex characteristics": the fact that many of the frequently applied sets of equations for two-phase gas-liquid flows are ill-posed and the finding that this ill-posedness is related tot the incorrect modeling of added mass and bubble-interaction effects in the momentum equation for the gas phase. These effects govern the propagation of voidage waves and indeed it can be shown that instabilities, related to the ill-posedness, are associated with voidage waves. As a reminder to the reader a brief account of the phenomenon of complex characteristics is given in section 2. This is followed in section 3 by a rather general description of some aspects of voidage wave propagation. In particular, a stability criterion is derived and the role of bubble interactions and added mass is shown to be significant.
In the remaining sections we address the problem of determining analytically (macroscopic) properties of voidage waves through an analysis,

on a microscopic level, of the motion of the bubbles and their hydrodynamic interactions. The method is explained in the context of voidage shock waves, and results are given for the velocity of rise of the bubbles and the shock thickness.

2. THE PHENOMENON OF COMPLEX CHARACTERISTICS

The simplest way to describe gas/liquid flow is to formulate conservation of mass and momentum separately for both phases, thereby assuming that interaction forces depend on concentration and velocities but not on derivatives of these. In one dimension x this gives, again for incompressible phases, the velocity of liquid being indicated with U_ℓ, that of gas with U_g,

$$\frac{\partial}{\partial t}(1-\alpha) + \frac{\partial}{\partial x}(1-\alpha)\,U_\ell = 0 \tag{2.1}$$

$$\frac{\partial \alpha}{\partial t} + \frac{\partial}{\partial x}\,\alpha\,U_g = 0 \tag{2.2}$$

$$\rho_\ell\left[\frac{\partial}{\partial t}\{(1-\alpha)\,U_\ell\} + \frac{\partial}{\partial x}\{(1-\alpha)\,U_\ell^2\}\right] + (1-\alpha)\frac{\partial p}{\partial x} = I. \tag{2.3}$$

$$\rho_g\left\{\frac{\partial}{\partial t}\alpha U_g + \frac{\partial}{\partial x}\alpha U_g^2\right\} + \alpha\frac{\partial p}{\partial x} = -I. \tag{2.4}$$

In these relations, density is denoted with ρ and pressure with p. With a view to the development of transients in bubbly flows, it is useful to inspect the characteristic directions $dx/dt = \lambda$, say, of these equations. These appear to be determined by the equation

$$\kappa\,(\lambda-U_g)^2 + (\lambda-U_\ell)^2 = 0 \tag{2.5}$$

with

$$\kappa = \frac{\rho_g(1-\alpha)}{\rho_\ell\alpha}.$$

Solving (2.5) for λ gives

$$\lambda = \pm\,(1+\kappa)^{-1}\,\{-\kappa\,(U_g - U_\ell)^2\}^{1/2}. \tag{2.6}$$

The characteristics are apparently imaginary, which means that the Cauchy initial value problem for the equations (2.1) - (2.4) is ill-posed. This has for good reasons worried scientists and engineers dealing with two-phase flow, ever since attention was drawn to it by Gidaspow (1974). Attempts to make equations of the type (2.3) and (2.4) more realistic by specifying I, thereby admitting terms with derivatives also, led to complex, instead of purely imaginary, characteristics. This leaves the danger of growing instabilities when codes based on (2.1) - (2.4) or similar equations are used to compute the evolution of transient phenomena. For some time it was not known with what kind of physical phenomenon these complex characteristics correspond. At the end of the 1970's experimental work on voidage waves was started in Grenoble by Bouré and his students

Mercadièr, Van Schaik, Micaelli (Bouré and Mercadier 1982), and at Caltech by Brennen and his student Bernier (Bernier 1981). It was realized that the mentioned characteristics are associated with voidage waves. According to the models as represented by equations (2.1) - (2.4) they cannot even exist. The experiments, however, carried out in Grenoble and Pasadena showed beyond doubt that stable concentration waves in bubbly flows do exist. An account of systematically obtained experimental results can be found in the thesis by Micaelli (1982).

Our own work on concentration waves was started because of this discrepancy between theory and experiment. The idea was that proper equations of motion must be obtained by taking interactions into account and by taking averages over the whole fluid instead of formulating conservation equations for the separate phases. Another good reason to study concentration waves is that they offer, as we will see in the following, an excellent opportunity for experimental verification of theoretically obtained results for interaction effects.

In our laboratory measurements are carried out basically of the same type as by the investigators mentioned above, however with a more controlled bubble size. This, because we want to test theoretical predictions and these are most easily done for uniform bubble size. Kapteyn will discuss the experiments during his contribution to this Seminar.

3. THE EFFECTS OF INTERACTIONS

Hydrodynamic equations for liquid/bubble mixtures were given by us (Biesheuvel & Van Wijngaarden 1984) in the dilute limit. Ensemble averages are taken or volume averages. In the latter the averaging is over a volume large with respect to bubble size but with linear dimensions small with respect to any macroscopic length scale. In such a volume the averaged quantities are supposed to be constant. From ensemble and/or volume averaging of results of equations at the particle level, the following momentum equations result for one dimensional flow and bubbles of constant volume $4/3\pi a^3$,

$$\rho_\ell(1-\alpha)\ \{\ \frac{\partial U_\ell}{\partial t} + U_\ell\ \frac{\partial U_\ell}{\partial x}\ \}\ -\frac{\partial p}{\partial x} - \rho_\ell\ (1-\alpha)g + \frac{1}{2}\ \rho_\ell\ \frac{\partial}{\partial x}\,\alpha\ (U_g-U_o)^2 = 0 \quad (3.1)$$

$$1/2\ \frac{D}{Dt}\ 4/3\pi\rho_\ell a^3(U_g-U_o) = \frac{DU_o}{Dt} - 12\pi\mu a(U_g-U_o) + 4/3\pi a^3\rho_\ell g\ . \quad (3.2)$$

The quantitiy U_o in (3.2) is the volume flow, the second term on the right-hand side is the frictional force at moderately high Reynolds numbers, whereas the material derivative D/Dt refers to the gas motion

$$D/Dt = \partial/\partial t + U_g\partial/\partial x.$$

In Biesheuvel & Van Wijngaarden (1984) it was found that, together with the conservation laws (2.1) and (2.2) the equations (3.1) and (3.2) have real characteristics viz. twice the directions

$$\frac{dx}{dt} = U_g\ .$$

Two things are of interest here. In the first place, the characteristics are real, in the second place we see that disturbances travel with the gas velocity irrespective of the amplitude. This is a consequence of the neglect of interactions.
When interactions are taken into account, the added mass which is $2/3\pi\rho_\ell a^3$ for an isolated sperical bubble, is a function of concentration $M(\alpha)$, say. Also the frictional force depends on concentration an becomes $f(\alpha)(U_g - U_o)$. For random structures and small α the first correction is a term linear in α, both in $M(\alpha)$ and $f(\alpha)$,

$$M(\alpha) = M(o) + \alpha M'(o) + O(\alpha^2) = \frac{2}{3}\pi\rho_\ell a^3 + \alpha M'(o) + O(\alpha^2)$$

$$f(\alpha) = f(o) + \alpha f'(\alpha) = 12\pi\mu a + \alpha f'(o) + O(\alpha^2) .$$

The coefficient $M'(o)$ has been calculated for random free arrays by Van Wijngaarden (1976) and by Biesheuvel (1984) for a random fixed array. In the former case $M'(o)$ is 2.78, in the latter case 3.37. The coefficient $f'(o)$ has not been calculated exactly. Ways to do this are discussed, among other items, during this Seminar by Kok. For random fixed arrays Biesheuvel (1984) finds the value 1.11. The ratio between $M(\alpha)$ and $f(\alpha)$ is a characteristic time. We denote this time with τ

$$\tau = M(\alpha)/f(\alpha).$$

In particular, for an isolated bubble we have

$$\bar{\tau} = \tau(o) = M(o)/f(o) = \frac{a^2}{18\nu} , \qquad (3.3)$$

where ν is the kinematic viscosity of the liquid.
When interactions are allowed for and the volume flow U_o is constant, a bubbly flow rising at concentration α under buoyancy in a liquid, is governed by, in stead of (3.2)

$$\frac{D}{Dt}\{M(\alpha)\ (U_g-U_o)\} = -f(\alpha)(U_g-U_o) + 4/3\pi\rho_\ell g a^3 . \qquad (3.4)$$

This equation and the continuity equation for the gas phase

$$\frac{\partial\alpha}{\partial t} + U_g\frac{\partial\alpha}{\partial x} + \alpha\frac{\partial U_g}{\partial x} = 0 \qquad (3.5)$$

contain all the necessary information for the study of voidage waves. Taking $U_o = 0$, for convenience, we find for the characteristic directions of (3.8) and (3.9)

$$dx/dt = C_+ = U_g \qquad (3.6)$$

$$dx/dt = C_- = U_g \{1 - \frac{\alpha M'(\alpha)}{M(\alpha)}\} . \tag{3.7}$$

Another important speed is the equilibrium speed, which occurs when everywhere in the α profile the frictional force equals buoyancy. Defining U_∞ by

$$U_\infty = \frac{4/3\pi\rho_\ell g a^3}{f(o)} = \frac{ga^2}{9\nu} = 2g\bar{\tau} \quad ,$$

we find that in equilibrium

$$U_g = \frac{f(o)U_\infty}{f(\alpha)} .$$

Inserting this into (3.9) gives a characteristic speed

$$C_0 = U_g[\ 1 - \frac{\alpha f'(\alpha)}{f(\alpha)}\ ] \tag{3.8}$$

The theory of hyperbolic equations of this type, see for a detailed account Whitham (1974), learns that for times $t \ll \tau$ waves are dominated by C_- and C_+ whereas for large times the asymptotic wave speed is C_0. Indeed, linearization of (3.4) and (3.5) around a given state gives with use of (3.6), (3.7) and (3.8)

$$\tau\ [\{\ \frac{\partial}{\partial t} + C_-\ \frac{\partial}{\partial x}\ \}\ \{\frac{\partial}{\partial t} + C_+\ \frac{\partial}{\partial x}\ \}\alpha]\ +\ [\ \frac{\partial \alpha}{\partial t} + C_0\ \frac{\partial \alpha}{\partial x}\ ]\ = 0, \tag{3.9}$$

which is the standard form used by e.g. Whitham (1974), Liu (1982) and Kluwick (1983) to discuss properties of solutions of this type of equations. Notably stability of these solutions is guaranteed as long as C_0 is in between C_+ and C_-. Using (3.6), (3.7) and (3.8) this is found to be equivalent with, for $M'(\alpha)$, $f'(\alpha) > 0$,

$$1 - \frac{\alpha M'(\alpha)}{M(\alpha)}\ <\ 1 - \frac{\alpha f'(\alpha)}{f(\alpha)}\ <\ 1\ , \tag{3.10}$$

which is equivalent to stating that for stability the value of the kinematic wave speed should lie in between those of the dynamic waves (Whitham 1974)

$$C_- < C_0 < C_+.$$

Relation (3.10) shows how important it is for the stability of voidage waves in what way friction and added mass depend on void fraction. It also shows a remarkable difference between voidage wave propagation in bubbly liquids and in gas-fluidized beds, the latter being described by an equation similar to (3.9), but with terms due to particle collisions, and leading to diffusion and dispersion, added to the right-hand side (Liu 1982). In gas-fluidized beds the particle mass (the counterpart of the added mass of the bubbles) does not depend on the particle concentration and (3.10) shows that without the diffusive and dispersive terms no stable gas-fluidized beds could exist. In bubbly liquids stability of voidage waves can arise solely due to the added mass of the bubbles, in particular its dependancy, through hydrodynamic interaction, upon the void fraction!

4. SHOCK WAVES

In the same way as waves are possible, one can generate smooth transitions form one constant voidage level to another (Nicklin 1962). Also here, there are plenty of opportunities to verify theoretical work on interaction effects. From (3.4) and (3.5) we find, putting $\partial/\partial t = - U_s\, \partial/\partial x$, and after some manipulation

$$\{(U_g - U_s)\}^2 \frac{d^2}{dx^2} \{M(\alpha)\, U_g\} + \{U_\infty f(o) - U_s f(\alpha)\} \frac{dU_g}{dx} = 0 \; . \qquad (4.1)$$

This shows again that the structure of voidage waves, in this case shock waves, is determined by $M(\alpha)$ and $f(\alpha)$. By measuring the voidage profile $\alpha(x)$ in experiments, important information on these quantities can be obtained. Kapteyn will discuss this further in his contribution to the Seminar.

A new and interesting effect arises when the lowest α level is zero. Since the quantity $\alpha(U_g - U_s)$ must be the same on both sides (cf 3.5 with $\partial/\partial t = - U_s\, \partial/\partial x$) we must have U_g = constant = U_s throughout the region where $\alpha \neq 0$. This is confirmed by (4.1) and it means that de D/Dt term in (3.4) is zero. This is, in fact, an often used method of determining the rise velocity of bubbles as a function of concentration (Nicklin 1962).

The description based on (3.4) and (3.5) does not allow a transition from level zero to a nonzero α level. Still, in the laboratory one can rather easily generate these type of shocks, just by cutting off the air supply! As Kapteyn will show, the thickness of these de-aeration shocks, as we will call them, is much smaller than the thickness of shocks connecting two levels of nonzero α. The conclusion must be that some diffusive mechanism arising from bubble interactions is acting to achieve a smooth transition. An early attempt to calculate a diffusive flux from binary interactions is made in Biesheuvel (1984). His idea was to calculate, following Batchelor (1972), first the force exerted on one bubble in a pair on the other and subsequently average this over all values which the distance between them can take. In a suspension of uniform number density the net force is zero because, when we consider a test bubble, there is for every position of a second bubble an exactly opposite one from which an opposite force is exerted. With a gradient in the number density, the possibility of a net force exists. Biesheuvel (1984) did this calculation for bubbles, moving all of them with the same speed and having a random spatial distribution. The conditional probability density, i.e. the probability of finding a bubble in $\underline{x} + \underline{r}$, given that there is one in $\underline{x}$ is given in his work by

$$P(\underline{x}+\underline{r}/\underline{x}) = n + (\nabla n)_{\underline{x}} \cdot \underline{r} \qquad (4.2)$$

for $r \geq 2a$, while P is zero for $r < 2a$.

The, disappointing, result is a diffusion up the gradient of α in stead of down. The reason for this must be that the probability density distribution

P cannot be chosen arbitrarily but must be determined from the dynamics of the interaction. Moreover, it is not sufficient to average over positions because in bubble pairs relative velocity does not depend uniquely on relative positions. This is an important difference with suspensions in which the flow field is given by Stokes's equations. The determination of the probability density is, like in other branches of suspension theory, a formidable problem even in the case of a macroscopically homogeneous suspension. In the following section an outline is given of an approximate calculation of the probability density distribution in the case of a de-aeration shock.

5. PROBABILITY DENSITY DISTRIBUTION AND DIFFUSION IN A DE-AERATION SHOCK

An outline is given here. A much more detailed account will be published elsewhere. With use of the potential for the flow around a bubble pair, as given e.g in Biesheuvel (1984), both the inertia and the frictional forces can be calculated. Biesheuvel and Van Wijngaarden (1982) calculated trajectories of bubble pairs under the neglect of frictional forces. Kok has extended these computations and has taken viscous and gravitational forces in account as well. He will present some of his result during his lecture at this Seminar. From his computations for bubble pairs released under buoyancy from arbitrary initial relative positions, it appears that they tend, from many of these positions, to a configuration in which the line connecting the bubble centres is at right angles with the gravity vector. In this configuration the bubbles either move away from each other or approach each other.
This result of computations with the full equations can be made plausible in an approximation in which only doublets are taken into account of the singularities by which the motion of the bubbles can be represented. Supplementing the pertinent equations, as given in Biesheuvel & Van Wijngaarden (1982) with frictional terms we have, 2R being the length of the vector connecting two bubbles in a pair and θ its inclination with respect to the vertical direction (See figure 1.)

$$\frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2 = \frac{9a^3U_\infty^{\ 2}}{32R^4}(3\cos^2\theta-1) - \tau^{-1}\frac{dR}{dt} \qquad (5.1)$$

$$R\frac{d^2\theta}{dt^2} + 2\frac{dR}{dt}\frac{d\theta}{dt} = \frac{9a^3U_\infty^2}{16R^4}\sin\theta\cos\theta - \tau^{-1}R\frac{d\theta}{dt} \quad . \qquad (5.2)$$

The first term on the right-hand side of both equations is the inertia force acting between two bubbles, while the second is the frictional force. (A bubble with velocity $\underline{v}_1$ experiences in a liquid with velocity $\underline{U}_0$ a frictional force proportional to $\underline{v}_1-\underline{U}_0$, likewise the frictional force on a bubble with velocity $\underline{v}_2$ is proportional to $\underline{v}_2 - \underline{U}_0$. The equation for the relative motion is obtained by subtracting the equations for the two bubbles in a pair which leaves a force proportional to $\underline{v}_2 - \underline{v}_1$. The components of this force are the terms τ^{-1} dR/dt and τ^{-1} dθ/dt in (5.1) and (5.2)). Equation (5.2), for the θ direction, has two equilibrium positions, $\theta=0$ and $\theta=\pi/2$. Analysis shows that the first is an unstabe equilibrium, while $\theta=\pi/2$ is a stable equilibrium. Computations with the approximate equations (5.1) and (5.2) show that pairs which start at arbitrary θ show a preference for $\theta=\pi/2$, thereby maintaining relative motion in R direction. It can also be shown analytically that this

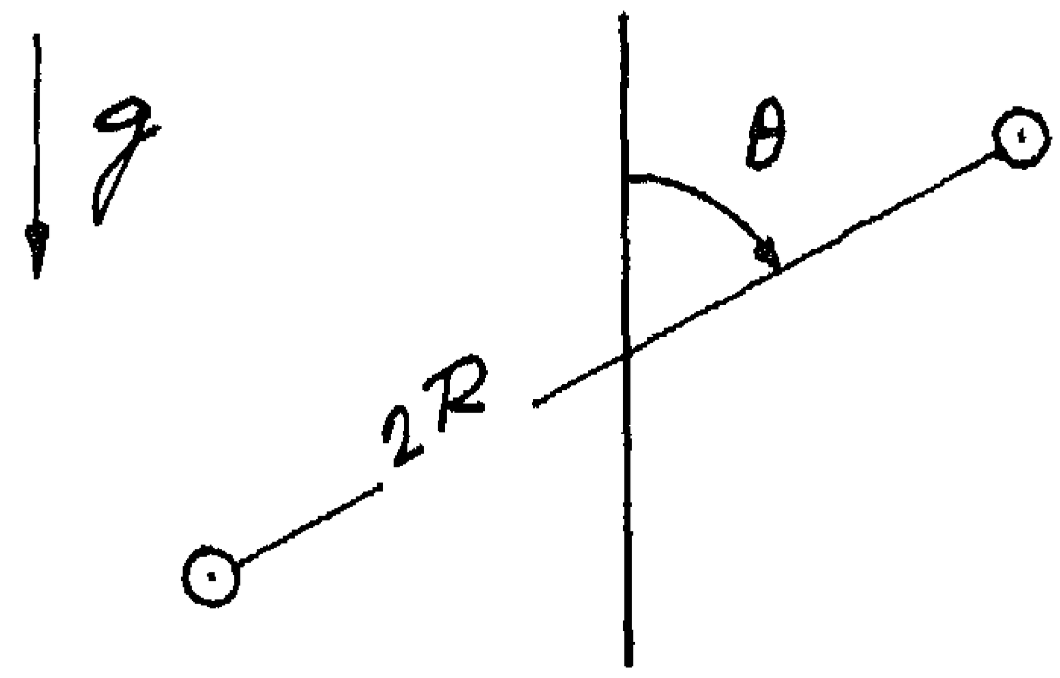

Figure 1 Coordinates used in (5.1) and (5.2) do describe relative motion.

relative motion is an oscillation with the particles slowing down until a maximum separation R_m is reached and next approaching each other again thereby regaining speed. It turns out that given some value of R_m there is a unique relation between the particle separation, the relative particle velocities and the vertical velocity of the particle pair. For simplicity, we assume in the following that in all pairs the angle θ equals $\pi/2$.
The mechanism which we envisage is the following. Bubbles move with respect to each other in the above described way. The point now is that both the virtual mass and the frictional force increase when the bubbles in a pair approach each other but decrease when they move away from each other. Next we consider the motion of the pair in vertical direction, distinct form the relative motion which takes place in horizontal planes. The way in which added mass and drag vary with distance means that bubbles, approaching each other slow down in vertical direction while a pair is speeding up when the constituent bubbles move away. In an homogeneous suspension, in which the number density is uniform, this has no macroscopic effect. However, when the number density has a gradient, the result is a net flow down the gradient because, at a given location, there are more pairs coming from above, while slowing down than pairs that go up while increasing their speed.
The programme to determine this diffusive flux quantitatively involves first the probability density distribution in an inhomogeneous suspension, and subsequently the associated volume flux.

6. THE PROBABILITY DENSITY DISTRIBUTION

We restrict ourselves to interactions between two bubbles and need to know the pair probability distribution $P(\underline{r}_1,\underline{r}_2|R_m)$ which is the probability of finding one bubble centre at $\underline{r}_1$ and another one at $\underline{r}_2$, given that the maximum value of possible separation of the bubble pair is R_m (wherever no confusion can arise we shall suppress the use of $|R_m$ in the following). This probability distribution is related to the conditional probability $P(\underline{r}_1/\underline{r}_2)$, that is the probalitiy of finding a bubble centre in $\underline{r}_1$ given that there is one in $\underline{r}_2$, by

$$P(\underline{r}_1,\underline{r}_2) = P(r_2)P(r_1/r_2) = N\ P(\underline{r}_1/\underline{r}_2) ,$$

when the configuration consists of N bubbles and the probability of finding a single bubble is the same everywhere.

When interactions higher than pair interactions are excluded $P(\underline{r}_1,\underline{r}_2)$ and $P(\underline{r}_1/\underline{r}_2)$, which we indicate in the following simply with P, for convenience, obey the Liouville equation

$$\frac{\partial P}{\partial t} + \nabla \cdot (P\dot{\underline{R}}) = 0 ,$$

where $\dot{\underline{R}}$ is the relative velocity in a pair. We now apply this to the steady de-aeration shock. When this is moving with speed U_s we have in a frame moving with this speed and recalling that we have relative motion in a horizontal plane

$$- U_s \frac{\partial P}{\partial z} + \frac{1}{R}\frac{\partial}{\partial R}(P\ R\ \dot{R}) = 0 . \qquad (6.1)$$

We introduce dimensionless coordinates with help of the bubble radius a and the (as yet unknown) shock thickness d,

$$R = a\eta;\ z = d\xi . \qquad (6.2)$$

From the experiments we know that the shock covers many bubbles, or

$$a/d = \varepsilon \ll 1 . \qquad (6.3)$$

Inserting (6.2) and (6.3) in (6.1) gives

$$\frac{1}{\eta}\frac{\partial}{\partial \eta}(\eta P\dot{R}) - \varepsilon U_s \frac{\partial P}{\partial \xi} = 0 . \qquad (6.4)$$

In analogy with the Chapman-Enskog theory in the kinetic theory of gases (Chapman & Cowling 1952) we expand P in a series in ascending powers of ε

$$P = P_o + \varepsilon P_1 + \varepsilon^2 P_2 + \ldots.. \qquad (6.5)$$

Inserting this into (6.4), gives for P_o

$$\frac{1}{\eta}\frac{\partial}{\partial \eta}(\eta P_o\dot{R}) = 0 , \qquad (6.6)$$

while P_1 obeys

$$\frac{1}{\eta}\frac{\partial}{\partial \eta}(\eta P_1\dot{R}) - U_s \frac{\partial P_o}{\partial \xi} = 0 . \qquad (6.7)$$

The radial relative velocity is found from the (approximate) solution of

(5.1) with $\theta=\pi/2$. The result is substituted in (6.6) and (6.7) whereupon these equations can be solved for P_0 and P_1. Once P_0 and P_1 are known, the vertical flow can be calculated.
During the calculation of P_0 and P_1 an important parameter arises, which we call σ, defined as

$$\sigma = \frac{U_\infty \bar{\tau}}{a} , \tag{6.8}$$

where τ is the time defined in (3.3). This can be interpreted as the time elapsing in an acceleration or deceleration before friction becomes important (cf 5.1). Accordingly σ measures how many times its own size a bubble travels in the time $\bar{\tau}$. In the circumstances of our experiments σ is of the order 10^2, whence it is justified to assume

$$\sigma >> 1 . \tag{6.9}$$

7. THE VERTICAL VELOCITY AND SHOCK THICKNESS

Let u be the vertical velocity of a pair of bubbles. The ensemble average over all configurations C_N, where C_N includes all possible relative positions and velocities, of N bubbles is

$$\langle u \rangle = \frac{1}{N!} \int P(C_N|\underline{r})\ u\ (\underline{r},\ C_N) dC_N . \tag{7.1}$$

We limit ourselves to configurations consisting of a test bubble and one other bubble. Accordingly u is the velocity of the test bubble in the presence of one other bubble, the integration is over all possible positions of the latter. The velocity of rise of a bubble pair is determined by an equation of the type (3.4). Whereas in that equation a bubble is considered amidst many others, to a concentration α, we need now an equation for a pair. This is

$$\frac{D}{Dt} \{M(R)u\} + f(R)u = \rho g \frac{4}{3}\pi a^3 , \tag{7.2}$$

where M(R) and f(R) determine the virtual mass and friction force, respectively, of a bubble in the presence of another one at distance 2R. Since in (5.1) and (5.2) only dipoles are taken into account, we need in $M(\eta)$ and $f(\eta)$ no greater accuracy than in terms of order η^{-3}. To this accuracy

$$M(\eta) = M(o)\ (1 + \frac{3}{16}\ \eta^{-3}) , \tag{7.3}$$

$$f(\eta) = f(o)\ (1 + \frac{1}{8}\ \eta^{-3}) . \tag{7.4}$$

When we restrict ourselves to binary interactions, the ensemble average (7.1) reduces to

$$\langle u\rangle = \int_a^{\bar{R}} 4\pi^2 R_m dR_m \int_a^{R_m} P_0(\eta, R_m)\eta u(\eta, R_m) d\eta \ , \qquad (7.5)$$

where $P(\eta, R_m)$ is the probability density for given R_m as described by (6.5). The range of integration of R_m is from the minimal possible value a to $\bar{R}$, where $\bar{R}$ is determined by the macroscopic boundaries of the flow and conveniently can be taken to be ∞. We write D/Dt in (7.2) as $a^{-1}d\dot{R}/d\eta$, use the solution for $\dot{R}$ of (5.1), insert this in (7.2) to obtain $u(\eta)$ with help of (7.3) and (7.4) for $M(\eta)$ and $f(\eta)$. The resulting expression for $u(\eta)$, together with that for P, when introduced in (7.5), finally give us $\langle u\rangle$. When in the series expansion (6.5) for P only the first term P_0 is used, we get $\langle u\rangle_0$, the uniform average rising velocity of N bubbles with radius a.

Of some interest here is the connection between N and the concentration α. Since the relative motion of the bubbles takes place in horizontal planes, the distribution is in two dimensions. Imagine a circular cylinder of large radius $\bar{R}$ in which the N bubbles find themselves (fig. 2). From a horizontal cross section $\pi\bar{R}^2$ of this cylinder a part $\alpha\pi\bar{R}^2$ is cut out by bubbles, when these have a volume concentration α. The number of the bubbles which are cut by the plane is, at number density n, $2a\pi\bar{R}^2 n$.
Hence

$$N = 2an\pi\bar{R}^2 \ .$$

And since

$$\alpha = \frac{4}{3}\pi a^3 n \ ,$$

we have

$$\alpha = \frac{2}{3}\frac{Na^2}{\bar{R}^2} \ .$$

The result for $\langle u\rangle_0$ then is,

$$\langle u\rangle_0 = U_\infty(1 - 2{,}28\alpha) + 1{,}92\alpha U_\infty \sigma^{-\frac{1}{5}} + 0(\sigma^{-\frac{2}{5}}) \qquad (7.6)$$

For sufficiently large values of σ, we have the interesting result that the average velocity with which bubbles rise is less than the rising velocity of an isolated bubble by an amount $2{,}28\alpha U_\infty$.
In the situation of the de-aeration shock, N and therefore α change slowly

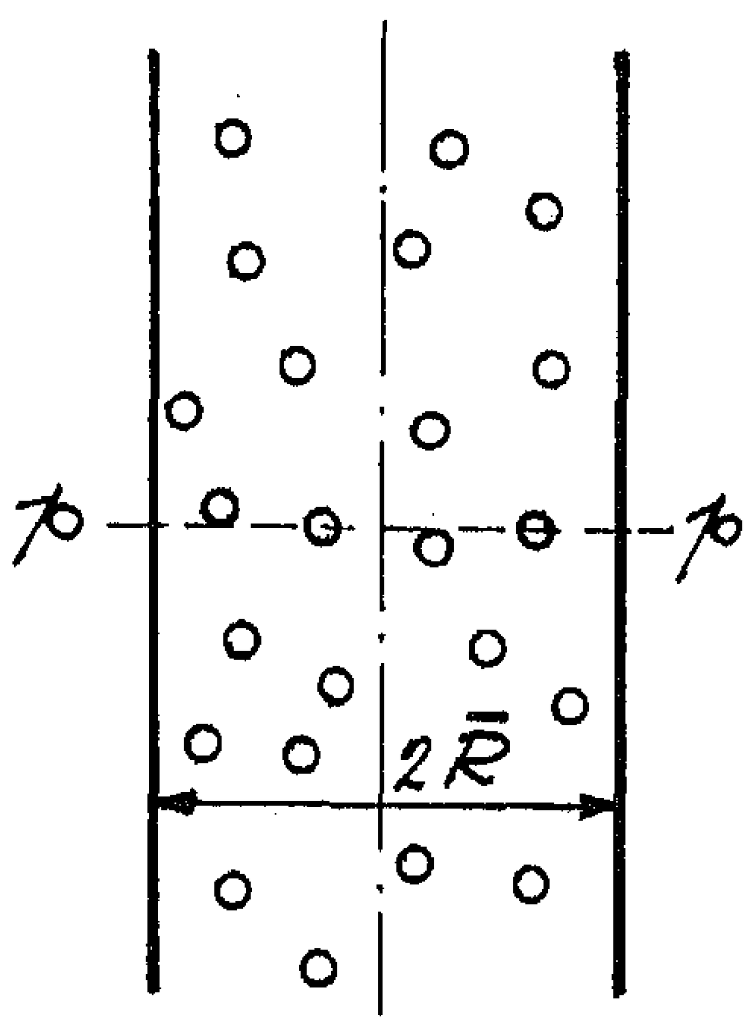

Figure 2 With number density n, and bubble radius a, the number of bubbles N cut by the horizontal plane p-..p through the cylinder of radius $\bar{R}$, is $2an\pi\bar{R}^2$.

with z. The additional flux caused by this is

$$\langle u\rangle_1 = 4\pi^2\varepsilon \int_a^{\bar{R}} R_m dR_m \int_a^{R_m} P_1(\eta, R_m)\eta u(\eta, R_m) d\eta \; . \tag{7.7}$$

The dependence of u on η and R_m is the same as with the uniform α distribution. Using the results for $P_1(\eta)$, as described in the foregoing section, we obtain

$$\langle u\rangle_1 = - \frac{3^{3/2}}{2} a\beta\sigma^{1/5} U_\infty \frac{d\alpha}{dz} , \tag{7.8}$$

where

$$\beta = 1 - \frac{8}{9}\sigma^{-\frac{1}{5}} - 0,23\,\sigma^{-\frac{4}{5}} .$$

Applying this again to the de-aeration shock we see that at the upstream end, where $\alpha = \alpha_\infty$ we have, according to (7.6) and for large enough σ, a flux $U_\infty(1 - 2,28\alpha_\infty)$. The total flux equals this expression along the shock because of volume conservation. It is made up by $\langle u\rangle_0$ and $\langle u\rangle_1$ as calculated in the preceding section, which gives, using (7.6) and (7.8)

$$2{,}28\ (\alpha_\infty - \alpha) + \frac{1}{2}\, 3^{1/2}\ a\ \sigma^{4/5} \frac{d}{dz} (\alpha_\infty - \alpha) = 0\ .$$

Upon integration we obtain

$$\alpha = \alpha_\infty \{1 - \exp - \frac{z}{\ell}\}\ ,$$

with

$$\ell = 0{,}38 a \sigma^{4/5}\ .$$

When we insert numbers we obtain for the shock thickness, 3ℓ, say values which are, compared to what we find in experiments, of the right order of magnitude. Of course, a more precise verification has to be made. This is currently done in our laboratory by Kapteyn.

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