CHAPTER 0

Matching in the method of controlled Lagrangians and IDA-passivity based control

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This paper reviews the method of controlled Lagrangians and the interconnection and damping assignment passivity based control (IDA-PBC) method. Both methods have been presented recently in the literature as means to stabilize a desired equilibrium point of an Euler-Lagrange system, respectively Hamiltonian system, by searching for a stabilizing structure preserving feedback law. The conditions under which two Euler-Lagrange or Hamiltonian systems are equivalent under feedback are called the matching conditions (consisting of a set of nonlinear PDEs).

Both methods are applied to the general class of underactuated mechanical systems and it is shown that the IDA-PBC method contains the controlled Lagrangians method as a special case by choosing an appropriate closed-loop interconnection structure. Moreover, explicit conditions are derived under which the closed-loop Hamiltonian system is integrable, leading to the introduction of gyroscopic terms.

The λ-method as introduced in recent papers for the controlled Lagrangians method transforms the matching conditions into a set of linear PDEs. In this paper the method is extended, transforming the matching conditions obtained in the IDA-PBC method into a set of quasi-linear and linear PDEs.

1. Introduction

Recently there has been a lot of interest in the stabilization of underactuated mechanical systems using methods that preserve the mathematical structure of the system. A mechanical system is called underactuated if the number of control inputs is stricly less that the number of degrees of freedom of the system. Such systems often occur e.g. in robotics, and are generally difficult to control. While fully actuated mechanical systems admit an arbitrary shaping of the potential energy by means of feedback, and therefore a stabilization to any desired equilibrium, such a strategy is in general not possible for underactuated systems. Indeed, underactuation puts a severe restriction on the possibilities to shape the potential energy. In certain cases this problem can be overcome by also modifying the kinetic energy of the system, thus leading to a new mechanical system with a *modified total energy*. A well known example is given by the inverted pendulum on a cart. This is an underactuated system since only the horizontal position of the cart can be controlled directly by a force in this direction, whereas by the absence of a torque the angle of the pendulum is uncontrolled. For this system it is not possible to stabilize the upright position of the pendulum by potential energy shaping only. However, allowing in addition the shaping of kinetic energy does stabilize the upright position of the pendulum, as well as the horizontal position of the cart. The closed-loop system is again described by a mechanical system, with a modified positive definite total energy function.

1.1. *Controlled Lagrangians*

The idea of kinetic energy shaping has led to a method for stabilizing underactuated mechanical systems, called the *method of controlled Lagrangians.* This method was introduced by [1,2,3] for the stabilization of relative equilibria of mechanical systems with symmetry. Starting point is an underactuated mechanical control system described by the forced Euler-Lagrange equations with a Lagrangian being the difference of the kinetic and potential energy of the system. The system is assumed to admit a symmetry, in fact, the Lagrangian is assumed to be invariant under the action of an Abelian Lie group (in the case of a cart and pendulum this means that the Lagrangian is independent of the horizontal position of the cart). The idea now is to stabilize a relative equilibrium of the system (i.e., the upright position of the pendulum, irrespective of the horizontal position of the cart) by searching for a suitable (stabilizing) closed-loop system which is again in Euler-Lagrange format *and* preserves the symmetry of the system. This is done by proposing a class of Lagrangians, called *controlled Lagrangians*, which preserve the symmetry of the system, and investigating which of these Lagrangians can possibly be obtained as a closed-loop Lagrangian by choosing a suitable feedback law for the original system. The conditions under which such a feedback law exists are called *matching conditions*, and in case these conditions are satisfied the original control system and the closed-loop Euler-Lagrange system are said to *match*. The feedback law can be calculated by using the symmetry properties of the system. The class of controlled Lagrangians proposed by Bloch et al. consists of Lagrangians being the difference of a *shaped kinetic energy* and the potential energy of the original system. That is, the kinetic energy is modified (in a certain restricted way), whereas the potential energy of the system remains unchanged. In general, the matching conditions for this class of controlled Lagrangians are described by a set of nonlinear partial differential equations to be solved for the closed-loop Lagrangian. In special cases, the so-called simplified matching assumptions [3], defining a restrictive but useful class of possible closed-loop controlled Lagrangians, these PDEs are automatically solved. The desired relative equilibrium is locally stabilized by finding a controlled Lagrangian, satisfying the matching assumptions, such that the total energy of the closed-loop system is (usually negative) definite around this equilibrium. This method has proved to work well for the examples of stabilization of an inverted pendulum on a cart or an inverted spherical pendulum and the stabilization of a satellite with an internal rotor, see [1,2,3] for more details.

The method of Bloch et al. concerning mechanical systems with symmetry, has been refined in the work of [4,5,6] to describe the stabilization of equilibria of general mechanical systems, see also the work of [7]. The idea is to stabilize a desired equilibrium by searching for a closed-loop Euler-Lagrange system with a *modified total energy*, i.e., in addition to the shaping of kinetic energy also the shaping of potential energy is allowed. Again, the matching conditions are described by a set of nonlinear PDEs.

In [4,8] the so-called λ-method is presented to convert these nonlinear PDEs into a set of *linear* PDEs. The method is designed for general mechanical systems and does not require any symmetry of the system. In fact, in general the symmetries present in the original system will be destroyed by the shaping of the potential energy in order to stabilize a desired equilibrium *point*. For the cart and pendulum this means that besides stabilizing the upright position of the pendulum, as in the method of Bloch et al., simultaneously the position of the cart is stabilized towards a desired horizontal position. We remark that the need for potential energy shaping to stabilize an equilibrium point has also been recognized in [9,10], where the term *symmetry-breaking potential* has been used.

The method of controlled Lagrangians has been extended in the work of [11] to describe the matching of general Euler-Lagrange systems. These systems are not restricted to be of a mechanical nature, that is, the Lagrangian is not necessarily given by the difference of a kinetic and a potential energy. Under a regularity assumption on the Lagrangian the matching conditions define a set of nonlinear PDEs, generalizing the PDEs described previously for mechanical systems.

Finally, we would like to remark that recently some results have been obtained in [12,13] extending the method of controlled Lagrangians to also include the matching and stabilization of Euler-Lagrange systems with (nonholonomic) constraints.

1.2. *Interconnection and damping assignment*

At the same time, on the Hamiltonian side a method has been developed to stabilize port-controlled Hamiltonian systems, [14,15]. Port-controlled Hamiltonian systems have shown to be instrumental in the network modeling of energy conserving physical systems. They strictly contain the class of Euler-Lagrange systems. See [16] and the references therein for more information on the development and the use of port-controlled Hamiltonian systems. Analogously to the method of controlled Lagrangians, the idea is to stabilize a desired equilibrium point of the system by searching for a suitable closed-loop system which is again in port-controlled Hamiltonian format. The closed-loop system is defined by changing the internal interconnection structure (i.e., the skew-symmetric structure matrix corresponding to the

Poisson bracket of the system) and the Hamiltonian (i.e., energy) function of the system. The conditions under which these changes lead to a system that can possibly be obtained as a closed-loop system of the original system, by choosing a suitable feedback law, constitute a new set of matching conditions. These are a set of nonlinear PDEs to be solved for the closed-loop Hamiltonian *and* the closed-loop interconnection structure. The principal (energy) concept used to stabilize the system is *passivity*, and since the closed-loop system is defined by shaping the internal interconnection structure of the system, the term *interconnection and damping assignment passivity based control (IDA-PBC)* has been coined to describe this method.[a] We refer to [14,15] for more details on the method and on the underlying passivity concept. It is important to notice that the possibility of also changing the interconnection structure, in addition to changing the Hamiltonian function, gives an extra degree of freedom to the IDA-PBC method with respect to the controlled Lagrangians method. Furthermore, since the class of port-controlled Hamiltonian systems strictly contains the class of forced Euler-Lagrange systems, the IDA-PBC method is more generally applicable than the controlled Lagrangians method. In [14,15] it has been shown that the method can be used to stabilize electrical systems such as power converters, electromechanical systems, e.g. synchronous motors, and mass-balance systems. The application of IDA-PBC to mechanical systems has been described in [15,17]. The method has been extended to systems with constraints in [18]. We refer to [19] for a recent survey on the IDA-PBC method.

1.3. *Contributions and outline of the paper*

In section 2 we discuss the matching of general Euler-Lagrange systems. Necessary and sufficient conditions are derived for two Euler-Lagrange systems to match, resulting in a set of nonlinear PDEs to be solved for the closed-loop Lagrangian. The method of [3] for mechanical systems with symmetry is reviewed, and the matching conditions obtained in that method are given an interpretation in terms of the matching of kinetic and poten-

[a]The method described in [14,15] allows additionally the shaping of the damping structure of the system. However, in this paper we will not consider this possibility, see the remarks afterwards.

tial energy. Section 3 recalls the matching of port-controlled Hamiltonian systems, as used in the IDA-PBC method. In section 4 both methods, applied to the class of mechanical systems, are compared. It is shown that the controlled Lagrangians method is strictly included in the IDA-PBC method (see however Remark 16 for a novel extension of the controlled Lagrangians method, yielding equivalence of both methods). Furthermore, the λ-method as described in [4] for the controlled Lagrangians method is extended to the IDA-PBC method. It is shown that the matching conditions, consisting of a set of nonlinear PDEs, can be transformed into an equivalent set of one quasi-linear and two linear PDEs, to be solved recursively. In section 5 the extra degree of freedom provided by the IDA-PBC method, i.e., the shaping of the internal interconnection structure, is used to discuss the integrability of the closed-loop Hamiltonian system. Necessary and sufficient conditions are given for the closed-loop system to be integrable, leading to the introduction of gyroscopic terms in the closed-loop system. Section 6 contains the conclusions.

Some further details and proofs can be found in the journal version of this paper [20]. A brief survey was presented in [21].

Important remarks: Before continuing with the technical part of the paper it is important to make the following two remarks. Firstly, notice that this paper is not concerned with the actual stabilization of equilibrium points of Euler-Lagrange or Hamiltonian systems. The (asymptotic) stabilization of equilibria is the aim of the papers [3,10,14,15,17] where the controlled Lagrangians method and the IDA-PBC method are introduced. In this paper we are merely interested in the matching of Euler-Lagrange, respectively Hamiltonian systems, which is the fundamental concept underlying both stabilization methods.

Secondly, for simplicity of exposition we do not consider any natural damping to be present in the control system, nor the introduction of energy dissipation by feedback in the closed-loop system. That is, we consider all systems to be energy conserving. The introduction of damping by feedback, called damping injection or damping assignment, is a very important issue in the methods as described in [3,10,14,15,17] to *asymptotically* stabilize an equilibrium which is made stable by shaping the Lagrangian, respectively the Hamiltonian and the internal interconnection structure, of the system. The inclusion of damping assignment in the results of this paper should be straightforward. Indeed, for mechanical systems with no natural damping feeding back the passive output results (under some detectability condition) in an asymptotically stable system. In this case the damping does not appear in the matching conditions, see [17].

Notation: Let $L(q, \dot{q})$ be a smooth function, then $\partial_q L$ denotes the partial derivative of L with respect to q and $\partial_{\dot{q}} L$ denotes the partial derivative of L with respect to $\dot{q}$ (these are $n \times 1$ matrices). The second order derivatives of L (which are $n \times n$ matrices) are denoted by $\partial_{qq} L, \partial_{q\dot{q}} L$ etc. Furthermore, if $\Theta(q, \dot{q}) \in \mathbb{R}^n$ is a smooth vector-valued function of $(q, \dot{q})$, then $\partial_q \Theta$ denotes the $n \times n$ matrix with (i, j)-th entry being $\partial_{q_j} \Theta_i(q, \dot{q})$.

2. Matching of Euler-Lagrange systems

In this section we describe the matching of Euler-Lagrange systems.

2.1. *General matching conditions*

Consider a forced Euler-Lagrange system with configuration space $\mathcal{Q}$, taken for simplicity to be equal to $\mathbb{R}^n$, and described by a Lagrangian $L : T\mathcal{Q} \to \mathbb{R}$,

$$\frac{d}{dt} \partial_{\dot{q}} L(q, \dot{q}) - \partial_q L(q, \dot{q}) = G(q) u. \tag{1}$$

The matrix $G(q) : \mathbb{R}^m \to T_q^* \mathcal{Q} \simeq \mathbb{R}^n$, with rank $G = m$, defines the force fields corresponding to the input $u \in \mathbb{R}^m$. Note that if $m = n$, then (1) describes a *fully actuated* Euler-Lagrange system, whereas the system is *underactuated* if (and only if) $m < n$. Consider a second, autonomous Euler-Lagrange system, defined by a Lagrangian $L_c : T\mathcal{Q} \to \mathbb{R}$ (the subscript c suggestively stands for closed-loop),

$$\frac{d}{dt} \partial_{\dot{q}} L_c(q, \dot{q}) - \partial_q L_c(q, \dot{q}) = 0. \tag{2}$$

The question we ask ourselves is whether the system (2) can be obtained as a possible closed-loop system corresponding to (1) by choosing a suitable control law u. If (2) is a possible closed-loop system of (1) then we say that the systems (1) and (2) *match*.

Now, consider the system (1), and let $G^{\perp}(q) : (\mathbb{R}^{n-m})^T \to (\mathbb{R}^n)^T$ denote a full rank left annihilator of $G(q)$, i.e., $G^{\perp}(q)G(q) = 0$, $\forall q \in \mathcal{Q}$. Note that from (1) it follows that

$$G^{\perp}(q)\left(\frac{d}{dt}\partial_{\dot{q}}L(q,\dot{q}) - \partial_q L(q,\dot{q})\right) = 0. \tag{3}$$

Consider the system (2). First notice that $\mathbb{R}^n = \mathrm{Im}\, G(q) \oplus \mathrm{Im}\, (G^{\perp})^T(q)$. This implies that (2) is equivalent to the following two equations:

$$G^T(q)\left(\frac{d}{dt}\partial_{\dot{q}}L_c(q,\dot{q}) - \partial_q L_c(q,\dot{q})\right) = 0, \tag{4}$$

$$G^{\perp}(q)\left(\frac{d}{dt}\partial_{\dot{q}}L_c(q,\dot{q}) - \partial_q L_c(q,\dot{q})\right) = 0. \tag{5}$$

The first of these two equations can always be obtained from (1) by choosing the control

$$u = (G^TG)^{-1}G^T\left[\left(\frac{d}{dt}\partial_{\dot{q}}L - \partial_q L\right) - \left(\frac{d}{dt}\partial_{\dot{q}}L_c - \partial_q L_c\right)\right] \in \mathbb{R}^m, \tag{6}$$

where we left out the arguments $(q,\dot{q})$ for clarity (notice that indeed G^TG is square and has full rank m). This leads to the following proposition.

Proposition 1: *The systems (1) and (2) match if and only if equation (5) holds along solutions of the system (1, 6) (equivalently (3, 4)).*

Remark 2: If rank $G = n$ then $G^{\perp} = 0$ and equation (5) is trivially satisfied, for any arbitrary closed-loop Lagrangian L_c. This corresponds to the well known fact that in case the system is fully actuated, its dynamics can be modified arbitrarily.

Equation (5) is refered to as the matching conditions. Following common terminology we call the closed-loop Lagrangian L_c the *controlled Lagrangian.*

Recall that the matching conditions (5) have to be satisfied *along solutions* of the system $(1,6)$, or equivalently $(3,4)$. Now take into account the regularity of the Lagrangians L and L_c, that is $\partial_{\dot{q}\dot{q}}L$ and $\partial_{\dot{q}\dot{q}}L_c$ are invertible. Then by eliminating the accelerations, the matching conditions (5) can be written as a set of nonlinear partial differential equations, to be satisfied for all $(q,\dot{q})$. Furthermore, the control law (6) is seen to be a state feedback control law. The construction is as follows:

Writing out the system (1) gives

$$(\partial_{\dot{q}\dot{q}}L)\ddot{q} + (\partial_{q\dot{q}}L)\dot{q} - \partial_q L = Gu. \tag{7}$$

Assuming that the Lagrangian is regular the system can be written as

$$\ddot{q} = -(\partial_{\dot{q}\dot{q}}L)^{-1}(\partial_{q\dot{q}}L)\dot{q} + (\partial_{\dot{q}\dot{q}}L)^{-1}\partial_q L + (\partial_{\dot{q}\dot{q}}L)^{-1}Gu. \tag{8}$$

Equivalently, the system (2) can be written as (assuming regularity)

$$\ddot{q} = -(\partial_{\dot{q}\dot{q}}L_c)^{-1}(\partial_{q\dot{q}}L_c)\dot{q} + (\partial_{\dot{q}\dot{q}}L_c)^{-1}\partial_q L_c. \tag{9}$$

The systems (1) and (2) match, for some suitably defined control law u, if the solutions of both systems are the same. That is, $(q(t), u(t)))$ is a solution of (1) if and only if $q(t)$ is a solution of (2), or equivalently, $(q(t), u(t)))$ satisfies (8) if and only if $q(t)$ satisfies (9). It follows that (1) and (2) match if and only if

$$\begin{aligned} &-(\partial_{\dot{q}\dot{q}}L)^{-1}(\partial_{q\dot{q}}L)\dot{q} + (\partial_{\dot{q}\dot{q}}L)^{-1}\partial_q L + (\partial_{\dot{q}\dot{q}}L)^{-1}Gu = \\ &-(\partial_{\dot{q}\dot{q}}L_c)^{-1}(\partial_{q\dot{q}}L_c)\dot{q} + (\partial_{\dot{q}\dot{q}}L_c)^{-1}\partial_q L_c, \end{aligned} \tag{10}$$

which can be written as

$$Gu = \{\partial_{q\dot{q}}L - (\partial_{\dot{q}\dot{q}}L)(\partial_{\dot{q}\dot{q}}L_c)^{-1}(\partial_{q\dot{q}}L_c)\}\dot{q} - \partial_q L + (\partial_{\dot{q}\dot{q}}L)(\partial_{\dot{q}\dot{q}}L_c)^{-1}\partial_q L_c. \tag{11}$$

Using the left annihilator $G^\perp$ of G, (11) can be equivalently written as

$$\begin{aligned} &G^\perp\Big(\{\partial_{q\dot{q}}L - (\partial_{\dot{q}\dot{q}}L)(\partial_{\dot{q}\dot{q}}L_c)^{-1}(\partial_{q\dot{q}}L_c)\}\dot{q} - \partial_q L + (\partial_{\dot{q}\dot{q}}L)(\partial_{\dot{q}\dot{q}}L_c)^{-1}\partial_q L_c\Big) \\ &= 0. \end{aligned} \tag{12}$$

Proposition 3: *The systems (1) and (2) match if and only if the matching conditions (12) hold. In that case, the state feedback control law is explicitly given by*

$$u = (G^T G)^{-1}G^T(\text{rhs of (11)}). \tag{13}$$

Remark 4: Writing out (6) and using (9) it is easy to show that the control laws defined in (6) and (13) are the same. Notice that the control law is a state feedback law, depending only on q and $\dot{q}$.

Equation (12) is equivalent to the matching conditions of [11], eq. (5). Furthermore, notice that (12) defines a set of nonlinear PDEs, where L is given and L_c acts as the unknown variable. The set of solutions L_c of (12) describes all the possible Euler-Lagrangian closed-loop systems (2) that can be obtained from (1) by a suitable choice (i.e., (13)) of the control law.

2.2. *Mechanical systems*

In case the Euler-Lagrange systems (1) and (2) both describe a mechanical system, then the matching conditions (12) can be split into two parts. The first part describes the shaping of kinetic energy, whereas the second part describes the shaping of potential energy.

Assume that (1) describes an (under)actuated mechanical system, that is, L is the difference of kinetic and potential energy

$$L(q,\dot q) = \frac{1}{2}\dot q^T M(q)\dot q - V(q), \tag{14}$$

where $M = M^T$ describes the generalized mass matrix of the system. We assume that M is invertible, which is equivalent to L being regular (the usual assumption is that M is positive definite.) We consider control laws which render the closed-loop system to be a mechanical system, that is, of the form (2) with controlled Lagrangian being of the form

$$L_c(q,\dot q) = \frac{1}{2}\dot q^T M_c(q)\dot q - V_c(q), \tag{15}$$

for some *shaped* generalized mass matrix $M_c = M_c^T$ (assumed to be invertible) and potential energy function V_c. In this case, the matching conditions (12) become

$$\begin{aligned} G^\perp(q)\Big[&\{\partial_q(M(q)\dot q) - M(q)M_c^{-1}(q)\partial_q(M_c(q)\dot q)\}\dot q - \partial_q(\frac{1}{2}\dot q^T M(q)\dot q) \\ &+ \partial_q V(q) + M(q)M_c^{-1}(q)[\partial_q(\frac{1}{2}\dot q^T M_c(q)\dot q) - \partial_q V_c(q)]\Big] = 0 \end{aligned} \tag{16}$$

Collecting the terms dependent, respectively independent, on $\dot q$ we see that (16) can be equivalently written as a set of two nonlinear PDEs in $M_c(q)$ and $V_c(q)$,

$$\begin{aligned} G^\perp(q)\Big[&\{\partial_q(M(q)\dot q) - M(q)M_c^{-1}(q)\partial_q(M_c(q)\dot q)\}\dot q \\ &- \partial_q(\frac{1}{2}\dot q^T M(q)\dot q) + M(q)M_c^{-1}(q)[\partial_q(\frac{1}{2}\dot q^T M_c(q)\dot q)]\Big] = 0 \end{aligned} \tag{17}$$

and

$$G^\perp(q)\big[\,\partial_q V(q) - M(q)M_c^{-1}(q)\partial_q V_c(q)\,\big] = 0. \tag{18}$$

Equation (17) matches the *kinetic* energy and is independent of the potential energy, whereas equation (18) matches the *potential* energy of the closed-loop system and depends on the shaped generalized mass matrix M_c. Notice that (17) defines a homogeneous polynomial in $\dot{q}$, whereas (18) is independent of $\dot{q}$.

The $\boldsymbol{\lambda}$-method of Auckly et al. The equations (17, 18) constitute a set of two *nonlinear* PDEs in M_c and V_c. In [4,5,6,8] a method has been presented to solve (17, 18) by recursively solving a set of three *linear* PDEs, thereby greatly reducing the complexity of finding solutions. Let us translate this method into our notation.

Consider equation (17) and notice that this equation has to hold for all points $(q, \dot{q}) \in T\mathcal{Q}$, whereby q and $\dot{q}$ should be seen as independent variables (i.e., the state of the system). This means that (17) can be equivalently written as (at a point $q_0 \in \mathcal{Q}$)

$$\begin{aligned} &G^{\perp}(q_0)M(q_0)\Big[M^{-1}(q_0)\partial_q(M(q)v)|_{q_0}v - M^{-1}(q_0)\partial_q(\frac{1}{2}v^T M(q)v)|_{q_0} \\ &+ (\partial_q X)|_{q_0}v - M_c^{-1}(q_0)\partial_q(M_c(q)v)|_{q_0}v + M_c^{-1}(q_0)\partial_q(\frac{1}{2}v^T M_c(q)v)|_{q_0} \\ &- (\partial_q X)|_{q_0}v\Big] = 0, \end{aligned} \tag{19}$$

for all vector fields $X \in T\mathcal{Q}$ with $X(q_0) = v \in T_{q_0}\mathcal{Q}$. In (19) we recognize the expression for the covariant derivative, see e.g. [22]. The covariant derivative, denoted by ∇, assigns to two vector fields $X, Y \in T\mathcal{Q}$ a third one denoted by $\nabla_X Y \in T\mathcal{Q}$, called the covariant derivative of Y with respect to X. It is uniquely defined by the kinetic energy metric $g(X,Y)(q) = X(q)^T M(q)Y(q),\ X, Y \in T\mathcal{Q}$.[b] (The symbol ∇ is also called the Levi-Civita connection corresponding to the metric g.) Let $\hat{\nabla}$ denote the covariant derivative corresponding to the metric defined by the matrix M_c. Then (19) can be written as (suppressing the argument q_0)

$$G^{\perp}M\left[\nabla_X X - \hat{\nabla}_X X\right] = 0, \quad \forall X \in T\mathcal{Q}. \tag{20}$$

This is exactly the matching condition as given in [4], eq. 1.4 (where $G^{\perp}M$ is denoted by P), see also [5,6]. Writing out the expression for the covariant

[b] In our notation: $2\nabla_X Y = M^{-1}\partial_q(MX)Y + M^{-1}\partial_q(MY)X - M^{-1}\partial_q(X^T MY) + [X, Y] + M^{-1}(\partial_q X)^T Y + M^{-1}(\partial_q Y)^T X$, with $X, Y \in T\mathcal{Q}$.

derivative in the coefficients of X using the Christoffel symbols results in the matching conditions as given in [7], Theorem 1. Furthermore, the control law given in [7], Theorem 1, equals the control law defined by (13).

We can polarize (20) to get

$$\begin{aligned} 0 &= \frac{1}{2} G^{\perp} M \Big[\nabla_{X+Y}(X+Y) - \hat{\nabla}_{X+Y}(X+Y) \\ &\qquad - (\nabla_X X - \hat{\nabla}_X X) - (\nabla_Y Y - \hat{\nabla}_Y Y) \Big] \\ &= \frac{1}{2} G^{\perp} M \left[\nabla_X Y + \nabla_Y X - \hat{\nabla}_X Y - \hat{\nabla}_Y X \right] \\ &= G^{\perp} M \left[\nabla_X Y - \hat{\nabla}_X Y \right], \quad \forall X, Y \in TQ, \end{aligned} \tag{21}$$

where we used that $\nabla_X Y - \nabla_Y X = [X, Y] = \hat{\nabla}_X Y - \hat{\nabla}_Y X$, which follows easily from the formula for the covariant derivative. Recall that $G^{\perp}$ denotes a full rank left annihilator of G (i.e., normalizing G to $[0 \quad I]^T$ this means that $G^{\perp} = [I \quad 0]$). Instead, let $\bar{G}^{\perp}$ denote an orthogonal projection matrix, i.e., $(\bar{G}^{\perp})^T = \bar{G}^{\perp}$ and $(\bar{G}^{\perp})^2 = \bar{G}^{\perp}$, such that $\bar{G}^{\perp} G = 0$. Normalizing G to $[0 \quad I]^T$ this means that

$$\bar{G}^{\perp} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \tag{22}$$

Then (21) still holds when one writes $\bar{G}^{\perp}$ instead of $G^{\perp}$. Now introduce a 'new' matrix variable by $\lambda = M_c^{-1} M$. Then a linear PDE in λ is obtained by taking $X = \lambda \bar{G}^{\perp} M X'$ and $Y = Y'$ and premultiplying (21) by $(X')^T M$. After some algebra, eliminating Y', this results in the following equation (suppressing the prime and writing X for X'):

$$\begin{aligned} 0 &= X^T M \bar{G}^{\perp} \lambda^T \Big\{ [\partial_q (M \bar{G}^{\perp} M X)]^T - [\partial_q (\bar{G}^{\perp} M X)]^T M - M \partial_q (\bar{G}^{\perp} M X) \Big\} \\ &\quad + X^T M \bar{G}^{\perp} \Big\{ [\partial_q (\lambda \bar{G}^{\perp} M X)]^T M + M \partial_q (\lambda \bar{G}^{\perp} M X) - [\partial_q (M \lambda \bar{G}^{\perp} M X)]^T \Big\}, \\ &\forall X \in TQ. \end{aligned} \tag{23}$$

Observe that (23) is a *linear* PDE in λ. However, notice that a solution is only defined with respect to the image of $\bar{G}^{\perp}$, i.e., a solution is only defined for $\lambda \bar{G}^{\perp} M$. Equation (23) is called the λ-*equation* and corresponds to equation 1.11 in [4].

The complete solution λ (or, equivalently, M_c) of the kinetic energy matching condition (17) can be found by solving another linear PDE. Indeed, premultiply (17) by M to get

$$0 = M\bar{G}^{\perp}\lambda^T\big\{\partial_q(\frac{1}{2}\dot{q}^T M_c\dot{q}) - \partial_q(M_c\dot{q})\dot{q}\big\} + M\bar{G}^{\perp}\big\{\partial_q(M\dot{q})\dot{q} - \partial_q(\frac{1}{2}\dot{q}^T M\dot{q})\big\}, \tag{24}$$

$\forall(q,\dot{q}) \in T\mathcal{Q}$. Given a solution $\lambda\bar{G}^{\perp}M$ of (23) this is a *linear* PDE in M_c. Equation (24) corresponds to equation 1.12 in [4] (with $Z = \dot{q}$ and eliminating X from 1.12).

Finally, given M_c, the potential energy matching condition (18) is a *linear* PDE in V_c. It can also be written in terms of a solution $\lambda\bar{G}^{\perp}M$ of (23) by premultiplying (18) by M to obtain:

$$0 = M\bar{G}^{\perp}\partial_q V - M\bar{G}^{\perp}\lambda^T\partial_q V_c. \tag{25}$$

This equation corresponds to equation 1.13 in [4].

In [4,5] it is shown that the matching conditions (17, 18) can be solved by solving the equivalent set of three linear PDEs (23, 24, 25). That is, first solving (23) for $\lambda\bar{G}^{\perp}M$, then (24) for M_c, and finally (25) for V_c.

2.3. *Mechanical systems with symmetry*

In this section we review the controlled Lagrangians method as introduced by [1,2,3] for mechanical systems with symmetry. In particular, we interpret the matching conditions obtained in those papers in terms of the matching of kinetic and potential energy as described by the PDEs (17, 18).

Consider a mechanical system with configuration space an n-dimensional manifold $\mathcal{Q} \simeq \mathbb{R}^n$. Let the configuration coordinates be denoted by $q = (x,\theta) \in \mathbb{R}^n$. Here $x \in \mathbb{R}^{n-m}$ are called the *shape variables* and $\theta \in \mathbb{R}^m$ are called the *group variables.* We assume that the group variables are fully actuated, whereas the shape variables are unactuated, this corresponds to $G = [0 \quad I_m]^T$. Furthermore, we assume that the Lagrangian of the system does not depend on the variables θ (we call θ *cyclic variables*).

Remark 5: The mathematical construction used in [3] is to consider a principal fiber bundle $\mathcal{Q} \to \mathcal{Q}/\mathcal{G}$ corresponding to the regular action of an Abelian (i.e., commutative) Lie group $\mathcal{G}$ on $\mathcal{Q}$. Then $x \in \mathcal{Q}/\mathcal{G}$ and $\theta \in \mathcal{G}$, and the Lagrangian L being cyclic in θ is equivalent to assuming that L is invariant under the action of the group G.

The forced Euler-Lagrange equations become

$$\frac{d}{dt}\partial_{\dot{x}}L - \partial_x L = 0, \tag{26}$$

$$\frac{d}{dt}\partial_{\dot{\theta}}L = u, \tag{27}$$

with

$$L(x, \dot{x}, \dot{\theta}) = \frac{1}{2}\dot{q}^T M(x)\dot{q} - V(x), \qquad \dot{q} = (\dot{x}, \dot{\theta}). \tag{28}$$

As explained in [3] quite a large class of mechanical systems fall within this description. The goal of the controlled Lagrangians method described in [3] is to stabilize a *relative equilibrium*[c] ($x = x_e, \dot{x} = 0, \dot{\theta} = 0$) of the system. This is done by searching for a stabilizing closed-loop Euler-Lagrangian system which preserves the symmetry of the system. In [3] a class of controlled Lagrangians is proposed which have the property that θ is a cyclic variable for L_c. This class can be described as follows: First, decompose the generalized mass matrix M as follows

$$M = \begin{bmatrix} M^{xx} & M^{x\theta} \\ M^{\theta x} & M^{\theta\theta} \end{bmatrix}, \tag{29}$$

according to the decomposition $q = (x, \theta)$. Define the shaped generalized mass matrix as follows

$$M_c = \begin{bmatrix} M^{xx} + M^{x\theta}\tau + \tau^T M^{\theta x} + \tau^T(M^{\theta\theta} + \sigma)\tau & M^{x\theta} + \tau^T M^{\theta\theta} \\ M^{\theta x} + M^{\theta\theta}\tau & M^{\theta\theta} \end{bmatrix}. \tag{30}$$

Here, $\tau(x) \in \mathbb{R}^{m\times n}$ and $\sigma(x) \in \mathbb{R}^{m\times m}$ are matrices only depending on the shape variables. In [3] τ is called a 'Lie algebra valued horizontal one-form', which means that it works only on vectors in the shape space $\mathbb{R}^{n-m}$ and takes values in $\mathbb{R}^m$. The matrix σ is called the 'changed metric acting on horizontal vectors', which means that it changes the mass matrix in the direction of the shape variables. The controlled Lagrangian is then defined by, corresponding to formula (2.11) in [3],

$$L_c(x, \dot{x}, \dot{\theta}) = \frac{1}{2}\dot{q}^T M_c(x)\dot{q} - V(x), \qquad \dot{q} = (\dot{x}, \dot{\theta}). \tag{31}$$

[c]The term relative equilibrium is used in reduction theory. It denotes an equilibrium in the shape variables, whereas motion with constant velocity (or better, momentum) in the group variables is allowed. In our case the relative equilibrium has velocity zero in the group variables. The configuration θ of the group variables however is unspecified.

It is important to notice that only the kinetic energy is changed whereas the potential energy of the system is left unchanged. Since the controlled Lagrangian preserves symmetry, i.e., L_c does not depend on θ, the corresponding Euler-Lagrange system looks like

$$\frac{d}{dt}\partial_{\dot{x}}L_c - \partial_x L_c = 0, \tag{32}$$

$$\frac{d}{dt}\partial_{\dot{\theta}}L_c = 0. \tag{33}$$

The idea of the method of [3] is to shape the kinetic energy, by choosing suitable matrices τ and σ, in order to obtain a closed-loop Euler-Lagrangian system $(31, 32, 33)$ for which the desired relative equilibrium is stable. The conditions under which L_c can be obtained as a possible closed-loop Lagrangian by choosing a suitable control law for the system $(26, 27, 28)$ are the matching conditions of [3]. In general, they consist of a set of nonlinear PDEs in the components of the matrices τ and σ. In the next paragraph the derivation of these matching conditions is described.

The matching conditions of Bloch et al. In [3] the result of proposition 1 is used to deduce conditions under which the systems $(26, 27, 28)$ and $(31, 32, 33)$ match. That is, they give conditions under which (32) holds along solutions of $(26, 33)$. Towards this objective denote the x-component of the Euler-Lagrange equations as:

$$\mathcal{E}_x(L_c) = G^{\perp}\left(\frac{d}{dt}\partial_{\dot{q}}L_c - \partial_q L_c\right) = \frac{d}{dt}\partial_{\dot{x}}L_c - \partial_x L_c. \tag{34}$$

Subtracting (3), equivalently (26), this becomes

$$\begin{aligned}\mathcal{E}_x(L_c) &= G^{\perp}\left(\frac{d}{dt}\partial_{\dot{q}}L_c - \partial_q L_c - \frac{d}{dt}\partial_{\dot{q}}L + \partial_q L\right)\\ &= G^{\perp}\Big((I - MM_c^{-1})M_c\ddot{q} + \partial_q(M_c\dot{q})\dot{q} - \partial_q(M\dot{q})\dot{q}\\ &\qquad - \partial_q(\frac{1}{2}\dot{q}^T M_c\dot{q}) + \partial_q(\frac{1}{2}\dot{q}^T M\dot{q})\Big),\end{aligned} \tag{35}$$

assuming M_c is invertible.

Now notice that (33) defines the first integral $\partial_{\dot{\theta}}L_c$ of the controlled Lagrangian system. Decompose M_c, defined in (30), according to the decomposition $q = (x, \theta)$ and write

$$M_c = \begin{bmatrix} M_c^{xx} & M_c^{x\theta} \\ M_c^{\theta x} & M_c^{\theta\theta} \end{bmatrix}, \tag{36}$$

Then

$$\partial_{\dot\theta} L_c = M_c^{\theta x}\dot x + M_c^{\theta\theta}\dot\theta, \tag{37}$$

which gives by (33), taking into account that θ is a cyclic variable,

$$M_c^{\theta x}\ddot x + M_c^{\theta\theta}\ddot\theta + \partial_x(M_c^{\theta x}\dot x)\dot x + \partial_x(M_c^{\theta\theta}\dot\theta)\dot x = 0. \tag{38}$$

Assuming that $M_c^{\theta\theta}$ is invertible (notice that a sufficient condition for $M_c^{\theta\theta}$ to be invertible is that M_c is definite) this results in

$$\ddot\theta = -(M_c^{\theta\theta})^{-1}M_c^{\theta x}\ddot x - (M_c^{\theta\theta})^{-1}\left(\partial_x(M_c^{\theta x}\dot x)\dot x + \partial_x(M_c^{\theta\theta}\dot\theta)\dot x\right). \tag{39}$$

Using (39) we can calculate

$$\begin{aligned} M_c\ddot q &= \begin{bmatrix} M_c^{xx}\ddot x + M_c^{x\theta}\ddot\theta \\ M_c^{\theta x}\ddot x + M_c^{\theta\theta}\ddot\theta \end{bmatrix} \\ &= \begin{bmatrix} S_c\ddot x - M_c^{x\theta}(M_c^{\theta\theta})^{-1}\left(\partial_x(M_c^{\theta x}\dot x)\dot x + \partial_x(M_c^{\theta\theta}\dot\theta)\dot x\right) \\ -\left(\partial_x(M_c^{\theta x}\dot x)\dot x + \partial_x(M_c^{\theta\theta}\dot\theta)\dot x\right) \end{bmatrix}, \end{aligned} \tag{40}$$

where $S_c := M_c^{xx} - M_c^{x\theta}(M_c^{\theta\theta})^{-1}M_c^{\theta x}$ is precisely the Schur-complement of the matrix M_c. Since we assume that M_c is invertible, it follows that S_c is invertible, see e.g. [23], p. 46.

Now substitute (40) into (35). The only terms of $\mathcal{E}_x(L_c)$ involving accelerations are given by

$$G^\perp(I - MM_c^{-1})\begin{bmatrix} I \\ 0 \end{bmatrix} S_c\ddot x \tag{41}$$

Bloch et al.[3] define their first matching condition, Assumption M-1, in such a way as to cancel all the terms in $\mathcal{E}_x(L_c)$ that involve the accelerations $\ddot x$. Since S_c is invertible, we have the following proposition, valid with respect to the class of controlled Lagrangians (30, 31) considered in [3]. (Recall that $G^\perp = [I_{n-m} \quad 0]$.)

Proposition 6: *The matching condition M-1 of* [3] *is equivalent to the condition*

$$[I_{n-m} \quad 0](I_n - MM_c^{-1}) \begin{bmatrix} I_{n-m} \\ 0 \end{bmatrix} = 0. \tag{42}$$

Condition (42) is an algebraic condition on the kinetic energy metric defined by M_c. Assuming (42) holds, let us calculate $\mathcal{E}_x(L_c)$. First calculate that

$$-\partial_q(M_c\dot{q})\dot{q} = \begin{bmatrix} * \\ -\Big(\partial_x(M_c^{\theta x}\dot{x})\dot{x} + \partial_x(M_c^{\theta\theta}\dot{\theta})\dot{x}\Big) \end{bmatrix}. \tag{43}$$

Then after substitution of (40) into (35) and using (42) and (43), equation (35) becomes

$$\begin{aligned}\mathcal{E}_x(L_c) &= G^{\perp}\big(-(I - MM_c^{-1})\partial_q(M_c\dot{q})\dot{q} + \partial_q(M_c\dot{q})\dot{q} - \partial_q(M\dot{q})\dot{q} \\ &\qquad - \partial_q(\frac{1}{2}\dot{q}^T M_c\dot{q}) + \partial_q(\frac{1}{2}\dot{q}^T M\dot{q})\big) \\ &= G^{\perp}\left(MM_c^{-1}\partial_q(M_c\dot{q})\dot{q} - \partial_q(M\dot{q})\dot{q} - \partial_q(\frac{1}{2}\dot{q}^T M_c\dot{q}) + \partial_q(\frac{1}{2}\dot{q}^T M\dot{q})\right)\end{aligned} \tag{44}$$

From the fact that θ is a cyclic variable for L_c it follows using (42) that

$$G^{\perp}\partial_q(\frac{1}{2}\dot{q}^T M_c\dot{q}) = G^{\perp}\begin{bmatrix} I \\ 0 \end{bmatrix}\partial_x(\frac{1}{2}\dot{q}^T M_c\dot{q}) = G^{\perp}MM_c^{-1}\partial_q(\frac{1}{2}\dot{q}^T M_c\dot{q}). \tag{45}$$

Finally, this results in the following equation for $\mathcal{E}_x(L_c)$:

$$\begin{aligned}\mathcal{E}_x(L_c) = G^{\perp}\Big(&\{MM_c^{-1}\partial_q(M_c\dot{q}) - \partial_q(M\dot{q})\}\dot{q} \\ &- MM_c^{-1}\partial_q(\frac{1}{2}\dot{q}^T M_c\dot{q}) + \partial_q(\frac{1}{2}\dot{q}^T M\dot{q})\Big).\end{aligned} \tag{46}$$

This corresponds to equation (2.25) in [3]. Bloch et al.[3] proceed by giving two conditions, i.e., Assumption M-2 and Assumption M-3, under which $\mathcal{E}_x(L_c)$ is identically zero, thereby accomplishing matching.

Interpretation of the matching conditions According to section 2.2 the systems (26, 27, 28) and (31, 32, 33) match if and only if the two PDEs (17, 18) hold. Notice that (18), describing the matching of the potential energy, in this case becomes the *algebraic equation*

$$G^{\perp}\left[\left(I - M(x)M_c^{-1}(x)\right)\partial_q V(x)\right] = 0. \tag{47}$$

In the sequel we will interpret the matching conditions obtained by [3] in terms of the conditions (17) and (47).

As described above the assumptions M-1, M-2 and M-3 accomplish matching for the class of controlled Lagrangians (30, 31) considered in [3]. According to proposition 6, condition M-1 is equivalent to (42). Now consider the matching condition (47) for the potential energy. Since θ is a cyclic variable for V, we have that

$$\partial_q V(x) = \begin{bmatrix} I \\ 0 \end{bmatrix} \partial_x V(x). \tag{48}$$

However, this means that (42) implies (47). Actually, this holds for any function V which is independent of the variables θ.

Proposition 7: *Assumption M-1 of* [3] *implies that the unchanged potential energy V matches.*

In other words, assumption M-1 takes care of the matching of potential energy. Notice that similarly to (47), assumption M-1 describes an *algebraic equation* on the kinetic energy matrix M_c.

Secondly, assuming that condition M-1 holds, we calculated $\mathcal{E}_x(L_c)$ to be as in (46). The condition that $\mathcal{E}_x(L_c)$ is equal to zero is precisely the matching condition (17) for the kinetic energy.

Proposition 8: *Assume that condition M-1 holds. Then assumptions M-2 and M-3 are equivalent to the matching condition (17) on the kinetic energy.*

In other words, assumptions M-2 and M-3 take care of the matching of kinetic energy. Notice that similar to (17), assumptions M-2 and M-3 define a set of nonlinear PDEs, to be solved for the kinetic energy matrix M_c (or its determining components τ and σ).

The above two propositions give an interpretation of the matching conditions as defined in [3] in terms of the matching of kinetic and potential energy.

Observe that to conclude if a certain controlled Lagrangian can be obtained as a closed-loop Lagrangian (i.e., matches) one needs to check the nonlinear PDEs (17, 18). In case one considers the class of systems and controlled Lagrangians as defined in [3] this comes down to checking the algebraic condition (42) and the nonlinear PDE (17) (or equivalently, checking assumptions M-1, M-2, M-3). In [3] a set of conditions, called the *simplified matching assumptions*, is given under which (42) and (17) automatically hold. Let us translate these conditions into the notation used in this paper.

Recall the decomposition of the matrix M as in (29) and denote $\Delta := M^{x\theta}(M^{\theta\theta})^{-1}M^{\theta x}$. The second and fourth of the simplified matching assumptions [3] can be translated as follows:

[SM-1] $M^{\theta\theta}(x) = M^{\theta\theta}$ is a constant (invertible) matrix,
[SM-2] $\partial_{x_j} M^{x_i\theta_k} = \partial_{x_i} M^{x_j\theta_k}, \quad i, j = 1, \ldots, n-m, \ k = 1, \ldots, m.$

As remarked in [3], these conditions imply that the mechanical connection corresponding to the system is flat, that is, the system lacks gyroscopic forces. The first and third of the simplified matching assumptions [3] can be translated into taking[d]

$$\tau = \kappa(M^{\theta\theta})^{-1}M^{\theta x}, \quad \sigma = -\frac{1}{\kappa}M^{\theta\theta}, \tag{49}$$

for some arbitrary nonzero constant $\kappa \in \mathbb{R}$, which can be seen as a design parameter. This results in the shaped kinetic energy matrix M_c

[SM-3] $$M_c = \begin{bmatrix} M^{xx} + \kappa(\kappa+1)\Delta & (\kappa+1)M^{x\theta} \\ (\kappa+1)M^{\theta x} & M^{\theta\theta} \end{bmatrix},$$

Now we can state the following proposition.[3]

Proposition 9: *Assume that the Lagrangian (28) satisfies assumptions SM-1 and SM-2. Take the controlled Lagrangian L_c to be of the form (31), with M_c as in SM-3 (for arbitrary κ). Then L_c is a matching Lagrangian, that is, the systems (26, 27, 28) and (31, 32, 33) match.*

Although the assumptions SM-1, SM-2 and SM-3 are quite restrictive[e], they seem to work well for the matching and stabilization of a number of interesting systems like the inverted pendulum on a cart and the spherical inverted pendulum. See [3] for worked examples.

[d]For $\kappa = 0$: take $\tau = 0$ and σ any matrix. Then $M_c = M$.
[e]However, in the case $n = 2,\ m = 1$ (e.g. inverted pendulum on a cart) assumptions M-1, M-2, M-3 and assumptions SM-1, SM-2, SM-3 are equivalent, as can easily be seen.

2.4. *The cart and pendulum*

In this section we want to make a few remarks on the matching methods we have described so far, taking as a guideline the example of an inverted pendulum on a cart. This system was first stabilized using the method of controlled Lagrangians by [1,3]. We described this method in the previous section. The method has two key features:

(I) The method stabilizes a *relative* equilibrium.

In the case of the cart and pendulum this means that the upright position of the pendulum is stabilized, irrespective of the horizontal position of the cart.

(II) The kinetic energy of the closed-loop system is *negative* definite.

This means that the closed-loop system simulates a mechanical system with *negative* masses and inertias, which is physically not very appealing.[f]

The first problem can easily be overcome by allowing also the shaping of potential energy (recall that in the method of [3] the potential energy was unchanged). This destroys the symmetry present in the system but in return stabilizes the group variables (i.e., the position of the cart) at a desired equilibrium point. Extending the above method by also including potential energy shaping was described in [9,10]. In those papers, the kinetic energy is still shaped according to assumptions SM-1, SM-2 and SM-3, and in addition the potential energy is also shaped (by introducing a new matching assumption). This solves the first problem, however, it cannot solve the second problem. In fact, for the cart and pendulum example, it can easily be checked that taking the shaped kinetic energy according to assumptions SM-1, SM-2 and SM-3, the potential energy can never be shaped in such a way that the stabilizing closed-loop kinetic energy is positive definite at the desired equilibrium (i.e., upright position of the pendulum, cart at a desired horizontal position). This seems to be a structural property of the method as described in [3,10].

[f] Besides, the problem of a negative definite kinetic energy becomes serious in the presence of physical damping. Indeed physical damping *dissipates* energy, pushing the state towards a *minimum* of the energy. This means that in order for the controlled Lagrangians method to work the (usually unknown) damping has to be compensated, see also [24,25].

On the other hand, if we consider the more general matching conditions as described in section 2.2, then problems (I) and (II) are absent. Indeed, as shown in [4,7], it is possible to stabilize the cart and pendulum system at the desired equilibrium point, such that the total energy of the closed-loop system is *positive* definite. This means that the closed-loop system corresponds to a physically existing mechanical system, with positive masses and inertias. Remark that indeed the corresponding shaped kinetic energy matrix does not have the form as in SM-3.

We conclude that although the controlled Lagrangians method, and the corresponding (simplified) matching assumptions, described in [3,10] and section 2.3, can be very helpful in solving the matching conditions and stabilizing a mechanical system, for a large class of examples it leads to closed-loop systems having a negative definite total energy, something which is physically not very appealing and can become problematic in the presence of damping. This problem does not occur when one shapes the energy according to the more general matching conditions described in section 2.2, see [4,7] for examples.

3. Matching of port-controlled Hamiltonian systems

In [14,15] a method has been developed to stabilize a desired equilibrium point of a port-controlled Hamiltonian system. The class of port-controlled Hamiltonian systems strictly contains the class of regular Euler-Lagrange systems. The method is called the *interconnection and damping assignment passivity based control (IDA-PBC)* method. Analogously to the method of controlled Lagrangians the basic idea is to search for a closed-loop system with stable desired equilibrium point which is again in port-controlled Hamiltonian format. As in the previously described method this leads to a set of matching conditions, described by a set of nonlinear PDEs. In this section we recall the method developed in [14,15] and its application to mechanical systems.

3.1. *General matching conditions*

Consider a port-controlled Hamiltonian system of the form

$$\dot{z} = J(z)\partial_z H(z) + g(z)u, \tag{50}$$

where $z \in \mathcal{M}$ (a manifold), $J(z) = -J^T(z) : T_z^*\mathcal{M} \to T_z\mathcal{M}$ is a skew-symmetric matrix (or better, vector bundle map) describing the internal interconnection structure of the system, $g(z) : \mathbb{R}^m \to T_z\mathcal{M}$ describes the input vector fields corresponding to the input $u \in \mathbb{R}^m$ and $H(z)$ is the Hamiltonian (or energy) function of the system. The objective of IDA-PBC is to stabilize a desired equilibrium point of the system. Analogously to the method of controlled Lagrangians this goal is pursued by considering static state feedback laws which render the closed-loop system in port-controlled Hamiltonian format. That is, the closed-loop system is described by the equations

$$\dot{z} = J_d(z)\partial_z H_d(z). \tag{51}$$

Here, $J_d(z) = -J_d^T(z)$ denotes the closed-loop interconnection matrix and $H_d(z)$ the closed-loop Hamiltonian function. The system (51) can be obtained from (50) by state feedback $u = u(z)$ if and only if

$$J_d(z)\partial_z H_d(z) = J(z)\partial_z H(z) + g(z)u(z). \tag{52}$$

Let $g^\perp(z)$ denote a full rank left annihilator of $g(z)$, then (52) can be equivalently written as

$$g^\perp(z)\big[J_d(z)\partial_z H_d(z) - J(z)\partial_z H(z)\big] = 0, \tag{53}$$

which are the matching conditions of the IDA-PBC method [14,15]. Notice that the matching conditions (53) define a set of nonlinear PDEs, to be solved for the shaped Hamiltonian H_d *and* the shaped interconnection matrix J_d. If the matching conditions are satisfied, i.e., the systems (50) and (51) match, then the corresponding state feedback law is explicitly given by

$$u(z) = (g^T(z)g(z))^{-1}g^T(z)\big\{J_d(z)\partial_z H_d(z) - J(z)\partial_z H(z)\big\}. \tag{54}$$

Remark 10: In [14,15] the following equivalent form of the matching conditions can be found: Write $J_a = J_d - J$ and $H_a = H_d - H$, then equation (52) becomes

$$(J(z) + J_a(z))\partial_z H_a(z) = -J_a(z)\partial_z H(z) + g(z)u(z), \tag{55}$$

and the matching conditions (53) get the form

$$g^\perp(z)\big[(J(z) + J_a(z))\partial_z H_a(z) + J_a(z)\partial_z H(z)\big] = 0, \tag{56}$$

which is a set of nonlinear PDEs to be solved for H_a and J_a.

Remark 11: Suppose (50) represents a *linear* port-controlled Hamiltonian system, i.e., $\dot{z} = JQz + gu$ for constant matrices $J = -J^T$, g, and Hamiltonian function $H(z) = \frac{1}{2}z^TQz$, $Q = Q^T$, and suppose that also the closed-loop system (51) is a linear system. It has been shown in [26] that in this case the matching conditions (53), as well as the conditions for stability of the closed-loop system, can be transformed into a set of linear matrix inequalities (LMIs). Powerful algorithms for solving these LMIs are available in several software packages.

Remark 12: *Equivalence under state feedback.* The closed-loop system (51) does not include the description of external inputs. This stems from the fact that the IDA-PBC method is designed to construct feedback controllers $u = u(z)$ which stabilize an assigned equilibrium point z^*, that is, the closed-loop system (51) has a stable equilibrium point at z^*. The addition of external inputs to the closed-loop system, yielding

$$\dot{z} = J_d(z)\partial_z H_d(z) + g(z)v, \quad v \in \mathbb{R}^m, \tag{57}$$

can be of importance in reaching additional control objectives. For instance, feeding back the passive output $y = g^T\partial_z H_d$ by $v = -Ky,\ K > 0$, yields under suitable assumptions *asymptotic* stability, see e.g. [17]. However, the addition of external inputs to the closed-loop system does not change the matching conditions (53). The systems (50) and (51) are *equivalent* under state feedback $u(z, v) = \alpha(z)+v$ if and only if (53) holds. The corresponding control law $\alpha(z)$ is defined by (54). Of course, an analogous remark can be made for the controlled Lagrangians method.

3.2. *Mechanical systems*

In this section we apply the method described above to mechanical systems, see [17]. A mechanical system can be described by a port-controlled Hamiltonian system of the form (50),

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \partial_q H \\ \partial_p H \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u, \tag{58}$$

where (q, p) (consisting of configuration coordinates q and momenta p) denote coordinates for the state space $\mathcal{M} = T^*\mathcal{Q} \simeq \mathbb{R}^{2n}$, with $\mathcal{Q} \simeq \mathbb{R}^n$

denoting the configuration space of the mechanical system. The matrix $G(q) : \mathbb{R}^m \to T_q^*\mathcal{Q} \simeq \mathbb{R}^n$ defines the force fields corresponding to the input $u \in \mathbb{R}^m$. The Hamiltonian function $H(q,p)$ is given by the total, i.e., kinetic plus potential, energy in the system

$$H(q,p) = \frac{1}{2}p^T M^{-1}(q)p + V(q), \tag{59}$$

where $M = M^T$ describes the generalized mass matrix of the system, and is assumed to be invertible (for most physical systems M will be positive definite). Note that from (58) and (59) it follows that the momenta are defined as usual by $p = M(q)\dot{q}$. Following [17] we propose the shaped Hamiltonian function $H_d(q,p)$ to be again of the form (59),

$$H_d(q,p) = \frac{1}{2}p^T M_d^{-1}(q)p + V_d(q), \tag{60}$$

for some shaped generalized mass matrix $M_d = M_d^T$ (assumed to be invertible) and potential energy function $V_d(q)$. The shaped interconnection matrix is taken to be in the most general form

$$J_d(q,p) = \begin{bmatrix} 0 & M^{-1}(q)M_d(q) \\ -M_d(q)M^{-1}(q) & J_2(q,p) \end{bmatrix}, \tag{61}$$

for some skew-symmetric matrix $J_2(q,p)$. Then, system (51) becomes

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & M^{-1}M_d \\ -M_d M^{-1} & J_2 \end{bmatrix} \begin{bmatrix} \partial_q H_d \\ \partial_p H_d \end{bmatrix}. \tag{62}$$

Remark 13: Since $\dot{q}$ is a nonactuated coordinate, it follows that the relationship $\dot{q} = M^{-1}(q)p$ should also hold in closed-loop. Fixing (51) and (60) this explains the first row of the matrix J_d.

In this case the matching conditions (53) become

$$G^{\perp}\left[\partial_q H - M_d M^{-1}\partial_q H_d + J_2 M_d^{-1} p\right] = 0. \tag{63}$$

Using (59) and (60) and collecting terms dependent, respectively independent, of p we see that (63) can be equivalently written as a set of two nonlinear PDEs

$$\begin{aligned} G^{\perp}(q)\Big[\partial_q(\frac{1}{2}p^T M^{-1}(q)p) - M_d(q)M^{-1}(q)\partial_q(\frac{1}{2}p^T M_d^{-1}(q)p) \\ + J_2(q,p)M_d^{-1}(q)p\Big] = 0, \end{aligned} \tag{64}$$

and

$$G^{\perp}(q)\big[\partial_q V(q) - M_d(q)M^{-1}(q)\partial_q V_d(q)\big] = 0. \tag{65}$$

Like in the Lagrangian case, equation (64) matches the kinetic energy and is independent of the potential energy, whereas equation (65) matches the potential energy of the closed-loop system (and depends on M_d). The PDEs contain the unknown variables M_d and V_d, whereas the matrix J_2 acts as a free parameter which can be suitably chosen to allow the PDEs to be solvable for specific choices of M_d and V_d (directed by the stabilization objective). In case of matching the corresponding feedback law is given by (54)

$$u = (G^T G)^{-1} G^T \big\{\partial_q H - M_d M^{-1} \partial_q H_d + J_2 M_d^{-1} p\big\}. \tag{66}$$

Again remark that (64) and (65) define a set of nonlinear PDEs, which are in general not easy to solve. However, for a special class of systems these PDEs can be transformed into a set of nonlinear ODEs which are much easier to solve. This is described in [27]. The class of systems for which this transformation is possible is defined by the following assumptions: i) the system is assumed to have n degrees of freedom and $n-1$ actuators (i.e., there is only one unactuated coordinate), and ii) the kinetic energy matrix M is assumed only to depend on the unactuated coordinate. This class of systems is quite common in underactuated mechanical systems and includes for instance the cart and pendulum example. By choosing the shaped kinetic energy matrix M_c to only depend on the unactuated coordinate, it can be shown that the set of PDEs (64, 65) can be transformed into an equivalent set of ODEs. In [27] the method is applied to the examples of a cart and pendulum system and a ball and beam system. For general systems we will show in section 4.2 that the λ-method as described in section 2.2 can also be used to simplify the process of solving the matching conditions (64) and (65), by transforming them into a set of quasi-linear and linear PDEs.

4. Comparison between the two methods

In sections 2 and 3 we described the matching of Euler-Lagrange systems, respectively of port-controlled Hamiltonian systems. Since the class of regular Euler-Lagrange systems is strictly contained in the class of port-controlled Hamiltonian systems, the method of section 2 should be a special case of the more general method described in section 3. In this section we consider both methods as applied to mechanical systems, see sections 2.2 and 3.2, and show that Euler-Lagrange matching is a special case of port-controlled Hamiltonian matching. Notice that the IDA-PBC method has an extra degree of freedom with respect to the controlled Lagrangians method, in the sense that, in addition to shaping the total energy of the system, it is also possible to shape the internal interconnection structure of the system. This extra freedom means that the IDA-PBC method results in a larger class of matching closed-loop systems than the controlled Lagrangians method described in section 2.2. This can be an important point in finding suitable stabilizing feedback controllers. Furthermore, the λ-method described in section 2.2 is shown to be useful in solving the matching conditions obtained in the IDA-PBC method.

4.1. *The controlled Lagrangians case of IDA-PBC*

Consider a mechanical system described by the Euler-Lagrange system $(1, 14)$. This system is equivalent via the Legendre transformation to the Hamiltonian system $(58, 59)$. In section 2.2 we gave conditions under which the autonomous Euler-Lagrange system $(2, 15)$ matches with the system $(1, 14)$. The system $(2, 15)$ is equivalent to a canonical Hamiltonian system in the following way: Define the momenta to be

$$p_c = \partial_{\dot{q}} L_c = M_c(q)\dot{q}, \tag{67}$$

and the Hamiltonian by the Legendre transformation,

$$H_c(q, p_c) = \frac{1}{2} p_c^T M_c^{-1}(q) p_c + V_c(q). \tag{68}$$

Then the Euler-Lagrange system $(2, 15)$ can be equivalently written as the Hamiltonian system

$$\begin{bmatrix} \dot{q} \\ \dot{p}_c \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}}_{J_c} \begin{bmatrix} \partial_q H_c \\ \partial_{p_c} H_c \end{bmatrix}. \tag{69}$$

It follows that in the particular case that we choose M_d and J_d such that the closed-loop Hamiltonian system $(60, 62)$ is equivalent (by a coordinate transformation) to the Hamiltonian system (68, 69), then the IDA-PBC method effectively results in the controlled Lagrangians method. Indeed, we will show that for a certain choice of M_c (or equivalently, for M_d) and J_2 the systems (68, 69) and (60, 62) are equivalent, as well as the corresponding matching conditions (17, 18) and (64, 65). This means that for this particular choice of J_2 (and therefore of the shaped interconnection structure J_d) the IDA-PBC and the controlled Lagrangians method are equivalent.

The systems (68, 69) and (60, 62) are equivalent (by a coordinate transformation) if and only if the Hamiltonians H_c and H_d are equivalent and in addition the structure matrices J_c and J_d are equivalent. Notice that $p_c = M_c M^{-1} p$, and calculate H_c in the coordinates (q, p) to obtain

$$H_c(q,p) = \frac{1}{2} p^T M^{-1}(q) M_c(q) M^{-1}(q) p + V_c(q). \tag{70}$$

The Hamiltonians H_c and H_d are equivalent if and only if

$$M_c(q) = M(q) M_d^{-1}(q) M(q) \quad \text{and} \quad V_c(q) = V_d(q). \tag{71}$$

Notice that there is a one-to-one relation between M_c and M_d. (71) implies

$$p_c = M(q) M_d^{-1}(q) p. \tag{72}$$

The structure matrices J_c and J_d are the same if and only if J_d becomes in the coordinates (q, p_c) the canonical matrix J_c (in that case we call (q, p_c) *canonical coordinates* for the matrix J_d). This means that the Poisson brackets of the coordinates (q, p_c) should satisfy

$$\{q, q\}_d = 0, \quad \{q, p_c\}_d = I_n \quad \text{and} \quad \{p_c, p_c\}_d = 0, \tag{73}$$

where $\{\cdot, \cdot\}_d$ denotes the Poisson bracket corresponding to the structure matrix J_d. It is easy to check that the first two conditions in (73) are satisfied, while for the last one:

$$\begin{aligned} \{p_c, p_c\}_d &= \{M M_d^{-1} p, M M_d^{-1} p\}_d \\ &= -[\partial_q (M M_d^{-1} p)]^T + \partial_q (M M_d^{-1} p) + M M_d^{-1} J_2 M_d^{-1} M. \end{aligned} \tag{74}$$

Thus $\{p_c, p_c\}_d$ is equal to zero if and only if

$$J_2(q,p) = M_d M^{-1} \big[[\partial_q (M M_d^{-1} p)]^T - \partial_q (M M_d^{-1} p) \big] M^{-1} M_d. \tag{75}$$

(For clarity we left out the argument q of the matrices M and M_d.) Note that J_2 is clearly skew-symmetric. In conclusion, the Hamiltonian systems

(68, 69) and (60, 62) are equivalent if and only if conditions (71) and (75) hold.

Remark 14: The entries of the matrix J_2 in (75) can equivalently be written as

$$(J_2)_{ij}(q,p) = -p^T M_d^{-1} M \left[(M^{-1}M_d)_i, (M^{-1}M_d)_j\right], \quad i,j = 1,\ldots,n. \quad (76)$$

($[\cdot,\cdot]$ denotes the Lie bracket of vector fields.) This formulation was suggested in [15], although with swapped indices due to an unfortunate typo.

Since under conditions (71, 75) the Euler-Lagrange system (2, 15) and the Hamiltonian system (60, 62) are equivalent, the corresponding matching conditions (17, 18) and (64, 65) should also be equivalent. Indeed, it is easy to see that (71) implies that the matching conditions (18) and (65), describing the matching of potential energy, are the same. Furthermore, after some lengthy computations it can be shown that (64) is equal to (17) if J_2 is defined as in (75). Since under conditions (71, 75) the matching conditions (17, 18) (or equivalently (12)) and (64, 65) (or equivalently (63)) are equal, it follows immediately that also the corresponding feedback laws (13) and (66) are equal. In conclusion, we have the following proposition:

Proposition 15: *Consider the controlled Lagrangians method described in section 2 and the IDA-PBC method described in section 3, both applied to the class of mechanical systems (see section 2.2, respectively 3.2). The IDA-PBC method is equivalent to the controlled Lagrangians method if and only if the shaped interconnection structure is chosen as in (75). The controlled Lagrangian L_c and the shaped Hamiltonian H_d are related by (71).*

Remark 16: Proposition 15 states that the controlled Lagrangians method as described in section 2.2 is a special case of the more general IDA-PBC method (namely, with J_2 choosen equal to (75)). Independently from the present paper, the controlled Lagrangians method has been extended in [28] in such a way that for mechanical systems it becomes *equivalent* with the IDA-PBC method. Essentially, instead of restricting to systems of the form (2), they also allow to include some external forces into the closed-loop Euler-Lagrange system (i.e., the right hand side of (2) is not necessarily equal to zero, but can be any external force). In this way, it is possible to write any mechanical Hamiltonian system in Euler-Lagrange format by including the non-integrable part of the Hamiltonian system (corresponding to the failure of the Jacobi identity by the Poisson bracket) as an external (gyroscopic) force into the Euler-Lagrange system. Notice that this method only works for the class of simple mechanical systems (i.e., with total energy consisting of kinetic plus potential energy). Considering this larger class of closed-loop Euler-Lagrange systems in [28] it is shown that for simple mechanical systems the controlled Lagrangians method is equivalent to the IDA-PBC method.

4.2. *The λ-method for Hamiltonian matching*

In section 2.2 we described the λ-method of [4]. This method describes a way to solve the matching condition (17), a nonlinear PDE in M_c, by recursively solving the two linear PDEs (23) and (24). In this section we will show that the method can also be used to solve the matching condition (64) obtained in the IDA-PBC procedure. However, instead of recursively solving two linear PDEs, we now have to solve one *quasi-linear* PDE and afterwards a linear PDE. Solving the quasi-linear PDE might be simplified by using the freedom in J_2.

Without loss of generality we may write the skew-symmetric matrix J_2 as

$$J_2(q,p) = M_d M^{-1}\big[[\partial_q(MM_d^{-1}p)]^T - \partial_q(MM_d^{-1}p)\big]M^{-1}M_d + U(q,p), \quad (77)$$

where $U(q,p)$ is a skew-symmetric matrix, free to choose by the designer. According to the results in the previous section, equation (64) then results in

$$G^\perp\Big[\{\partial_q(M\dot q) - MM_c^{-1}\partial_q(M_c\dot q)\}\dot q - \partial_q(\frac{1}{2}\dot q^T M\dot q) + MM_c^{-1}[\partial_q(\frac{1}{2}\dot q^T M_c\dot q)] + U(q,M\dot q)M^{-1}M_c\dot q\Big] = 0, \quad \forall (q,\dot q)\in TQ. \quad (78)$$

As explained in section 2.2 this can be equivalently written as

$$G^\perp M\Big[\nabla_X X - \hat\nabla_X X + M^{-1}U(q,MX)M^{-1}M_c X\Big] = 0, \quad \forall X\in TQ. \quad (79)$$

Equations (78) and (79) clearly show the extra freedom, represented by U, obtained in the IDA-PBC method with respect to the controlled Lagrangians method (equations (17) resp. (20)). Consider (78) and notice that in order to satisfy the matching condition the term $G^{\perp}U(q, M\dot{q})M^{-1}M_c\dot{q}$ has to be quadratic in $\dot{q}$. Therefore we take $U(q,p)$ to be linear in its second component. In that case we can write

$$U(q,p) = \sum_{k=1}^{n} p_k U_k(q), \quad U_k^T = -U_k, \tag{80}$$

where p_k denotes the k-th component of the vector p.

Remark 17: In general U can also be chosen to include terms independent of p. These terms however will not be present in the quadratic (in $\dot{q}$) part of matching condition. Indeed, they should satisfy a matching condition of their own (see section 5.3). Terms in U independent of p come up in the matching of *integrable* Hamiltonian systems, see section 5.

Next we will show that the nonlinear PDE (78), or equivalently (79), can be solved by first solving a quasi-linear PDE in $\lambda = M_c^{-1}M$ and afterwards a linear PDE in M_c. First, define the skew-symmetric matrices W_k by

$$U_k = 2\lambda^T W_k \lambda, \quad \text{i.e., } U(q,p) = 2\sum_{k=1}^{n} p_k \lambda^T W_k(q)\lambda. \tag{81}$$

Then (79) becomes

$$G^{\perp}M\left[\nabla_X X - \hat{\nabla}_X X\right] + 2\sum_{k=1}^{n} G^{\perp}(MX)_k \lambda^T W_k X = 0, \quad \forall X \in T\mathcal{Q}, \tag{82}$$

where $(MX)_k$ denotes the k-th component of the vector MX. We can polarize this equation to obtain the equivalent condition

$$G^{\perp}M\left[\nabla_X Y - \hat{\nabla}_X Y\right] + \sum_{k=1}^{n} G^{\perp}\left[(MX)_k \lambda^T W_k Y + (MY)_k \lambda^T W_k X\right] = 0,$$
$$\forall X, Y \in T\mathcal{Q}. \tag{83}$$

As in the original method of [4], see section 2.2, consider (83) with the orthogonal projection matrix $\bar{G}^{\perp}$ instead of $G^{\perp}$. Furthermore, take $X = \lambda\bar{G}^{\perp}MX'$ and $Y = Y'$ and premultiply (83) by $(X')^T M$. Then the summation on the left hand side of (83) becomes

$$\sum_{k=1}^{n}\Big(\underbrace{(M\lambda\bar{G}^{\perp}MX')_k}_{\in\mathbb{R}}(X')^TM\bar{G}^{\perp}\lambda^TW_kY' \\ +\underbrace{((X')^TM\bar{G}^{\perp}\lambda^TW_k\lambda\bar{G}^{\perp}MX')}_{\in\mathbb{R}}M_{k*}Y'\Big), \tag{84}$$

where M_{k*} denotes the k-th row of the matrix M. As described in section 2.2 the first term of the left hand side of (83) will result in the right hand side of the λ-equation (23). Then by eliminating Y' the nonlinear PDE (83) becomes (suppressing the prime and writing X for X'):

$$0 = X^TM\bar{G}^{\perp}\lambda^T\Big\{[\partial_q(M\bar{G}^{\perp}MX)]^T - [\partial_q(\bar{G}^{\perp}MX)]^TM - M\partial_q(\bar{G}^{\perp}MX)\Big\} \\ + X^TM\bar{G}^{\perp}\Big\{[\partial_q(\lambda\bar{G}^{\perp}MX)]^TM + M\partial_q(\lambda\bar{G}^{\perp}MX) - [\partial_q(M\lambda\bar{G}^{\perp}MX)]^T\Big\} \\ + \sum_{k=1}^{n}\Big((M\lambda\bar{G}^{\perp}MX)_k\,X^TM\bar{G}^{\perp}\lambda^TW_k + (X^TM\bar{G}^{\perp}\lambda^TW_k\lambda\bar{G}^{\perp}MX)\,M_{k*}\Big) \\ \forall X\in T\mathcal{Q}. \tag{85}$$

This is a quasi-linear PDE in the sense that the derivatives of λ appear linear in the equation but the summation contains terms quadratic in the components of λ. Equation (85) can be regarded as the λ-equation for the matching of port-controlled Hamiltonian systems. Analogously to (23) it can be solved for $\lambda\bar{G}^{\perp}M$.

Remark 18: Remember that the skew-symmetric matrices W_k are designer chosen matrices. Exploiting the freedom in W_k might simplify the search for solutions of (85). Furthermore, notice that by taking $W_k = 0$, i.e., $U(q,p) = 0$, equation (85) results in the original λ-equation (23) (a *linear* PDE in λ), and the method reduces to the method of [4].

Once we have found a solution $\lambda\bar{G}^{\perp}M$ (together with some suitably chosen matrices W_k) of (85), the complete solution λ (or, equivalently, M_c) of the kinetic energy matching condition (78) can be found by solving a linear PDE. Indeed, premultiply (78) by M to obtain:

$$0 = M\bar{G}^{\perp}\lambda^T\{\partial_q(\frac{1}{2}\dot{q}^T M_c \dot{q}) - \partial_q(M_c\dot{q})\dot{q}\} + M\bar{G}^{\perp}\{\partial_q(M\dot{q})\dot{q} - \partial_q(\frac{1}{2}\dot{q}^T M\dot{q})\}$$
$$+ 2\sum_{k=1}^{n}(M\dot{q})_k \, M\bar{G}^{\perp}\lambda^T W_k \dot{q}, \quad \forall (q,\dot{q}) \in T\mathcal{Q}. \tag{86}$$

Given a solution $\lambda\bar{G}^{\perp}M$ of (85), this is a linear PDE in M_c.

In conclusion, this suggests the following approach for solving the nonlinear matching PDE (64): First solve the λ-equation (85) for $\lambda\bar{G}^{\perp}M$, thereby choosing suitable matrices W_k. Afterwards solve (86) for M_c. Then the solution of (64) is given by $M_d = MM_c^{-1}M = M\lambda$ and J_2 as in (77), where $U(q,p)$ is defined in (81).

5. Integrability

In the previous section we showed that if we choose J_2 to be equal to (75), or equivalently (76), then there exist canonical coordinates (q, p_c) such that in these coordinates the structure matrix J_d (61) becomes the canonical matrix J_c. By Darboux's Theorem the existence of canonical coordinates is equivalent to the Poisson bracket satisfying the Jacobi identity. In this case we call the Poisson bracket, or equivalently J_d, *integrable*.

5.1. *Integrability of the structure matrix*

In this section we give necessary and sufficient conditions for the structure matrix J_d to be integrable. Recall the structure matrix J_d (61):

$$J_d(q,p) = \begin{bmatrix} 0 & M^{-1}(q)M_d(q) \\ -M_d(q)M^{-1}(q) & J_2(q,p) \end{bmatrix}. \tag{87}$$

Assume the matrix J_d is integrable and let the canonical coordinates be denoted by $(q_c, p_c) = (q_c(q,p), p_c(q,p))$. without loss of generality we can assume that $q_c = q$. (See [20] for a precise statement and a proof of this.) Thus, let $(q_c, p_c) = (q, p_c(q,p))$ be canonical coordinates for J_d. This means that the relations (73) must be satisfied. Calculate

$$\{q, p_c\}_d = [I \quad 0] \begin{bmatrix} 0 & M^{-1}M_d \\ -M_d M^{-1} & J_2 \end{bmatrix} \begin{bmatrix} [\partial_q p_c]^T \\ [\partial_p p_c]^T \end{bmatrix} = M^{-1}M_d[\partial_p p_c]^T, \tag{88}$$

which is equal to I_n if and only if

$$p_c(q,p) = M(q)M_d^{-1}(q)p + Q(q), \tag{89}$$

with $Q(q)$ any smooth vector-valued function of the coordinates q. Secondly, use (89) to calculate

$$\begin{aligned}\{p_c, p_c\}_d = &-[\partial_q(MM_d^{-1}p)]^T - [\partial_q Q]^T + \partial_q(MM_d^{-1}p) + \partial_q Q \\ &+ MM_d^{-1}J_2M_d^{-1}M.\end{aligned} \tag{90}$$

This is equal to zero if and only if

$$\begin{aligned}J_2 = M_dM^{-1}&\left[[\partial_q(MM_d^{-1}p)]^T - \partial_q(MM_d^{-1}p)\right]M^{-1}M_d + \\ &M_dM^{-1}\left[[\partial_q Q]^T - \partial_q Q\right]M^{-1}M_d.\end{aligned} \tag{91}$$

We find it convenient to write $J_2(q,p) = J_2^\circ(q,p) + \hat{J}(q)$, with J_2° equal to (75) and

$$\hat{J}(q) = M_dM^{-1}\left[[\partial_q Q]^T - \partial_q Q\right]M^{-1}M_d. \tag{92}$$

So, if J_d is integrable then J_2 necessarily has the form (91). Conversely, if J_2 has the form (91), then clearly $q_c = q$ and p_c (89) are canonical coordinates for J_2. Notice that $Q(q) = 0$ yields $\hat{J} = 0$ and consequently $J_2 = J_2^\circ$, for which the canonical coordinates are $(q, p_c) = (q, MM_d^{-1}p)$ as we have seen in the previous section.

Proposition 19: *The structure matrix J_d defined in (87) is integrable if and only if J_2 has the form (91), for some smooth vector-valued function $Q(q)$.*

5.2. *Gyroscopic terms*

Consider the Hamiltonian H_d expressed in the canonical coordinates (q, p_c). For $(q, p_c) = (q, MM_d^{-1}p)$, corresponding to J_2°, the Hamiltonian H_d (60) becomes the canonical Hamiltonian H_c (68) with M_c and V_c defined by (71). Similar to H_d the canonical Hamiltonian H_c has the form of the sum of kinetic and potential energy. However, this is not the case anymore for $\hat{J} \neq 0$. Indeed, take $\hat{J}$ as in (92), then in the canonical coordinates the

Hamiltonian H_d becomes the canonical Hamiltonian H_c defined by (substituting $p = M_d M^{-1}(p_c - Q)$ into (60)):

$$H_c(q_c, p_c) = \frac{1}{2} p_c^T M^{-1} M_d M^{-1} p_c - p_c^T M^{-1} M_d M^{-1} Q + \frac{1}{2} Q^T M^{-1} M_d M^{-1} Q + V_d. \qquad (93)$$

The canonical Hamiltonian includes the *gyroscopic terms*

$$-p_c^T M^{-1} M_d M^{-1} Q, \qquad (94)$$

which are terms linear in the p-variables (the momenta). In addition the potential energy is augmented to be

$$V_c = \frac{1}{2} Q^T M^{-1} M_d M^{-1} Q + V_d. \qquad (95)$$

Thus in case $\hat{J}$ is defined as in (92), then the system (60, 61, 62) becomes in the canonical coordinates $q_c = q$ and p_c (89) the canonical Hamiltonian system (69, 93). If $Q(q)$ is chosen to be nonzero then gyroscopic terms are introduced into the system and in addition the potential energy is augmented.

Remark 20: The canonical Hamiltonian system (69, 93) corresponds via the inverse Legendre transformation to the Euler-Lagrange system (2) with Lagrangian defined by

$$L_c(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) M_d^{-1}(q) M(q) \dot{q} + \dot{q}^T Q(q) - V_d(q). \qquad (96)$$

An interesting question is if the gyroscopic terms introduced by $\hat{J}$ are intrinsic or not, defined in the following way:

Definition 21: The gyroscopic terms are called *intrinsic* if there does not exist a canonical transformation $(q_c, p_c) \mapsto (\bar{q}_c, \bar{p}_c)$ such that in the new coordinates $(\bar{q}_c, \bar{p}_c)$ the Hamiltonian (93) becomes the quadratic Hamiltonian

$$\bar{H}_c(\bar{q}_c, \bar{p}_c) = \frac{1}{2} \bar{p}_c^T \Lambda^{-1}(\bar{q}_c) \bar{p}_c + \bar{V}(\bar{q}_c), \qquad (97)$$

for some Λ and $\bar{V}$.

That is, the gyroscopic terms are intrinsic if they cannot be removed by a canonical coordinate transformation (and therefore the Hamiltonian cannot be transformed into the form of kinetic plus potential energy). The following proposition gives an answer to the above question.[20]

Proposition 22: *The gyroscopic terms are intrinsic to the closed-loop system if and only if* $[\partial_q Q]^T \neq \partial_q Q$ *(which is equivalent to* $\hat{J} \neq 0$*).*

5.3. *Integrability and matching*

Consider the matching condition (64) for the kinetic energy and plug in J_2 as defined in (91) to get

$$G^{\perp}(q)\Big[\partial_q(\frac{1}{2}p^T M^{-1}(q)p) - M_d(q)M^{-1}(q)\partial_q(\frac{1}{2}p^T M_d^{-1}(q)p) + J_2^{\circ}(q,p)M_d^{-1}(q)p\Big] + G^{\perp}(q)\left[\hat{J}(q)M_d^{-1}(q)p\right] = 0. \tag{98}$$

This equation has to hold for all $(q,p) \in T\mathcal{Q}$. Since the first part of (98) is quadratic in p (recall that J_2° is linear in p) and the second part is linear in p, it follows that (98) holds for all (q,p) if and only if the following two conditions hold:

$$G^{\perp}(q)\Big[\partial_q(\frac{1}{2}p^T M^{-1}(q)p) - M_d(q)M^{-1}(q)\partial_q(\frac{1}{2}p^T M_d^{-1}(q)p) + J_2^{\circ}(q,p)M_d^{-1}(q)p\Big] = 0, \tag{99}$$

and

$$G^{\perp}(q)\hat{J}(q)M_d^{-1}(q) = G^{\perp}M_d M^{-1}\left[[\partial_q Q]^T - \partial_q Q\right]M^{-1} = 0, \tag{100}$$

for all $(q,p) \in T\mathcal{Q}$. Equation (99) is nothing but the matching condition (64) with $J_2 = J_2^{\circ}$. Since it is equivalent to the matching condition (17), see section 4, it can be solved by the λ-method. Equation (100) defines a matching condition for $\hat{J}$. Given a solution M_d of (99), it is a linear PDE in Q. It can also be written in terms of a solution $\lambda\bar{G}^{\perp}M$ of the λ-equation (23) by premultiplying (100) with M to obtain (notice that $\lambda = M_c^{-1}M = M^{-1}M_d$):

$$M\bar{G}^{\perp}\lambda^T\left[[\partial_q Q]^T - \partial_q Q\right]M^{-1} = 0. \tag{101}$$

This result leads to the following parameterization of matching integrable Hamiltonian systems:

Proposition 23: *Assume that the Hamiltonian system (60, 61, 62) with* $J_2 = J_2^\circ$ *(75) satisfies the matching conditions (64, 65), i.e., matches with the port-controlled Hamiltonian system (58, 59). Then every Hamiltonian system (60, 61, 62, 91), with* $\hat{J}$ *satisfying condition (100), is integrable and matches with the port-controlled Hamiltonian system (58, 59). Furthermore, this class of systems (parametrized by* $\hat{J}$*) describes exactly all the possible integrable Hamiltonian systems with Hamiltonian (60) that match with (58, 59).*

We remark that the Hamiltonian matching described in proposition 23 can also be interpreted as Lagrangian matching with the closed-loop Lagrangian given by (96).

6. Conclusions

In this paper we reviewed two recently developed methods for the stabilization of underactuated mechanical systems. The first is the controlled Lagrangians method, defined for Euler-Lagrange systems. The second is the interconnection and damping assignment passivity based control (IDA-PBC) method, which considers port-controlled Hamiltonian systems. The fundamental idea underlying both methods is that of matching, that is, finding a suitable closed-loop Euler-Lagrange, respectively port-controlled Hamiltonian, system which stabilizes the desired equilibrium point (the conditions under which the corresponding control law exists are called matching conditions).

The controlled Lagrangians method as originally introduced in [3] for mechanical systems with symmetry is reviewed and the matching conditions obtained in that paper are interpreted in terms of kinetic and potential energy matching. Since the class of Euler-Lagrange systems is contained in the class of port-controlled Hamiltonian systems, the IDA-PBC method includes the controlled Lagrangians method as a special case. In fact, the possibility of shaping not only the energy function but also the interconnection structure of the system gives an extra degree of freedom to the IDA-PBC method. It is shown that for a particular choice of this interconnection structure the IDA-PBC method results in the controlled Lagrangians method. Furthermore the integrability of the closed-loop Hamiltonian system is investigated. Explicit (necessary and sufficient) conditions on the interconnection structure are given under which the closed-loop Hamiltonian system is integrable (i.e., corresponds to an Euler-Lagrange system). In general, this includes the introduction of intrinsic gyroscopic terms in the closed-loop system.

The matching conditions generally consist of a set of nonlinear PDEs, to be solved either for the closed-loop Lagrangian function (in the controlled Lagrangians method) or for the closed-loop Hamiltonian function and the interconnection structure (in case of the IDA-PBC method). The λ-method described in [4] for the controlled Lagrangians method converts these nonlinear PDEs into a set of linear PDEs, to be solved recursively. It is shown that the λ-method can also be applied to the PDEs obtained in the IDA-PBC method, leading to set of quasi-linear and linear PDEs to be solved recursively.

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