

Sound and Shock Waves in Bubbly Liquids

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Introduction

When small bubbles containing air or other gases are dispersed in water, the resulting acoustical properties differ a great deal from those of water even if the gas concentration by volume is only a few percent. The speed of sound is significantly lower than in pure water and effects of dispersion and attenuation brought about by the presence of the bubbles alter the propagation of sound waves. In what follows the most important features of propagation of sound in bubbly liquids are dealt with. In many practical applications such as propagation of pressure waves in steam-water mixtures or under water explosions, the amplitude of the waves is not small. Attention therefore has to be given to waves of finite amplitude as well. These finite amplitude waves may develop into shock waves the structure of which is discussed.

1. Acoustic Waves

When small air bubbles are dispersed in water the compressibility of the resulting liquid is much larger than that of the pure liquid. The compressibility in its turn determines the velocity of propagation of sound waves of low frequency, which can be calculated as follows. Denoting densities of gas and liquid with ρ_g and ρ_l respectively, concentration of air by volume with α , we have under the assumption of no relative velocity between bubbles and liquid

$$\frac{\alpha \rho_g}{\rho_l (1-\alpha) + \rho_g \alpha} = \text{const.} \quad (1.1)$$

which expresses that in a unit mass of the mixture the mass of gas is constant. In the denominator in (1.1) appears the density of the mixture, which we denote with ρ ,

$$\rho = \rho_l (1-\alpha) + \rho_g \alpha. \quad (1.2)$$

When the thermodynamic changes in the gas phase are isentropic and the frequency is low enough for the local pressure in the liquid to be equal to the local pressure in the gas we have

from (1.1)

$$\frac{\alpha p^{1/\gamma}}{\rho_L(1-\alpha) + \alpha \rho_g} = \text{const.}, \quad (1.3)$$

where γ is the ratio between specific heats in the gas. When taking $\rho_g \ll \rho_L$ and neglecting the liquid compressibility it follows from (1.1) that

$$\alpha \approx \frac{\rho_L - \rho}{\rho_L} \quad (1.4)$$

and that, from (1.3) and (1.4),

$$\frac{dp}{d\rho} = c_0^2 = \frac{\gamma p}{\rho_L \alpha (1-\alpha)}. \quad (1.5)$$

Of course this relation for the sound speed needs correction when α is very close either to zero or to unity but otherwise it provides an accurate prediction for the sound speed in liquid with dispersed gas. For water (1.5) gives at 1 bar for $\alpha = 4\%$, 8% and 10% the values $c_0 = 60$ m/s, 44 m/s and 40 m/s. These velocities are substantially lower than the sound velocities in either air or water. See Fig.1.

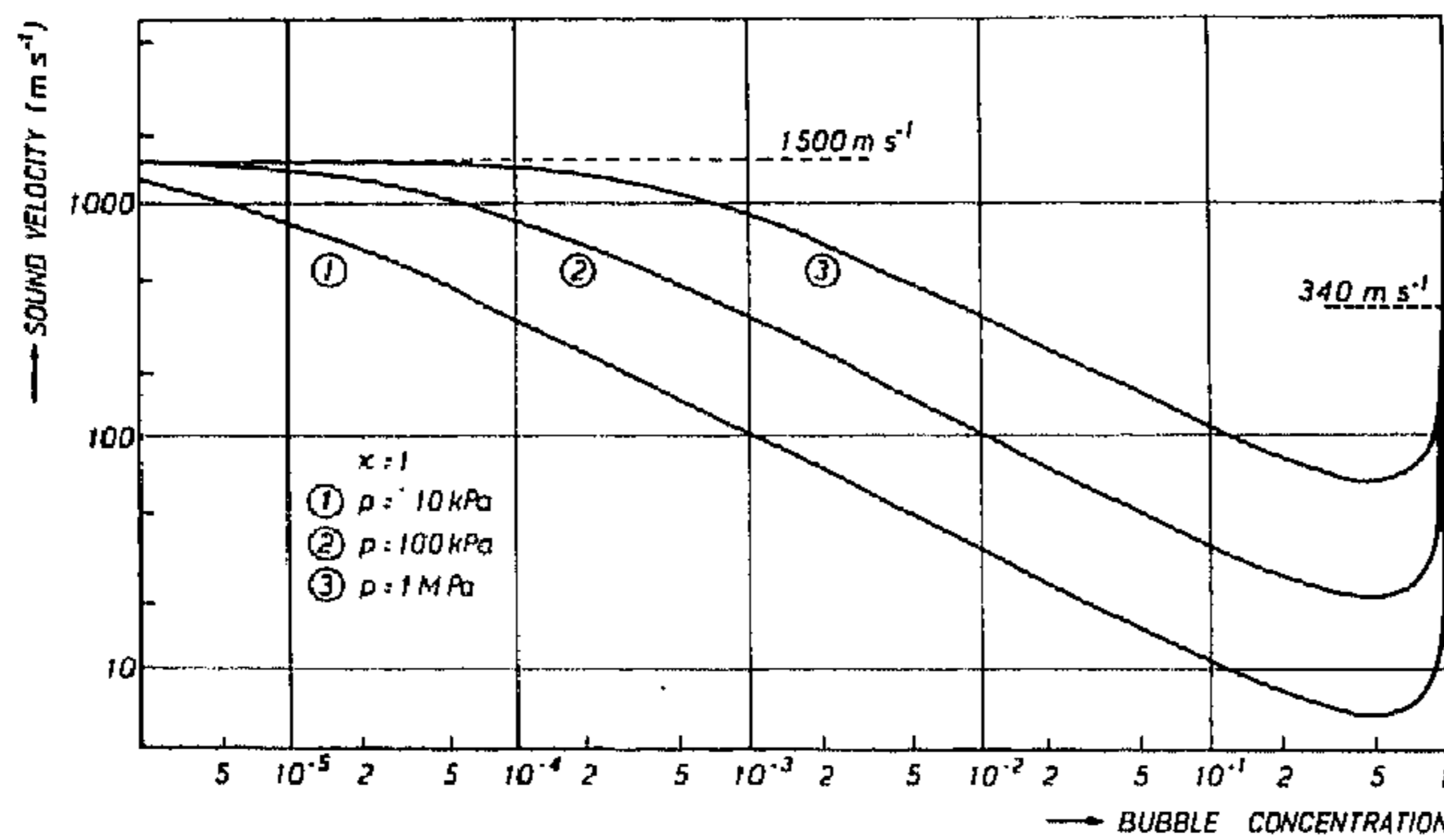


Fig.1 Sound velocity in water as function of free air concentration with pressure as parameter ($\kappa = \gamma$)

Equation (1.5) holds not only for bubbly flow but also for other topologies in which the liquid forms the continuous phase. There have been numerous experimental verifications of (1.5), for a survey see [1], and (1.5) may be considered as the fundamental relation for the propagation of sound waves in liquids containing gas. Corrections and modifications on (1.5) are necessary when physical processes need to be considered which are left out of account in the simple argument that leads to (1.5).

Relative Radial Motion

Brooks and other small free surface currents emit sound, first described by MINNAERT in his famous paper "Musical Air Bubbles and the Sound of Running water" [2]. The sound is due to volume vibrations of small air bubbles. We consider, for convenience, spherical bubbles in an incompressible viscous liquid. When $R(t)$ denotes the radius, motion in the liquid due to spherical vibrations $\dot{R}(t)$ of the bubble is described by the velocity potential

$$\phi = - \frac{R^2 \dot{R}}{r}, \quad (1.6)$$

where r is the distance of a point in the liquid to the centre of the bubble. The pressure in the liquid can be calculated with the aid of Bernoulli's theorem. The continuity of normal stresses at the interface between air and liquid gives, p_0 being the pressure in the liquid far away from the bubble and p_g being the pressure of the gas in the bubble,

$$p_g - p_0 = \rho_L \left\{ R \frac{d^2 R}{dt^2} + \frac{3}{2} \left(\frac{dR}{dt} \right)^2 + \frac{4\nu}{R} \frac{dR}{dt} \right\}. \quad (1.7)$$

In this equation ν is the kinematic viscosity, $10^{-6} \text{ m}^2/\text{s}$ in water. Further, surface tension has not yet been included¹. From this equation we can obtain the angular frequency of free volume oscillations of a bubble, by linearizing about an equilibrium state characterized by p_0 and R_0 . One finds in this way for isentropic behaviour of the gas content [2]

$$\omega_B = \frac{1}{R_0} \left(\frac{3\gamma p_0}{\rho_L} \right)^{\frac{1}{2}}. \quad (1.8)$$

The associated frequency $f_B = \omega_B/2\pi$ is for a bubble of 1 mm radius $3.3 \times 10^3 \text{ c/s}$. Bubbles with radii between 0.1 mm and 10 mm have natural frequencies within the audible range. The question whether turbulence will excite resonant bubble oscillations must be answered negatively [3], because at the resonance frequency of the small bubbles that persist, the length scale of pressure fluctuations is too small for the pressure oscillations to be coherent over the bubble surface.

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The occurrence of a flow potential as in (1.6) to describe the flow of a viscous liquid is no paradoxical situation: Such a flow exactly satisfies the Navier-Stokes equations. It does not, in general, satisfy the no slip condition at the surface of a body. In the present case only the radial velocity is prescribed at the interface, so that potential flow exactly satisfies the boundary conditions.

An important kind of small bubbles are cavitation bubbles, filled with vapour and air. When also surface tension is taken into account the quantity p_0 in (1.8) should be replaced by $(p_0 - p_v + 4/3 \frac{T}{R_0})$, T being the coefficient of surface tension and p_v the vapour pressure. No real value of the frequency exists when

$$p_0 < p_v - 4/3 \frac{T}{R_0}, \quad (1.9)$$

which is the wellknown threshold for vaporous cavitation. When the frequency of acoustic waves in bubbly liquids approaches the resonance frequency, as given in (1.8), the inertia of liquid radially accelerated and decelerated becomes important. As a result the velocity of propagation is no longer as in (1.5) but depends on the frequency. The radial motion is damped by viscosity, and taking (1.7) for the difference between the pressure in the liquid and the pressure in the bubbles leads for e.g. the velocity u in the mixture to

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \frac{\partial^2 u}{\partial x^2} + \frac{c_0^2}{\omega_B^2} \frac{\partial^4 u}{\partial x^2 \partial t^2} + \frac{4c_0^2 v}{\omega_B^2 R_0^2} \frac{\partial^3 u}{\partial x^2 \partial t}. \quad (1.10)$$

Inserting in (1.10) solutions of the form $\exp i(kx - \omega t)$ gives the dispersion equation

$$\frac{k^2}{\omega^2} = \frac{1}{c_l^2} + \frac{1 - \omega^2/\omega_B^2 + i\delta\omega/\omega_B}{c_0^2 \{ (1 - \omega^2/\omega_B^2)^2 + \delta^2 \omega^2/\omega_B^2 \}}. \quad (1.11)$$

The first term on the right hand side of (1.11) $1/c_l^2$, where c_l is the velocity of sound in pure liquid, does not follow from (1.10) but must be included when the liquid is no longer regarded as incompressible. The damping mechanisms are accounted for by the logarithmic decrement δ , which is for the viscous term included in (1.10)

$$\delta_{\text{visc}} = \frac{4\nu}{\omega_B R_0^2}. \quad (1.12)$$

In general damping of thermal nature with a corresponding δ_{th} , to be discussed below, is more important. Another cause for damping is the acoustic radiation of a bubble oscillating with frequency ω . The associated logarithmic decrement δ_{ac} is, see e.g. [4],

$$\delta_{\text{ac}} = \frac{\omega^2 R_0}{\{1 + (\frac{\omega R_0}{c_l})^2\} \omega_B c_l}. \quad (1.13)$$

Of course, if a mixture contains not bubbles of the same size but a distribution of bubbles of different size the expression

corresponding with (1.11) is much more complicated and results from integration of (1.11) weighed with the pertinent number density over the bubble distribution. The real part of ω/k as given by (1.11) constitutes the velocity of propagation of acoustic waves with angular frequency ω , the imaginary part gives the damping. They are represented for some particular values of R_0 and α for air bubbles in water in Fig.2 and Fig.3, taken from [5]

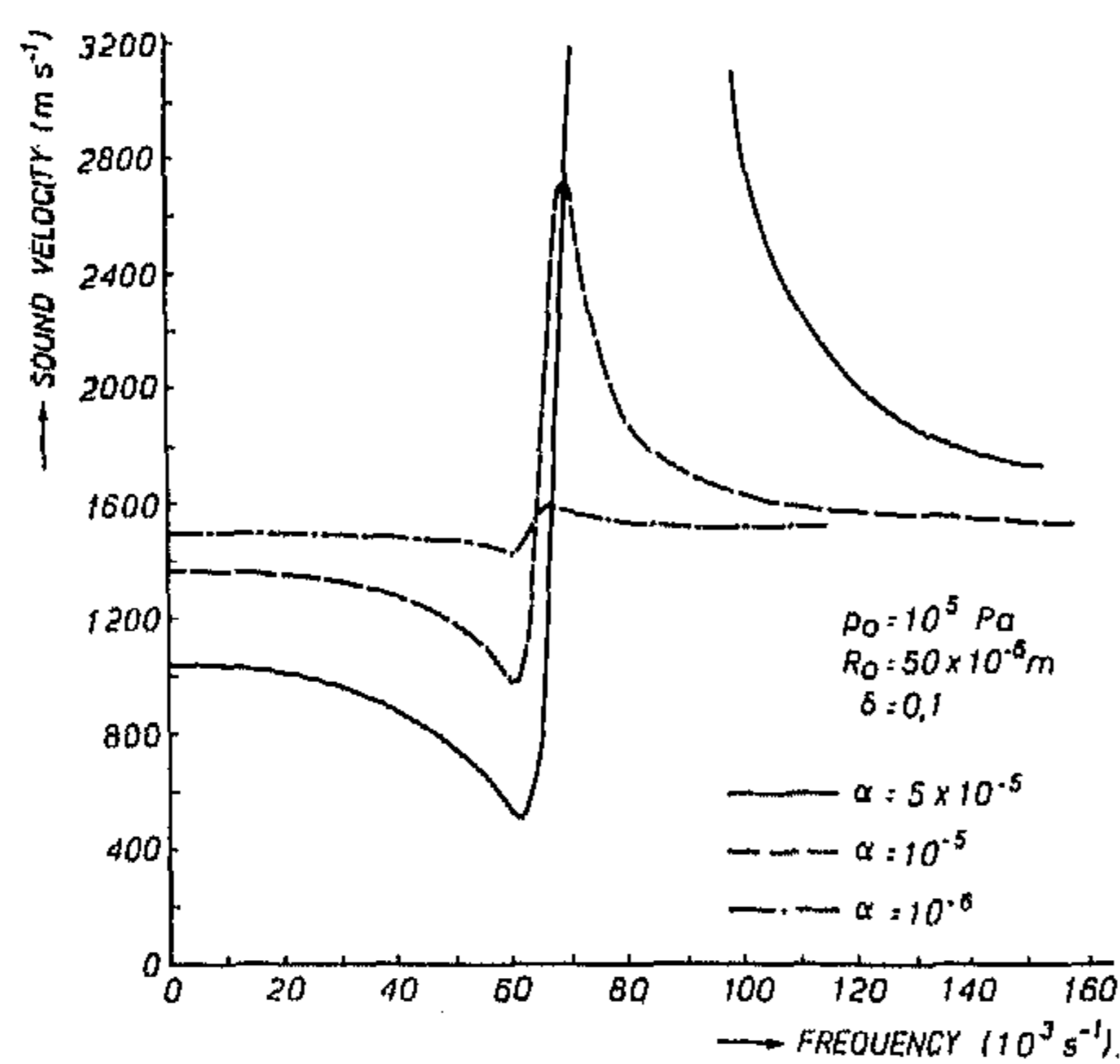


Fig.2 Sound speed in bubble mixture as function of frequency

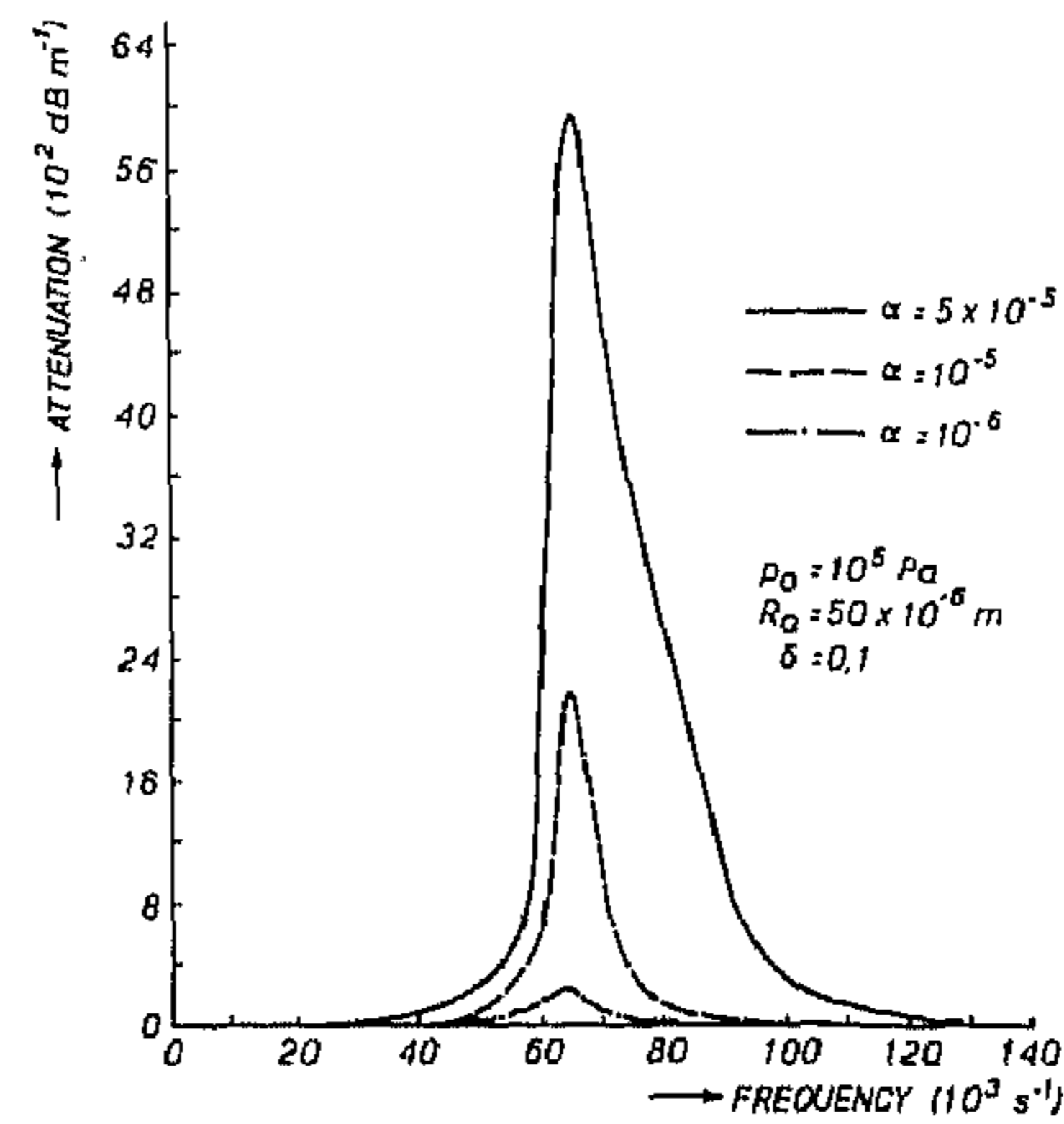


Fig.3 Attenuation of sound in bubble mixture

These relations appear, see [1] to be fairly well supported by experimental observations. Note that whereas the velocity of sound tends to the value given in (1.5) for $\omega \rightarrow 0$ it rises in the vicinity of $\omega = \omega_B$ above c_L and tends to c_L for $\omega \rightarrow \infty$. The strong dispersion of acoustic waves produced by the dynamic response of bubbles is of importance for cavitation noise [5] and also for the propagation of sound waves near the sea's surface where air is entrained mostly in the form of bubbles.

Relative Translational Motion

Only very tiny bubbles will move with the liquid locally. In general there will be as a result of the different densities a relative motion. This relative motion has profound effects on the mechanical behaviour of two phase flows in which only very partial insight has been gained till thus far. The study of these effects meets with severe difficulties even in the study of simple configurations. We will discuss here some relatively simple aspects. In establishing conservation equations for the suspension, the strategy is to deduce properties of the suspension on a scale comprising many bubbles from the relative flow around an individual constituent in the suspension. This relative flow depends apart from the particle under consideration also on all other particles. The difficulties referred to have to do with this interaction. The interaction can be neglected when the average distance between particles (or bubbles in our case) is so large that each bubble is unaware of the presence

of others. For number density n the average distance between bubble centres is $n^{-1/3}$ and interaction may be neglected when this is very large with respect to a representative bubble radius R ,

$$\begin{aligned} n^{-1/3} &\gg R, \text{ or} \\ \alpha = \frac{4}{3}\pi n R^3 &\ll 1. \end{aligned} \quad (1.14)$$

Under this restriction we have with bubble velocity v and liquid velocity u ,

$$\frac{d}{dt} m (v-u) = -\rho_l V \frac{du}{dt}, \quad (1.15)$$

m being the virtual mass of a bubble and V its volume,

$$m = \frac{1}{2}\rho_l V. \quad (1.16)$$

Equation (1.15) expresses that with $\rho_g \ll \rho_l$ the force on a bubble partly due to the pressure gradient in the uncoming flow, partly due to the relative motion, is zero. The validity of (1.15) can be rigorously proven when u is uniform in space. For nonuniform flow, for example brought about by the presence of other bodies only approximate results [5] are available. Insertion of (1.16) in (1.15) gives $v = 3u$ for rigid spheres in potential flow. If surface active agents are absent potential flow represents the flow around the bubble fairly accurately at high Reynolds numbers. The calculation of the velocity of sound leads with $v = 3u$ to [7]

$$c_f^2 = \frac{\gamma p (1+2\alpha)}{\rho_l \alpha (1-\alpha)}. \quad (1.17)$$

In the derivation (1.1) cannot be used, but the conservation of mass for the gas has to be formulated as

$$\frac{\partial}{\partial t} (\rho_g \alpha) + \frac{\partial}{\partial x} (\rho_g \alpha v) = 0. \quad (1.18)$$

Relation (1.17) is valid for $\alpha \ll 1$. Its verification cannot be done directly by measurement of the velocity of propagation of sound waves because for α of the order of a few percent the difference between the right hand sides of (1.5) and (1.17) is within the experimental accuracy. There is indirect verification possible however. The motion described by (1.15) is impeded by a viscous force. For an air bubble in water this is for low Reynolds numbers

$$W = 4\pi\mu R(v-u), \quad (1.19)$$

as follows from the Hadamard-Rybczynski relation [8] and for high Reynolds number

$$W = 12\pi\mu R(v-u) \quad (1.20)$$

based on potential flow around the bubble, see [7] and [8]. In these expressions μ is the viscosity of the liquid. The relations (1.19) and (1.20) hold when the bubbles are spherical, and for (1.20) the liquid must be devoid of surface active agents. When we adopt in spite of these restrictions (1.20) as resistance law, bearing in mind that in many practical circumstances the Reynolds number for the relative motion is large (1.15) becomes

$$\frac{d}{dt} m(v-u) + 12\pi\mu R(v-u) = -\rho_l V \frac{du}{dt}. \quad (1.21)$$

The competition between the inertia term with the virtual mass and the viscous term gives a relaxation time τ for adjusting the bubble velocity v to the velocity u of the liquid, given by

$$\tau = \frac{R^2}{18\nu}. \quad (1.22)$$

When relative motion is started at $t=0$ viscous resistance is un-effective at times

$$t \ll \tau. \quad (1.23)$$

During this time interval the relative motion obeys (1.15). Straight forward linearization of the governing equations including translatory motion as governed by (1.18), (1.21), the continuity and momentum equations for the mixture and the conservation of bubble numbers

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} (nv) = 0, \quad (1.24)$$

but excluding relative radial motion, gives, [9], for the acoustic pressure $p_g - p_0 = \tilde{p}$,

$$\frac{\partial}{\partial t} \left\{ c_f^2 \frac{\partial^2 \tilde{p}}{\partial x^2} - \frac{\partial^2 \tilde{p}}{\partial t^2} \right\} + \tau^{-1} \left\{ c_o^2 \frac{\partial^2 \tilde{p}}{\partial x^2} - \frac{\partial^2 \tilde{p}}{\partial t^2} \right\} = 0. \quad (1.25)$$

In unsteady essentially one dimensional flow, phenomena with a time scale less than τ are governed according to this relation by a wave equation with wave speed c_f . At larger time scales waves propagate with speed c_o . The influence of the higher order terms in this case can be appreciated by writing (1.25) as

$$\frac{\partial}{\partial t} \left\{ (c_f \frac{\partial}{\partial x} - \frac{\partial}{\partial t}) (c_f \frac{\partial \tilde{p}}{\partial x} + \frac{\partial \tilde{p}}{\partial t}) \right\} + \tau^{-1} \left\{ (c_o \frac{\partial}{\partial x} - \frac{\partial}{\partial t}) (c_o \frac{\partial \tilde{p}}{\partial x} + \frac{\partial \tilde{p}}{\partial t}) \right\} = 0.$$

Now we consider a wave which travels almost undisturbed to the right with velocity c_0 . Inserting $c_0 \partial/\partial x + \partial/\partial t = O(\epsilon)$, $\epsilon \rightarrow 0$, in the above expression and requiring $p = p_0$ at $x \rightarrow \infty$, gives upon integration

$$\frac{\partial \tilde{p}}{\partial t} + c_0 \frac{\partial \tilde{p}}{\partial x} - \frac{1}{2} \tau (c_f^2 - c_0^2) \frac{\partial^2 \tilde{p}}{\partial x^2} = 0. \quad (1.26)$$

This shows the interesting fact that the higher order terms have a diffusive effect on the wave. The diffusion coefficient is $\frac{1}{2} \tau (c_f^2 - c_0^2)$, which is, see (1.5) and (1.17) proportional to α for $\alpha \ll 1$, and therefore small. A linear wave will eventually diffuse completely as a result of this diffusion but in a non-linear wave this diffusion may be resisted by nonlinear effects as we shall see later.

Heat and Mass Transfer Between Bubbles and Liquid

In the derivation of a relation for the sound velocity (1.5) and for the natural frequency (1.8) it has been assumed that the thermodynamic changes within the bubbles are isentropic. Of course, this is a simplification because actually during compression and rarefaction of the gas inside the bubbles heat is exchanged with the surrounding liquid. Since the heat capacity of this liquid is huge in respect with the heat capacity of the bubbles, the liquid can be considered as being of constant temperature. Within the bubble appreciable temperature differences occur within a depth $(D_h/\omega)^{\frac{1}{2}}$ from the interface, where ω is again the angular frequency of the sound wave and D_h the thermal diffusivity, $2.10^{-5} \text{ m}^2/\text{s}$ for air. When this is very small with respect to the radius R of the bubble,

$$(D_h/\omega R^2) \ll 1 \quad (1.27)$$

the thermodynamic changes in the bubble are nearly adiabatic. In the opposite case where the depth of penetration of heat is large with respect to R no appreciable temperature differences occur inside a bubble which therefore behaves isothermally. In these limits the pressure p_g of the gas inside the bubbles is a function of ρ_g alone. In intermediate circumstances p_g is a function of ρ_g and the temperature T_g . In particular for an harmonic motion there is a phase difference between p_g and ρ_g . This makes in principle a representation as

$$p_g \sim \rho_g^n \quad (1.28)$$

impossible. Nevertheless for practical purposes the thermodynamic behaviour can [4] be represented as in (1.28). As a result of heat exchange between bubbles and liquid acoustic waves are

damped and hence there is a thermal contribution δ_{th} to the logarithmic decrement δ in (1.10). In fact this contribution to δ dominates over δ_{visc} in (1.12) and δ_{ac} in (1.13), as shown for $R_0 = 10^{-5} m$ and for $R_0 = 10^{-3} m$ in Fig. 4, taken from [4].

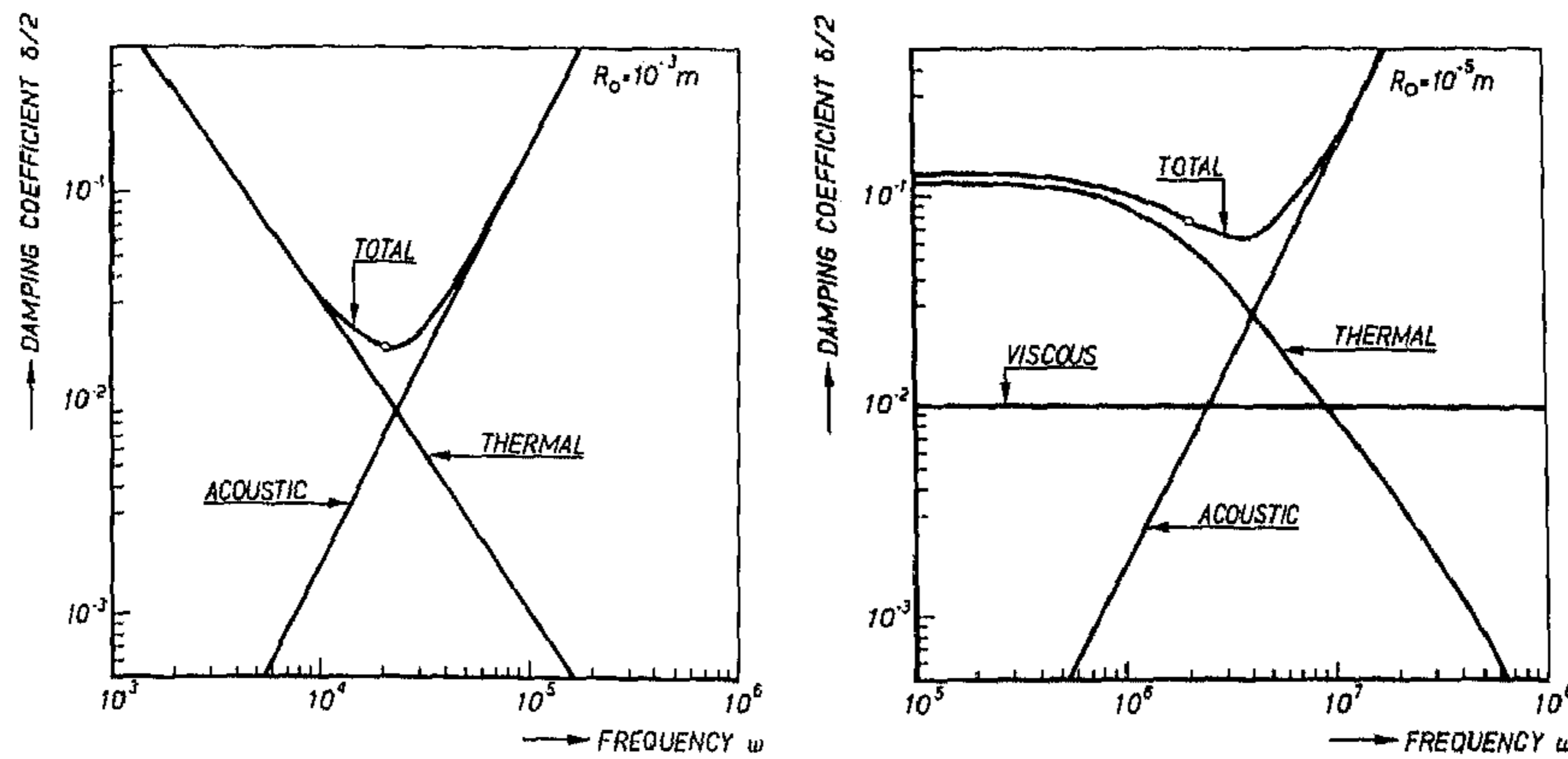


Fig.4 Damping coefficient δ for small amplitude forced oscillations of an air bubble in water (from [4])

In analogy with (1.12) the thermal contribution to δ is often represented by

$$\delta_{th} = \frac{4v_{th}}{\omega_B R_0^2} \quad (1.29)$$

It is [10] however not possible to give v_{th} as a simple analytical expression. An approximate expression is

$$\delta_{th} = \frac{3(\gamma-1)}{2(\omega/2D_h)^{1/2} R_0} \quad (1.30)$$

obtained by PFRIEM, see [1]. Mass transfer is of importance in various ways though perhaps less for acoustic waves as for cavitation phenomena. In vaporous cavitation condensation and vaporization are, of course, of essential interest, but also mass transfer by diffusion affects cavitation phenomena significantly. Mass transfer by diffusion in sound waves is not of very great importance. Rectified diffusion leads [4] to a very slow growth of a bubble in a sound wave. In [4] it is shown that for a bubble of 1 mm to grow to twice this radius 10^6 sec are needed with a relative pressure amplitude $(P_{max} - P_{average})/P_{average} = 0.25$.

2. Waves of finite amplitude

While we have been discussing till thus far waves of infinitesimal amplitude, one should be aware that in many practical situ-

ations like in the case of cavitation or under water explosions amplitudes are no longer small. For understanding the effects of finite amplitude, it is convenient to leave out momentarily the effects of relative motion and heat transfer. We then have a homogeneous mixture with a sound velocity given by (1.5). To such a mixture we may apply the theory of one phase gasdynamics as described for instance in [11]. With the quantity

$$\sigma = \int \frac{c_0(\rho) d\rho}{\rho} \quad (2.1)$$

one of the outcomes of this is that the equations of motion can be written in characteristic form as

$$\left\{ \frac{\partial}{\partial t} + (u \pm c_0) \frac{\partial}{\partial x} \right\} \{ u \pm \sigma \} = 0. \quad (2.2)$$

With the aid of (1.4) and (1.5) both ρ and c_0 can be expressed in terms of α and in this way σ can be found. With isothermal behaviour and $\alpha \ll 1$ we find

$$\sigma = c_0(\alpha_0 - \alpha). \quad (2.3)$$

Now we look in particular at a simple wave, that is a wave travelling towards an undisturbed medium. Along such a wave $u = \sigma$ because all 'left' characteristics, that is characteristics coming from the undisturbed region, carry along them a zero value of $u - \sigma$. Then along the right characteristics u and σ are constant, whence these characteristics are straight. The simple wave is described therefore, introducing

$$\frac{p - p_0}{p_0} = \tilde{p}, \quad \text{with} \quad (2.4)$$

$$\frac{\partial \tilde{p}}{\partial t} + c_0 \frac{\partial \tilde{p}}{\partial x} + c_0 \tilde{p} \frac{\partial \tilde{p}}{\partial x} = 0, \quad (2.5)$$

because $u + c_0 = \sigma + c_0 = c_0 \left(1 + \frac{\alpha_0 - \alpha}{\alpha}\right) = c_0 \left(1 + \frac{p - p_0}{p_0}\right)$, where we use that

with $\gamma = 1$ and $\alpha \ll 1$ (1.3) gives $p\alpha = \text{constant}$. This indicates that a wave of finite amplitude, while travelling forward, becomes steeper because each disturbance \tilde{p} travels with a speed $c_0(1 + \tilde{p})$. This is sometimes called amplitude dispersion. A wave of expansion continuously broadens by amplitude dispersion. Here, as with acoustic waves, relative motion and dissipation due to the presence of the bubble will affect the wave as it propagates forward. A complete analysis of this is far beyond analytical or numerical treatment but this becomes better if we restrict to an approximation, apart from $\alpha \ll 1$, in which the wave ampli-

tude is of moderate magnitude so that we may neglect terms which are of the third order in this amplitude. This has been done for acoustic waves in gases and gravity waves on water of finite depth [11],[12]. A similar analysis including amplitude dispersion and frequency dispersion by relative radial motion leads [1] to

$$\frac{\partial \tilde{p}}{\partial t} + c_0 \frac{\partial \tilde{p}}{\partial x} + c_0 \tilde{p} \frac{\partial \tilde{p}}{\partial x} + \frac{1}{2} \frac{c_0^3}{\omega_B} \frac{\partial^3 \tilde{p}}{\partial x^3} = 0. \quad (2.6)$$

In this equation c_0 is the value following from (1.5), with $\gamma=1$, in the undisturbed state. Equation (2.6) is known as the Korteweg-de Vries equation, originally [13] derived for water waves but later found to be of general validity for waves in which dispersion and nonlinearity compete [14]. Since all dissipative effects are disregarded in (2.6) there is little chance that in practice waves will obey this equation. Including dissipation as represented by δ in (1.11) for linear waves leads in an analogous way to

$$\frac{\partial \tilde{p}}{\partial t} + c_0 \frac{\partial \tilde{p}}{\partial x} + c_0 \tilde{p} \frac{\partial \tilde{p}}{\partial x} + \frac{1}{2} \frac{c_0^3}{\omega_B} \frac{\partial^3 \tilde{p}}{\partial x^3} - \frac{1}{2} \frac{\delta c_0^2}{\omega_B} \frac{\partial^2 \tilde{p}}{\partial x^2} = 0. \quad (2.7)$$

It is assumed here that the dissipative effect represented by the last term on the left hand side of (2.4) is small in the sense that it is comparable with the nonlinear term. Solutions of (2.7), evolving from a given \tilde{p} disturbance at $t=0$ behave roughly like those of (2.6) with the important difference that (2.7) has *steady* solutions in the form of a shock wave in which the tendency to steepen is balanced by dispersion. Such shock waves, weak because the formulation of (2.6) and (2.7) rests on the assumption of moderate amplitude, have the over all thickness of

$$d \sim \frac{R_0}{\alpha^{\frac{1}{2}}} \left(\frac{p_0}{\Delta p} \right)^{\frac{1}{2}}, \quad (2.8)$$

as follows from balancing the corresponding terms on the left hand side of (2.7), and the general behaviour [9] of an undular bore. The predictions of the extensive theory [11], [15] for (2.7), have been verified experimentally in the case of bubbly flows both for initial conditions in the form of a step, see [1] as for more general forms of initial pressure profile from which for example, [16], a finite number of solitons evolves. We have seen during the discussion of waves of small amplitude in section 2 that relative translational motion leads to the existence of a second characteristic speed, c_f in (1.18), next to c_0 defined in (1.5). Moreover we found that at times $t > \tau$ the higher order terms, associated with c_f , act as diffusion in the wave governed by the lower order terms as made plausible by (1.26).

A linear wave is eventually diffused by this but in a nonlinear wave this diffusion may be balanced by nonlinear steepening. Such a balance can in bubbly liquids only occur at small pressure differences over the wave, as the following argument shows. A steady uniform wave travels in an essentially adiabatic flow with velocity U given by its Mach number as

$$U^2/c_0^2 = \frac{p_1/p_0 - 1}{1 - (p_1/p_0)^{1/\gamma}}, \quad (2.9)$$

where p_1 and p_0 are the pressures at both ends of the wave, $p_1/p_0 > 1$. When this wave is not a shock wave but is smoothed out by diffusion the small wavelets in front, see Fig. 5,

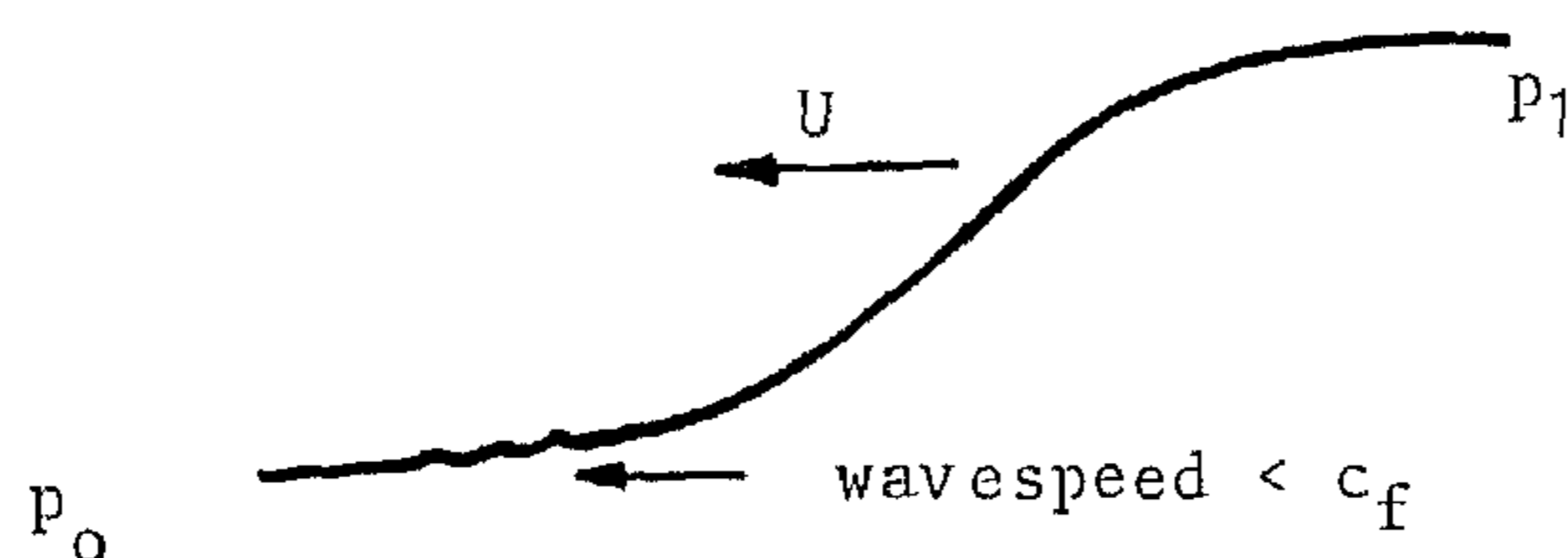


Fig.5 Smooth wave travelling from right (pressure p_1) to left (pressure p_0). The speed of the whole wave is U . The wavelets in front travel at maximum at velocity c_f .

travel at maximum at a speed c_f given in (1.17). Since these wavelets are stationary with respect to the wave as a whole, this can at most travel with $U=c_f$. Inserting c_f^2 , from (1.17) for U^2 in (2.9) gives with the aid of (1.5)

$$p_1/p_0 \leq 1 + \frac{4\alpha\gamma}{1+\gamma}. \quad (2.10)$$

For pressure ratios in the range indicated by (2.10) smooth waves have been observed experimentally [9]. The thickness is, as follows from balancing the nonlinear term $c_0^2 \partial^2 p / \partial x^2$ with the diffusion term $(c_f^2 - c_0^2) \tau \frac{\partial^2 p}{\partial x^2}$ and taking (2.10) into account, of the order of magnitude

$$\delta \sim c_0 \tau. \quad (2.11)$$

At pressure ratios larger than (2.10) the wave cannot be completely smooth. What happens provides an interesting possibility of verification of the existence of the two speeds of sound c_0 and c_f : A smooth wave of the type just discussed is preceded by a thin shock wave (with thickness of the order indicated in (2.8) which bridges the gap between the pressure p_0 and a

pressure p^+ , say, such that p_1/p^+ satisfies (2.10) with p^+ in stead of p_0 . In analogy with relaxation shocks in gasdynamics [12] this may be called a partly dispersed shockwave in contrast to a fully dispersed wave. In [9] examples of these are shown. In a shock tube of considerable length the propagation of shock waves was investigated. In Fig.6 a partly and a completely dispersed wave are shown taken from the results. The shocks are as in ordinary shock tubes produced by puncturing a seal separating a high pressure region from a low pressure region. Of course, for a dispersed or partly dispersed wave to occur the time elapsed since the puncturing of the seal must exceed τ . For times $t < \tau$ or, equivalently, distances $x < c_f \tau$, the shock wave is governed by (2.7) however with c_f instead of c_0 . Then the shock looks like in Fig. 6a.

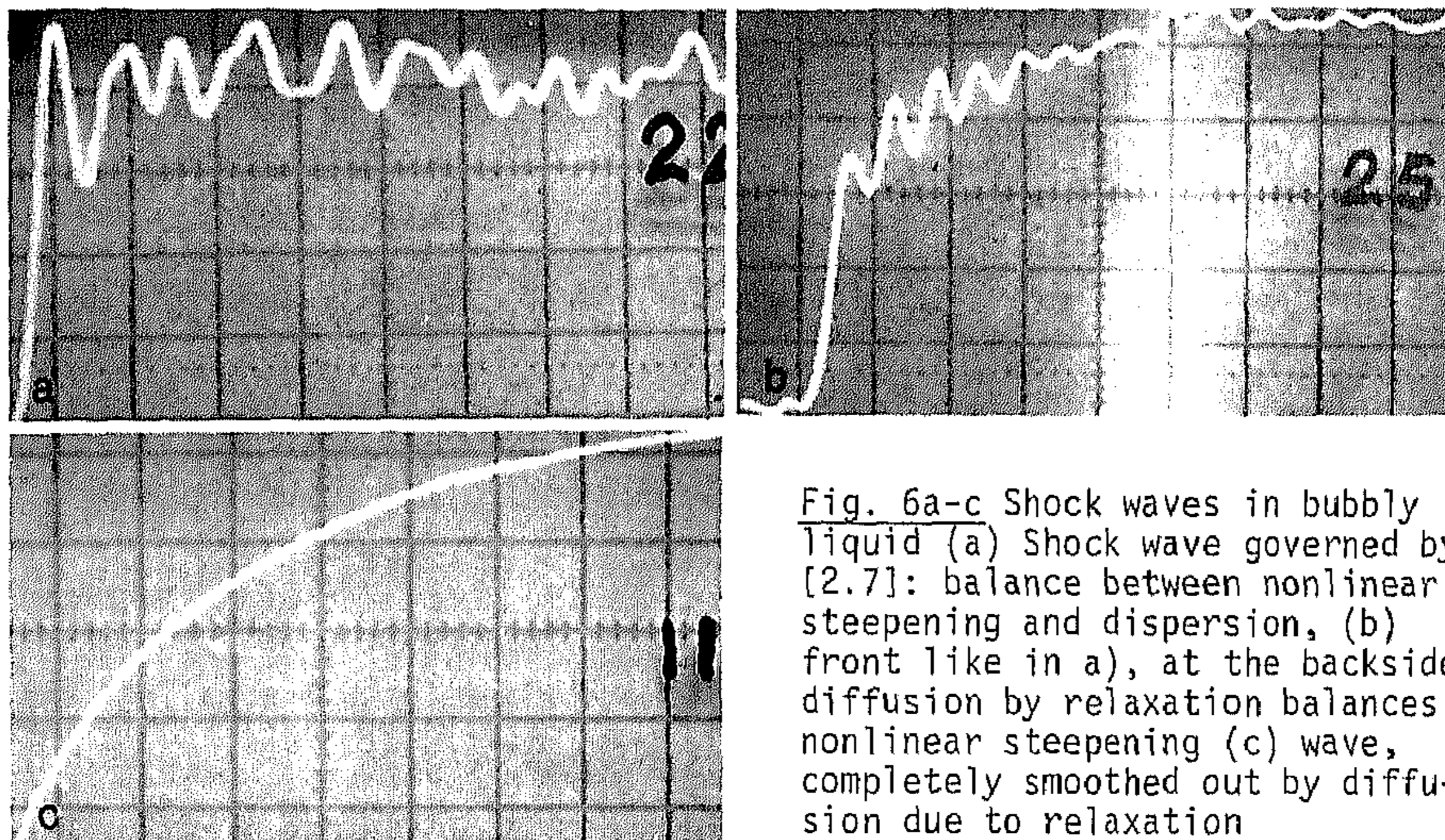


Fig. 6a-c Shock waves in bubbly liquid (a) Shock wave governed by [2.7]: balance between nonlinear steepening and dispersion, (b) front like in a), at the backside diffusion by relaxation balances nonlinear steepening (c) wave, completely smoothed out by diffusion due to relaxation

3. Conclusion

We have seen that under very restrictive circumstances acoustic waves in bubbly liquids can be analyzed. For waves of infinitesimal amplitude the restrictions are that all bubbles are spherical with the same radius in undisturbed conditions, that the concentration by volume is small, that the frictional force on a bubble can be calculated with Levich's model (equation (1.20), and the added mass is given by (1.16). For practical application it would be helpful if more were known about the shape which bubbles assume under the joint influence of surface tension, viscous and inertia forces. Dispersion and dissipation in mixtures that are not very dilute could be studied if more were known about effects of hydrodynamic and other interaction between bubbles constituting a bubbly suspension. The techniques for calculating transport properties in other areas of suspension technology [16] may be useful here. An example is the calculation [17] of the virtual mass of a bubble in the presence of other bubbles. From the foregoing sections it follows that,

apart from waves of infinitesimal amplitude, waves of moderate amplitude can be analyzed provided the effects of dispersion and dissipation are small. When this is no longer the case, very little can be said about the propagation of coherent waves because nonlinear bubble dynamics as reviewed in [4] must be coupled to nonlinear conservation equations in the mixture.

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