

# Fourier Spectral Solver for the Incompressible Navier-Stokes Equations with Volume-Penalization

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**Abstract.** In this study we use a fast Fourier spectral technique to simulate the Navier-Stokes equations with no-slip boundary conditions. This is enforced by an immersed boundary technique called volume-penalization. The approach has been justified by analytical proofs of the convergence with respect to the penalization parameter. However, the solution of the penalized Navier-Stokes equations is not smooth on the surface of the penalized volume. Therefore, it is not *a priori* known whether it is possible to actually perform accurate fast Fourier spectral computations. Convergence checks are reported using a recently revived, and unexpectedly difficult dipole-wall collision as a test case. It is found that Gibbs oscillations have a negligible effect on the flow evolution. This allows higher-order recovery of the accuracy on a Fourier basis by means of a post-processing procedure.

**Keywords:** Immersed boundary, volume-penalization, Fourier spectral methods, dipole-wall collision, vortices

## 1 Introduction

Fourier spectral methods are widely used in the CFD community to solve flow problems with periodic boundary conditions. Higher order accuracy can be achieved provided that the solution of the problem is sufficiently smooth. Moreover, these methods are fast, relatively easy to implement even for performing parallel computations. Incorporation of no-slip boundaries is, however, not straightforward. Therefore we use the volume-penalization method of Arquis & Caltagirone [2]. They model an obstacle or domain boundary as a porous object. By decreasing the permeability the penalized Navier-Stokes solution converge towards the Navier-Stokes solution with no-slip boundary conditions (see Ref.[1],[4]). A delicate issue is that a very steep velocity profile appears inside the porous obstacle. This can be a drawback for Fourier spectral methods as Gibbs oscillations might deteriorate the stability and accuracy of the scheme.

On the other hand, the numerical simulations of Kevlahan & Ghidaglia [10] and Schneider [11] have shown that it is possible to perform stable Fourier spectral computations with volume-penalization for flow around cylinders.

It is important to extend this analysis and to determine the accuracy of the Fourier spectral computations. An important issue is to fully quantify the role of the Gibbs effect on the flow dynamics: is it possible to recover higher-order accuracy of the Fourier spectral scheme? A very challenging dipole-wall collision experiment is used as a test problem. The Chebyshev spectral and finite differences computations of Clercx & Bruneau [5] have shown that a dipole colliding with a no-slip wall is a serious challenge for CFD methods. In particular, the formation and detachment of very thin boundary layers, containing high-amplitude vorticity, during the collision process and the subsequent formation of small-scale vorticity patches in the near-wall region can possibly deteriorate the accuracy of the flow computation. This dramatically affects the dynamics of the flow after the impact.

## 2 Fourier Collocation Scheme

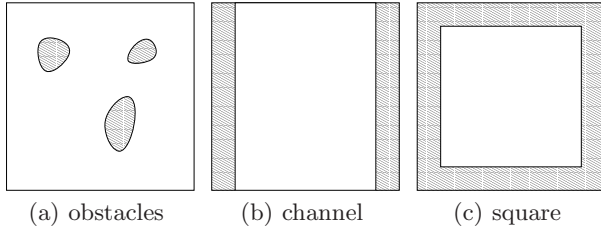
In the volume-penalization approach from Arquis & Caltagirone [2] an obstacle with a no-slip boundary is considered as a porous object with infinitely small permeability. The flow domain  $\Omega_f$  is embedded in a larger domain  $\Omega$ , such that  $\Omega_f = \Omega \setminus \overline{\Omega_s}$ , where  $\Omega_s$  represents the volume of the porous objects. The interaction between the flow and the obstacles is modelled by adding a Darcy drag term to the Navier-Stokes equations locally inside  $\Omega_s$ . This yields the penalized Navier-Stokes equations

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \nu \Delta \mathbf{u} + \frac{1}{\epsilon} H \mathbf{u} = 0 \quad \text{in} \quad \Omega \times [0, T], \quad (1)$$

where  $\mathbf{u} = (u(\mathbf{x}, t), v(\mathbf{x}, t))$  is the Eulerian velocity,  $p = p(\mathbf{x}, t)$  the scalar kinetic pressure,  $\nu$  the kinematic viscosity,  $\epsilon$  the penalization parameter and the mask function  $H$  is defined as

$$H = \begin{cases} 1 & \text{if } \mathbf{x} \in \overline{\Omega_s} \\ 0 & \text{if } \mathbf{x} \in \Omega_f. \end{cases} \quad (2)$$

Figure 1 shows some examples of different geometries. In this study we use the channel geometry (Fig. 1b) and the square bounded geometry (Fig. 1c). This allows a comparison with classical methods, such as Chebyshev spectral computations, that are well adapted to these geometries. The continuity condition  $\nabla \cdot \mathbf{u} = 0$  accompanies the penalized Navier-Stokes equations in  $\Omega$ . On the domain boundary  $\partial\Omega$  we take a periodic boundary condition, such that Fourier spectral methods can be applied. Carbou & Fabrie [4] have rigorously shown that the  $L_2$ -norm of the differences in the velocity and velocity gradients of the penalized Navier-Stokes equations and the Navier-Stokes equations with no-slip boundary conditions is proportional to  $\sqrt{\epsilon}$ . To ensure the  $C^1$  continuity of the velocity, an asymptotically thin boundary layer proportional to  $\sqrt{\nu\epsilon}$  appears



**Fig. 1.** Decomposition of a square computational domain  $\Omega$  into porous objects  $\Omega_s$  (dashed) and flow domain  $\Omega_f$  (white)

inside the obstacle. As a consequence, the limit  $\epsilon \rightarrow 0$  might result in two problems for a numerical approximation. *i*) The velocity gradient profile becomes extremely steep, which might result in Gibbs oscillations when using Fourier spectral schemes. *ii*) The validity of the  $\sqrt{\epsilon}$  error bound might be affected in case the details of the penalization boundary layer are not resolved. Furthermore, due to the slow convergence rate proportional to  $\sqrt{\epsilon}$  the time step has to be decoupled from the penalization parameter in order to achieve a stiffly stable time integration scheme. In this study this is achieved by a collocation approach of the penalized Navier-Stokes equations. A third-order extrapolated backward differentiation scheme (BDF) with exact differentiation of the diffusion term is applied,

$$\alpha_0 \mathbf{u}_N^{n+1} + \delta t \mathbf{L}_N(\mathbf{u}_N^{n+1}) = - \sum_{h=1}^3 (\delta t \beta_h \mathbf{G}_N(\mathbf{u}_N^{n+1-h}) + \alpha_h \mathbf{u}_N^{n+1-h}) e^{h\delta t \nu \Delta} \quad (3)$$

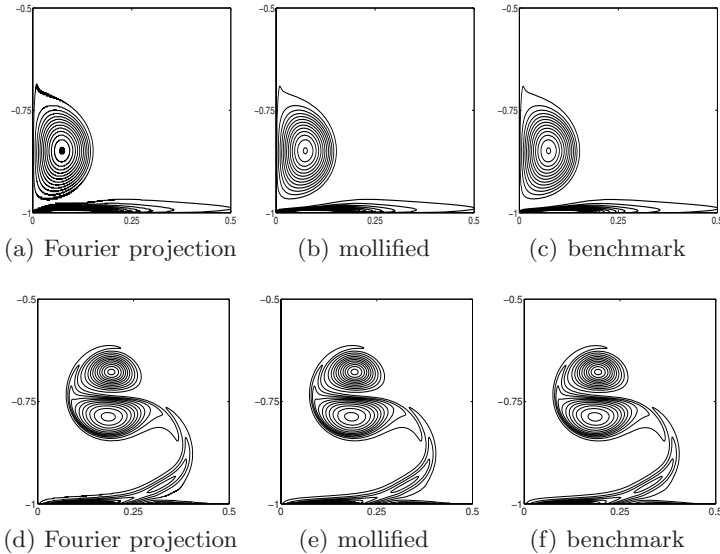
$$\tilde{\nabla} \cdot \mathbf{u}_N = 0 \quad (4)$$

where  $\mathbf{G}_N = (\mathbf{u}_N \cdot \tilde{\nabla})\mathbf{u}_N + \tilde{\nabla} p_N$ ,  $\mathbf{L}_N = \frac{1}{\epsilon} H \mathbf{u}_N$ ,  $e^{h\delta t \nu \Delta}$  is the semi-group of the heat kernel and  $\delta t$  the time step. A tilde marks that collocation derivatives are used here in order to distinguish from the exact Galerkin derivative. The coefficients  $\alpha_j$  and  $\beta_j$  are determined by third-order backward differentiation and extrapolation, respectively. The right-hand side of Eq. (3) is computed in transform space (Ref. [9]). Aliasing is avoided by applying the zero-padding technique introduced by Orszag, generally referred to as the 2/3-rule (see, for details Ref. [3]). Three forward and four backward FFTs are required in total.

### 3 Results

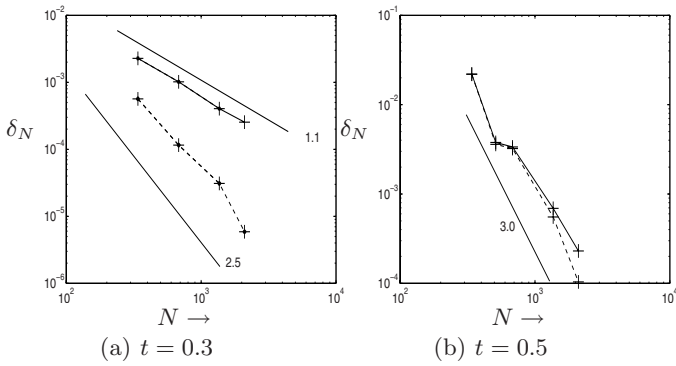
We consider two set of experiments. The first set concerns a normal dipole-wall collision where the dipole traverses from the center of the domain perpendicular to the wall. For these experiments a channel geometry Fig. 1b is considered. A high-resolution benchmark computation is achieved by using Chebyshev expansion in the direction perpendicular to the wall and Fourier expansion in the

periodic direction. For details of the Chebyshev-Fourier code see Kramer [8]. The second test problem is an oblique dipole-wall collision experiment in a square bounded geometry Fig. 1c. The benchmark computation for this case is conducted with Chebyshev expansions in both directions [6]. For details of the initial conditions and setup of the simulations see Clercx & Bruneau [5]. Fig. 2 show the result of a normal dipole-wall experiment at  $Re = 1000$ . The Reynolds number  $Re$  is based on the total kinetic energy of the flow  $E(t) = \frac{1}{2} \int_{-1}^1 \int_{-1}^1 \mathbf{u}^2(\mathbf{x}, t) dx dy$ , and the half width of the domain  $W = 1$ . As the dipole impinges the wall Gibbs oscillations in the vorticity isolines become apparent. This can be related to steep gradients of the velocity inside the porous obstacle. Note that only one half of the domain is shown because of the symmetry of a normal dipole-wall collision. The oscillations are more pronounced near the wall than in the interior of the flow domain. As the vortex moves into the interior of the flow domain at  $t = 0.5$  the wiggles disappear (see Fig. 2d). This indicates that the observed oscillations do not have a serious dynamical effect on the evolution of the dipole-wall collision. Fig 2 b,e show the result of a post-processing or smoothing technique developed by Tadmor & Tanner [12]. The mollified result in Fig. 2b shows that the Gibbs oscillations can safely be removed from the vorticity isolines. To determine the accuracy of the scheme we consider the normalized  $L_2$ -error  $\delta_N$  of the vorticity. The error is computed in a box  $\mathbf{x} \in [-0.99, 0.99]$  that is slightly smaller than



**Fig. 2.** Vorticity isolines of a normal dipole-wall collision with  $Re = 1000$  at  $t = 0.3$  (a,b,c) and  $t = 0.5$  (d,e,f).  $1364 \times 1364$  active Fourier modes,  $\delta t = 10^{-5}$  and  $\epsilon = 2.5 \times 10^{-5}$  are used. Benchmark computation is conducted with 1024 Chebyshev modes and 2048 active Fourier modes and  $\delta t = 10^{-5}$ . Contour levels are drawn for -270...,-50,-30,-10,10,30,50,..270.

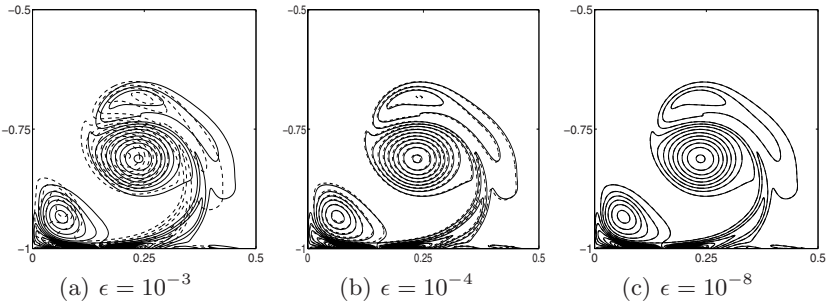
the computational domain  $\mathbf{x} \in [-1, 1]$ . This is motivated by the fact that the post-processing procedure of Tadmor & Tanner [12] is only second order in the close vicinity of a discontinuity, but is higher-order accurate at a sufficient distance. From Fig. 3 it can be deduced that the truncation error of the Fourier projection shows only first order behavior in case Gibbs oscillations are present around  $t = 0.3$ . The mollification procedure of Tadmor & Tanner [12] recovers, however, a higher-order accuracy rate of the Fourier spectral scheme. After the first collision around  $t = 0.5$  the Fourier projection without mollification shows a higher-order decay rate of the error in the vorticity. Mollification slightly improves the accuracy see Fig. 3b. An important issue is to check if it is possible



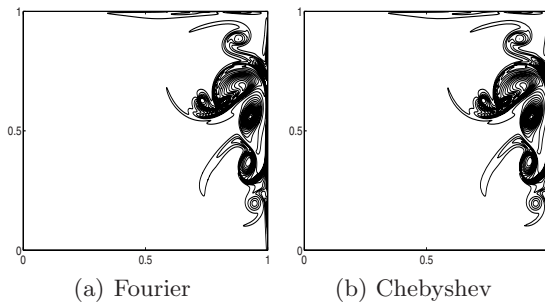
**Fig. 3.** Truncation error  $\delta_N$  computed with respect to the highest resolution run with  $N = 2730$  active Fourier modes in both directions. The time step  $\delta t = 2 \times 10^{-5}$  is fixed for all runs. The error of the Fourier projection (solid) and the mollified result (dashed).

to find a balance between the truncation error and the penalization error. Fig. 4 shows a set of computations where a balance between both error sources is achieved. By choosing an appropriate number of active Fourier modes  $N$  and value of the penalization parameter  $\epsilon$ , the vorticity isolines coincide (see Fig. 4c). By keeping  $\epsilon$  fixed and increasing  $N$  until a saturation level is reached in the error in the vorticity versus the high-resolution benchmark computation one obtains a measure for the penalization error. The decay rate (not shown here) is proportional to  $\epsilon^{0.7}$ , which is consistent with the theoretical upperbound proportional to  $\sqrt{\epsilon}$  derived by Carbou & Fabrie [4].

To increase the complexity of the flow problem we consider an oblique collision in the square bounded geometry Fig. 1c at a higher Reynolds number  $Re = 2500$ . For this problem it is necessary to perform mollification in both directions. Fig. 5 shows that the Fourier spectral scheme with post-processing is able to compute the isolines of the small-scale structures in this flow problem correctly as well. Some global measures are presented in Fig. 6. The total enstrophy  $Z(t)$  is the  $L_2$ -norm of the vorticity in the flow domain  $\Omega_f$ . The angular momentum of the

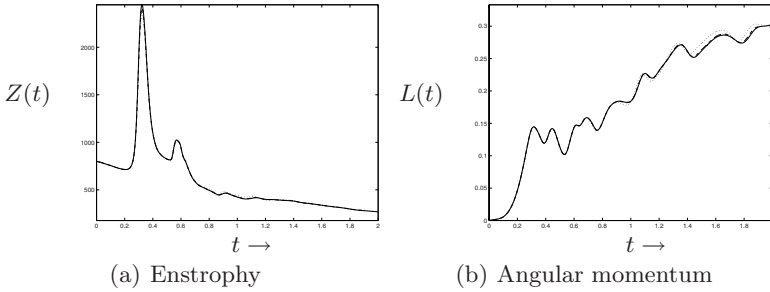


**Fig. 4.** Contour plots (dashed) of the vorticity after the second collision at  $t = 0.8$  for different values of the penalization parameter  $\epsilon$  with respect to the benchmark simulation (solid). Contour levels are drawn for  $-270, -250, \dots, -50, -30, -10, 10, 30, 50, \dots, 250, 270$ . Number of active Fourier modes  $N = 682$  (a),  $N = 1364$  (b) and  $N = 2730$  (c). Benchmark computation (solid) is conducted with 1024 Chebyshev modes and 2048 active Fourier modes and  $\delta t = 10^{-5}$ .



**Fig. 5.** Vorticity isolines of an oblique dipole-wall collision at  $t = 0.6$  obtained with a Fourier spectral method with volume-penalization with 2730 active Fourier modes and  $\delta t = 2 \times 10^{-5}$  (a) Benchmark computation conducted with 640 Chebyshev modes in both directions and  $\delta t = 1.25 \times 10^{-5}$  (b) Contour levels are drawn for  $-270, -250, \dots, -50, -30, -10, 10, 30, 50, \dots, 250, 270$

flow is computed with respect to the center of the container. It is found that the maximum error in the total enstrophy is smaller than one percent. Recall that the quality of post-processing procedure of Tadmor & Tanner [12] is only second order in the vicinity of the wall and cannot be applied on the wall itself. Therefore the accuracy in the total enstrophy is slightly limited. Clercx & Bruneau [5] found that is difficult to achieve convergence in the total angular momentum, especially after the second collision. Fig. 6b shows a good agreement between the high-resolution benchmark computation with 640 Chebyshev modes in both directions and the Fourier spectral computation with  $N = 2730$  active Fourier modes. Note that the Fourier spectral method corresponds to an equidistant grid, while the Chebyshev method uses a Gauss-Lobatto grid that is strongly



**Fig. 6.** Total enstrophy  $Z(t)$  and angular momentum  $L(t)$  with respect to the center of the domain versus time. Chebyshev benchmark computation 640 Chebyshev modes and  $\delta t = 1.25 \times 10^{-5}$  (solid), Fourier spectral computation with volume-penalization obtained with  $N = 1364$  (dots) and  $N = 2730$  active Fourier modes (dashed). For both runs  $\delta t = 2 \times 10^{-5}$  and penalization parameter  $\epsilon = 10^{-8}$  is used.

refined in the corners of the domain (see Ref. [6]). Therefore it is not surprising that the required number of active Fourier modes is larger than the number of Chebyshev modes to achieve mode convergence for this specific problem.

### 4 Conclusion and Discussion

The numerical results of a normal and an oblique dipole-wall collision demonstrate that it is possible to conduct stable and accurate Fourier spectral computations using volume-penalization. Gibbs oscillations are present in the Fourier projections but are not dynamically active. In addition, the accuracy can be recovered using post-processing. Similar results are reported for Fourier spectral computations of other non-smooth problems (see Ref. [7]). For instance on the simulation of shock development in the 2D Euler equations. In our case it is, however, not necessary to apply any artificial viscosity to stabilize the scheme. The penalization error can be controlled by an appropriate choice of the penalization parameter. The theoretical upperbound of the penalization error obtained by Carbou & Fabrie [4] proportional to  $\sqrt{\epsilon}$  is confirmed. It actually scales slightly better proportional to  $\epsilon^{0.7}$  for the dipole-wall problem. This is a remarkable result since the asymptotically thin boundary layer that appears inside the obstacle is unresolved in our simulations. This result may be related to the formal expansion of penalized Navier-Stokes solution (see Ref. [4] for details). Only the higher-order terms in the  $\sqrt{\epsilon}$  expansion rely on the details of the penalization boundary layer. As a consequence, a  $\sqrt{\epsilon}$  accuracy of the penalized Navier-Stokes equations is possible without computing the boundary layer components. In our opinion the combination of Fourier spectral methods and volume-penalization can be useful to pursue DNS of turbulence in complex geometries. Note that this can easily be achieved by a different choice of the mask function.

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