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Super-optimization for a class of four-block problems

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Abstract

The paper deals with super-optimization of the four-block problem via a dimension reducing technique based on the so-called equalizer principle.

1. Introduction

Optimization of control system performance with respect to the so-called $\mathcal{H}^\infty$ norm in the last decade has developed into a strikingly lively research field. A reason for this is the desire for design procedures for robust control systems. This leads to optimization problems whose criterion is the $\mathcal{H}^\infty$ norm of a rational transfer function. In a model-matching formulation the task becomes to find in a set

$$\Sigma := \{H \in \mathcal{H}^\infty_{m\times n} \mid H = F + BQC, Q \in \mathcal{H}^\infty_{p\times q}\} \qquad (1)$$

a function H_o of minimal $\mathcal{H}^\infty$ norm. Here $F \in \mathcal{H}^\infty_{m\times n}$, $B \in \mathcal{H}^\infty_{m\times p}$, $C \in \mathcal{H}^\infty_{q\times n}$ are fixed rational matrices depending only on the data of the original problem. Without loss of generality we assume that B is tall and of full column rank, and C wide and of full row rank. The case where B is strictly tall and C strictly wide corresponds to a genuine four-block problem, while the case where B and C both are square represents a one-block problem. The intermediate case with either B or C square, is a two-block problem.

In the multivariable case an $\mathcal{H}^\infty$ optimal compensator is in general far from unique. Simple examples suggest that the performance may be further improved by taking into consideration also sub-dominant singular values. An optimality criterion suitable for this was introduced in [14], where instead of minimizing only the largest singular values, the sequence $\{s_1^\infty(H), s_2^\infty(H), s_3^\infty(H), \ldots, s_n^\infty(H)\}$ of essential suprema of the n singular values of $H \in \Sigma$, arranged in decreasing order, is minimized with respect to the lexicographic ordering. This form of optimization has been called *super-optimization*, and has been studied in e.g. [14], [1], [3], [5], [12], [13], [8], [11], [7], and [9]. Most of the works done this far treat the one-block problem, but in some instances, e.g. [5], [7], [3], and [4], the two-block problem has been considered. Four-block extensions have recently been reported in [4], and [10].

A super-optimization method applicable to a reasonable large class of genuine four-block problems will briefly be outlined. It successively reduces the original problem to super-optimizations problems with steadily fewer singular values, until finally all freedom is exhausted. Each step amounts to an ordinary $\mathcal{H}^\infty$ optimization of the largest remaining singular value, and a subsequent removal of this singular value.

2. $\mathcal{H}^\infty$ optimization

For the $\mathcal{H}^\infty$ optimization part we need to find a true optimal solution, instead of only a sub-optimal. The reason for this is that the degree structure of an optimal solution in general differs from that of a sub-optimal. A method well suited for this purpose is the so-called *polynomial approach* by Kwakernaak [6]. This approach allows one, via two polynomial J-spectral factorizations, to compute not only sub-optimal, but indeed optimal solutions. Optimal solutions commonly have a reduced degree when compared with sub-optimal solutions. Moreover, a reduced degree solution admits a spectral density Φ such that the following so-called *equalizer principle* holds:

Suppose that $H_o = F + BQ_oC \in \Sigma$ is equalizing, i.e.

$$H_o^* H_o = \lambda_o^2 I - L_o^* L_o$$

for some nonnegative constant λ_o and some rational rank deficient matrix L_o, and that H_o and L_o minimize

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}\{[H^*(e^{i\theta})H(e^{i\theta}) + L^*(e^{i\theta})L(e^{i\theta})]\Phi(e^{i\theta})\}d\theta$$

with respect to all transfer matrices $H = F+BQC \in \Sigma$, and all rank deficient L. Then H_o is $\mathcal{H}^\infty$-optimal and $\|H_o\|_\infty = \lambda_o$.

For an arbitrary Φ, the set of transfer matrices H_o satisfying the equalizer principle either is empty or contains precisely the $\mathcal{H}^\infty$-optimal transfer matrices [9]. In the non-empty case Φ is called a *distinguishing* spectral density. It may always be chosen to have rank one, and is rather easely computed from any optimal H_o [10]. The class of problems Σ admitting a distinguishing spectral density contains all those for which the so-called Parrot type optimality of [2] does not occur.

3. Removing the largest singular value

In addition to the role it plays for the optimality criterion, a distinguishing spectral density exhibits features of considerable value for solving the super-optimization problem (cf. [5], [7], [9] and [10]). We briefly indicate the idea.

Suppose Σ admits a distinguishing spectral density. Let H_o be an $\mathcal{H}^\infty$-optimal solution in Σ, and let Φ be a corresponding distinguishing spectral density of rank one. Introduce the set $\mathcal{D}(\Sigma;\Phi) = \{H \in \Sigma \mid H^*H\Phi = H_o^*H_o\Phi\}$. From the equalizer principle it follows that H belongs to $\mathcal{D}(\Sigma;\Phi)$ whenever it is $\mathcal{H}^\infty$-optimal in Σ. Furthermore, it may be shown that the set $\mathcal{D}(\Sigma;\Phi)$ is a coset of type (1). More specifically, $\mathcal{D}(\Sigma;\Phi) = H_o + BQC$, where $\mathcal{Q}$ is a subspace of $\mathcal{H}^\infty_{p\times q}$ having one of the five forms $\mathcal{H}^\infty_{p\times q}$, $\mathcal{H}^\infty_{p\times(q-1)}P$, $R\mathcal{H}^\infty_{(p-1)\times q}$, $R\mathcal{H}^\infty_{(p-1)\times(q-1)}P$ or $\{0\}$, with $R \in \mathcal{H}^\infty_{m\times(p-1)}$ of full column rank, and $P \in \mathcal{H}^\infty_{(q-1)\times n}$ of full row rank. For $n > 1$, introduce a factorization $\Phi = \phi\phi^*$, where $\phi = V_a g$, with V_a a rational inner vector and g a rational scalar valued function. By completing V_a we obtain a square inner matrix $V = [V_a \quad V_b] \in \mathcal{H}^\infty_{n\times n}$, where V_b is an inner complement of V_a. Consider the set

$$\hat{\Sigma} = \mathcal{D}(\Sigma;\Phi)V_b$$

From the fact that $\mathcal{D}(\Sigma;\Phi)$ is a coset of type (1), it is easely concluded that the same holds for $\hat{\Sigma}$. But note that $\hat{\Sigma}$ is a coset in $\mathcal{H}^{\infty}_{m\times(n-1)}$ instead of $\mathcal{H}^{\infty}_{m\times n}$. The usefulness of $\hat{\Sigma}$ is further accentuated by the fact that the rule $H \rightarrow HV_b$ determines a one to one correspondence between the super-optimal functions in Σ and the super-optimal functions in $\hat{\Sigma}$. Hence if we find a super-optimal HV_b in $\hat{\Sigma}$, this immediately gives H as a super-optimal function in Σ.

4. Super-optimization

By the previous section, the super-optimization problem Σ has been reduced to that of finding a super-optimal solution in $\hat{\Sigma}$, a coset of precisely the same type as Σ, but consisting of functions with only $n-1$ singular values instead of n. We may therefore continue in the obvious way: Denote $\hat{\Sigma}$ by Σ_1. Apply the polynomial approach to Σ_1 to find a $\mathcal{H}^{\infty}$-optimal solution. If Σ_1 also admits a distinguishing spectral density, then as before we obtain a new coset $\Sigma_2 = \hat{\Sigma}_1 \in \mathcal{H}^{\infty}_{m\times(n-2)}$, having the same structure (1) as Σ and Σ_1, but consisting of functions with only $n-2$ singular values to be considered. Moreover, super-optimization over Σ_1 is reduced to super-optimization over Σ_2, and a super-optimal solution in Σ_2 gives in the obious way rise to a corresponding super-optimal solution in Σ. By repeating this we sooner or later obtain a coset Σ_k of type (1), together with a $\mathcal{H}^{\infty}$ optimal solution H_k in Σ_k and a corresponding solution H^k in Σ such that one following situations holds:

(1): Σ_k admits a distinguishing spectral density Φ_k but the set $\mathcal{D}(\Sigma_k;\Phi_k)$ contains only H_k. H^k is then the unique super-optimal solution in Σ.

(2): Σ_k admits no distinguishing spectral density, but is a scalar problem. Since all singular values have been minimized H^k is a super-optimal solution in Σ. Due to the lack of a distinguishing spectral density it need not however be unique.

(3): Σ_k admits no distinguishing spectral density, and is non-scalar. Optimality of H^k in Σ can then only be guaranteed with repsect to the $k+1$ largest singular values.

Situation (3) never occurs in one-block problems [9]. Moreover, two- and four-block problems arising in practical applications often admit distinguishing spectral densities (cf. [5], [6]). It is therefore believed that cases (1) and (2) define a reasonably large and interersting class of superoptimizable four-block problems.

5. Conclusions

The proposed procedure allows computation of super-optimal solutions for a large class of model-matching problems, including all one-block problems and a significant portion of genuine two- and four-block problems. For remaining problems optimality can, by the present method, be achieved only with respect to some first k largerst singular.

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